OPTIMALITY OF PAYOFFS IN LÉVY MODELS

ERNST AUGUST VON HAMMERSTEIN[†], EVA LÜTKEBOHMERT[†],
LUDGER RÜSCHENDORF[‡], VIKTOR WOLF[‡,*]

ABSTRACT. In this paper, we determine the lowest cost strategy for a given payoff in Lévy markets where the pricing is based on the Esscher martingale measure. In particular, we consider Lévy models where prices are driven by a normal inverse Gaussian (NIG)- or a variance Gamma (VG)-process. Explicit solutions for cost-efficient strategies are derived for a variety of vanilla options, spreads, and forwards. Applications to real financial market data show that the cost savings associated with these strategies can be quite substantial. The empirical findings are supplemented by a result that relates the magnitude of these savings to the strength of the market trend. Moreover, we consider the problem of hedging efficient claims, derive explicit formulas for the deltas of efficient calls and puts and apply the results to German stock market data. Using the time-varying payoff profile of efficient options, we further develop alternative delta hedging strategies for vanilla calls and puts. We find that the latter can provide a more accurate way of replicating the final payoff compared to their classical counterparts.

1. INTRODUCTION

In this paper, we study optimal investment decisions in incomplete markets where the prices of risky assets are driven by Lévy processes. In particular, we solve for the investment strategy with minimal cost that achieves a given payoff distribution. This strategy is called cost-efficient with respect to the given distribution. The problem of determining efficient strategies for a given payoff distribution was introduced by Dybvig (1988a,b) in the case of a discrete arbitrage-free and complete binomial model. Here the aim is to determine an investment strategy C that achieves the same payoff distribution F as a given claim X but at the same time minimizes the price. In a series of papers Bernard & Boyle (2010), Bernard *et al.* (2014), and Vanduffel *et al.* (2009, 2012) give a solution of the efficient claim problem in a fairly general setting. They calculate in explicit form efficient strategies for several options in Black–Scholes markets. Dana (2005), Föllmer & Schied (2004), Jouini & Kallal (2001), and Rüschendorf (2012) consider the extended problem to optimize the price under the condition that $C \leq_{\text{cx}} X$, i.e., C is smaller in convex

[†]DEPARTMENT OF QUANTITATIVE FINANCE, UNIVERSITY OF FREIBURG, PLATZ DER ALTEN SYNAGOGE, D-79098 FREIBURG IM BREISGAU, GERMANY

[‡]DEPARTMENT OF MATHEMATICAL STOCHASTICS, UNIVERSITY OF FREIBURG, ECKERSTRASSE 1, D-79104 FREIBURG IM BREISGAU, GERMANY

E-mail address: ernst.august.hammerstein@finance.uni-freiburg.de, eva.luetkebohmert@finance.uni-freiburg.de, ruschen@stochastik.uni-freiburg.de, wolf@stochastik.uni-freiburg.de.

Date: September 26, 2014.

2000 *Mathematics Subject Classification.* 60G51;60E15.

Key words and phrases. cost-efficient strategies; optimal payoffs; Lévy model; Esscher transform; delta hedging.

The first two authors gratefully acknowledge the financial support by the Excellence Initiative through the project "Pricing of Risk in Incomplete Markets" within the Institutional Strategy of the University of Freiburg.

*Corresponding author

order than X. The solution of this extended problem turns out to be identical to the solution of the efficient claim problem as formulated above. In general Lévy markets where the arbitrage-free pricing is based on the Esscher transform, Vanduffel *et al.* (2009) prove that path-dependent payoffs are inefficient w.r.t. the convex order $\leq_{\mathrm{cx}}$ and can be improved by conditioning on the price density process. The enhanced payoffs then are path-independent.

In this paper, we apply the above results on efficient payoffs to certain classes of exponential Lévy models. In particular, we consider variance Gamma (VG) and normal inverse Gaussian (NIG) processes and contrast them with the classical Black–Scholes model. We hypothesize that agents in the market agree on the Esscher martingale measure for pricing and suppose there exists a constant risk-free interest rate. For a variety of relevant financial derivatives we explicitly derive cost-efficient strategies. Based on the inefficiency results for path-dependent options mentioned above, we concentrate in this paper on path-independent payoffs. Further, we provide a cost-efficient version of the put-call parity stating that the cost-efficient strategy corresponding to a portfolio of a long call and a short put agrees with the cost-efficient strategy for a long forward.

Moreover, we investigate the impact of market behaviour on the level of cost reduction that can be achieved by switching to the efficient strategies. Roughly speaking, the overall market behaviour is characterized by the sign of the risk-neutral Esscher parameter, whereas the size of its absolute value determines the strength of the market trend. We show that the price differences between inefficient and optimal strategies are increasing when the market trend becomes more pronounced.

Furthermore, we explicitly determine hedging strategies for cost-efficient payoffs. Specifically, we provide formulas for delta hedging of cost-efficient strategies corresponding to European call and put options. This is particularly important for practical applications as the pricing formulas for cost-efficient strategies themselves are still unsatisfying if no hedging strategies exist. In a practical application using German stock price data we demonstrate that all derived formulas are numerically tractable and that cost-efficient puts can be hedged as accurately as the corresponding vanilla puts. For the latter, as well as for vanilla calls, we further develop alternative delta hedging strategies based on a series of efficient puts resp. calls with decreasing times to maturity. We prove that the magnitude of the associated deltas is in almost all cases smaller than that of the classical deltas. This suggests that also the hedging errors arising in discrete delta hedging should be smaller for the alternative than for the standard hedging strategies. Using the aforementioned market data, we demonstrate that this is indeed the case: The accumulated absolute hedge errors obtained from the alternative delta hedging strategy for vanilla puts on two German stocks are always smaller than those of the classical one. This shows that cost-efficient options not only provide a cheaper way of realizing a certain payoff distribution, but can also help to more accurately hedge related products.

The paper is structured as follows: Section 2 restates the basic results on price bounds and efficient claims for the case of Lévy models when pricing is based on the Esscher martingale measure and explains how the efficiency loss is influenced by market behaviour. Section 3 contains explicit calculations of cost-efficient strategies for a variety of derivative contracts using estimated parameters from German stock price data and discusses the put-call parity. Formulas for delta hedging of cost-efficient and vanilla call and put options and applications to put options on two German stocks are presented in Section 4 while Section 5 concludes. Proofs as well as detailed derivations of the risk-neutral Esscher parameters for the different Lévy models considered in the paper are provided in two appendices.

2. Cost-efficient strategies

2.1. **Cost-efficiency.** Consider a financial market on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ satisfying the usual conditions with finite trading horizon $[0,T]$, $T \in \mathbb{R}_+$. We use the simplifying notation $(Y_t)_{t\geq 0} := (Y_t)_{t\in[0,T]}$ for an arbitrary stochastic process Y on $[0,T]$. Further, we assume that the financial market is incomplete, but free of arbitrage, perfectly liquid and frictionless. Let $(S_t)_{t\geq 0}$ denote the price process of a risky asset and let r be the constant deterministic risk-free interest rate. We assume that all agents in the market agree on the same state price density $(Z_t)_{t\geq 0}$ for pricing, where the process $(Z_t)_{t\geq 0}$ is chosen such that the discounted process $(e^{-rt}Z_tS_t)_{t\geq 0}$ is a P-martingale. We assume this general setup throughout the whole paper and indicate whenever we concretize it.

In this paper, we are interested in strategies of European type yielding a given terminal payoff distribution, and among those, especially in the ones with minimal and maximal cost. Here the cost of a strategy with a given terminal payoff X_T, $T > 0$, is defined as the discounted expected payoff w.r.t. the state price density Z_T, i.e.,

$$c(X_T) = e^{-rT}E\big[Z_TX_T\big], \tag{2.1}$$

provided that the expectation exists. Note that here and in the following the expectation $E[\cdot] = E_P[\cdot]$ is always calculated w.r.t. the real-world measure P if not stated otherwise.

Definition 2.1 (Cost-efficient and most-expensive strategies).

(a) A strategy (or payoff) $\underline{X}_T \sim G$ is called *cost-efficient* w.r.t. the payoff-distribution G if any other strategy X_T that generates the same payoff-distribution G costs at least as much, that is,

$$c(\underline{X}_T) = e^{-rT}E[Z_T\underline{X}_T] = \min_{\{X_T\sim G\}} e^{-rT}E[Z_TX_T]. \tag{2.2}$$

(b) A strategy (or payoff) $\overline{X}_T \sim G$ is called *most-expensive* w.r.t. the payoff-distribution G if any other strategy X_T that generates the same payoff-distribution G costs at most as much, that is,

$$c(\overline{X}_T) = e^{-rT}E[Z_T\overline{X}_T] = \max_{\{X_T\sim G\}} e^{-rT}E[Z_TX_T]. \tag{2.3}$$

(c) The *efficiency loss* of a strategy with payoff $X_T \sim G$ at maturity T is defined as

$$c(X_T) - c(\underline{X}_T).$$

As a consequence of the definition one obtains (see Bernard *et al.* (2014)) that the net profit from investing into the cost-efficient strategy $\underline{X}_T$ is greater than that of X_T in the stochastic order $\leq_{\mathrm{st}}$, i.e.,

$$X_T - c(X_T)e^{rT} \leq_{\mathrm{st}} \underline{X}_T - c(\underline{X}_T)e^{rT}. \tag{2.4}$$

Thus, investors who only care about the distribution of terminal wealth will always prefer the cost-efficient strategies as the latter yield higher outcomes with greater probabilities.

From Eq. (2.2) we observe that the cost of a strategy with a given payoff distribution G is minimized when the expectation of the product of payoff X_T and the state price density Z_T is minimized. Since the marginal distributions G and F_{Z_T} have to be kept fixed, the problem of minimizing the cost is equivalent to finding a strategy $\underline{X}_T \sim G$ such that the covariance $\mathrm{Cov}(\underline{X}_T, Z_T)$ is minimized (we implicitly assume here that $E[X_T]$ and $E[X_TZ_T]$ are finite such that all expressions are well-defined). Then it is an immediate consequence of the Hoeffding formula (see Lehman (1966), Lemma 2) that the minimal covariance is obtained by setting the joint distribution of $\underline{X}_T$ and Z_T equal to the lower Fréchet bound, i.e., $\underline{X}_T$ and Z_T have to be countermonotonic. If at least one of the marginal distributions is continuous, this is equivalent to the fact that $\underline{X}_T$ a.s. is a non-increasing function of Z_T. Analogously, the most-expensive payoff is obtained by choosing the payoff $\overline{X}_T$ to be comonotonic with Z_T, i.e., $\overline{X}_T$ is a.s non-decreasing in Z_T. More precisely, the following result holds (see Bernard *et al.* (2014), Proposition 3, and the literature mentioned in the introduction).

Theorem 2.1 (Cost-efficient payoffs and price bounds). *For any given payoff X_T with distribution G it holds:*

(a)

$$\begin{aligned} \inf_{\{X_T\sim G\}} c(X_T) &= e^{-rT}\int_0^1 F_{Z_T}^{-1}(y)\, G^{-1}(1-y)\, dy \\ \sup_{\{X_T\sim G\}} c(X_T) &= e^{-rT}\int_0^1 F_{Z_T}^{-1}(y)\, G^{-1}(y)\, dy \end{aligned} \tag{2.5}$$

(b) *A random payoff $X_T \sim G$ is cost-efficient if and only if X_T and Z_T are countermonotonic. $X_T \sim G$ is most-expensive iff X_T and Z_T are comonotonic.*

(c) *If the distribution F_{Z_T} of the state price density Z_T is continuous, then*

$$\begin{aligned} &\underline{X}_T = G^{-1}(1-F_{Z_T}(Z_T)) \quad \text{is cost-efficient and} \\ &\overline{X}_T = G^{-1}(F_{Z_T}(Z_T)) \qquad \text{is most-expensive.} \end{aligned}$$

Explicit formulas for cost-efficient and for most-expensive payoffs can also be given in explicit form without assuming continuity of F_{Z_T} by means of the distributional transform (see Rüschendorf (2012, 2013)).

2.2. **Cost-efficiency in Lévy markets.** Suppose now that the asset price process $(S_t)_{t\geq 0} = (S_0 e^{L_t})_{t\geq 0}$ is driven by a Lévy process $(L_t)_{t\geq 0}$. Apart from the cases where $(L_t)_{t\geq 0}$ either is a Brownian motion or a Poisson process, such a Lévy market setting is incomplete and there exist infinitely many risk-neutral martingale measures. Thus, one has to rely on additional optimality criteria, preference assumptions, or calibration results to real data from option markets to choose a specific martingale measure for pricing. Throughout this paper we will use the Esscher martingale measure for this purpose which was introduced to option pricing by Gerber & Shiu (1994). Apart from the fact that the Esscher transform provides a transparent, unambigious, and numerically very tractable way to obtain a risk-neutral measure, this choice can also be motivated from a theoretical point of view. The Esscher approach can be obtained in a natural way from the assumption of the existence of a competitive equilibrium with respect to a power utility function (see e.g. Keller (1997), Chapter 1.4.3). Moreover, Chan (1997), Esche & Schweizer (2005), Goll & Rüschendorf (2002), and Miyahara (1999) prove that the Esscher martingale measure of the exponential transform of $(L_t)_{t\geq 0}$ coincides with the minimal entropy martingale measure. An extension of this result and a discussion on Esscher transforms of exponential Lévy models can be found in Hubalek & Sgarra (2006). Another useful feature of Esscher transforms is the preservation of the Lévy property: $(L_t)_{t\geq 0}$ remains a Lévy process under any Esscher measure Q^θ to be defined below. To properly define the Esscher martingale measure, the following basic assumption on the driving Lévy process $(L_t)_{t\geq 0}$ is made for the remainder of the paper.

Assumption 2.1. The random variable L_1 is nondegenerate and possesses a moment generating function (mgf) $M_{L_1}(u) = E[e^{uL_1}]$ on some open interval (a, b) with $a < 0 < b$ and $b - a > 1$.

This condition turns out to be necessary (but not always sufficient) for the existence of the risk-neutral Esscher measure.

Definition 2.2. We call an *Esscher transform* any change of P to a locally equivalent measure Q^θ with a density process $Z_t^\theta = \frac{dQ^\theta}{dP}|_{\mathcal{F}_t}$ of the form

$$Z_t^\theta = \frac{e^{\theta L_t}}{M_{L_t}(\theta)}, \tag{2.6}$$

where M_{L_t} is the mgf of L_t as before, and $\theta \in (a,b)$.

To emphasize the dependence of the Esscher measure Q^θ and its density process $(Z_t^\theta)_{t\geq 0}$ on the parameter θ, we shall always add the latter as superscript. Similarly, we will indicate by $E_\theta[\cdot]$ that the expectation is calculated with respect to Q^θ. Using the stationarity and independence of the increments of every Lévy process $(L_t)_{t\geq 0}$, which also imply the relation

$$M_{L_t}(u) = M_{L_1}(u)^t \quad \text{for all } u \in \mathbb{R} \text{ and } t \geq 0,$$

it is not hard to show that $(Z_t^\theta)_{t\geq 0}$ indeed is a density process for all $\theta \in (a,b)$ and $(L_t)_{t\geq 0}$ also is a Lévy process under Q^θ for all these θ. However, the discounted stock price process $(e^{-rt}S_t)_{t\geq 0}$ will not be a martingale under all Q^θ. The parameter $\bar\theta$ of the *risk-neutral Esscher measure* $Q^{\bar\theta}$, for which this property holds, has to fulfill the equation $S_0 = E_{\bar\theta}[e^{-rt}S_t]$, i.e., $\bar\theta$ has to solve the equation

$$e^r = \frac{M_{L_1}(\bar\theta + 1)}{M_{L_1}(\bar\theta)}. \tag{2.7}$$

This also explains why it is necessary to require M_{L_1} to be defined on an interval with length greater than one. But, as mentioned before, Assumption 2.1 alone does not guarantee the existence of a solution $\bar\theta$. The next lemma, taken from Raible (2000, Proposition 2.8), provides a sufficient condition for this and further shows that the solution, if existent, is unique such that we can define the risk-neutral Esscher measure or *Esscher martingale measure* without any ambiguity.

Lemma 2.1. *If Assumption* 2.1 *is in force, we have:*

(a) *For each $c > 0$, there is at most one $\theta \in \mathbb{R}$ such that*

$$\frac{M_{L_1}(\theta + 1)}{M_{L_1}(\theta)} = c.$$

(b) *If $\lim_{u\downarrow a} M_{L_1}(u) = \lim_{u\uparrow b} M_{L_1}(u) = \infty$, then the previous equation has exactly one solution $\theta \in (a, b-1)$ for each $c > 0$.*

Remark 2.1. Note that within this framework one cannot assume the state price density to be square integrable in general. The interval (a,b) on which the mgf M_{L_1} is well defined and finite, can be fairly small. Thus, it might happen that a solution $\bar\theta \in (a, b-1)$ of (2.7) exists, but $2\bar\theta \notin (a,b)$, implying that $E[e^{2\bar\theta L_t}]$ is infinite and hence $Z_t^{\bar\theta}$ is not square integrable for any $t > 0$.

Now we can reformulate Theorem 2.1 in terms of the driving Lévy process instead of the state price process.

Proposition 2.1 (Cost-efficient payoffs in Lévy models). *Let $(L_t)_{t\geq 0}$ be a Lévy process with continuous distribution function F_{L_T} at maturity $T > 0$, and assume that a solution $\bar\theta$ of* (2.7) *exists.*

(a) *If $\bar\theta < 0$, then the cost-efficient payoff $\underline{X}_T$ and the most-expensive payoff $\overline{X}_T$ with distribution function G are a.s. unique and are given by*

$$\underline{X}_T = G^{-1}(F_{L_T}(L_T)) \quad \text{and} \quad \overline{X}_T = G^{-1}(1 - F_{L_T}(L_T)). \tag{2.8}$$

Further, the following bounds for the cost of any strategy with terminal payoff $X_T \sim G$ hold:

$$c(X_T) \geq E\big[e^{-rT} Z_T^{\bar\theta} \underline{X}_T\big] = \frac{1}{M_{L_T}(\bar\theta)} \int_0^1 e^{\bar\theta F_{L_T}^{-1}(1-y) - rT} G^{-1}(1-y)\, dy,$$

$$c(X_T) \leq E\big[e^{-rT} Z_T^{\bar\theta} \overline{X}_T\big] = \frac{1}{M_{L_T}(\bar\theta)} \int_0^1 e^{\bar\theta F_{L_T}^{-1}(1-y) - rT} G^{-1}(y)\, dy.$$

(b) *If $\bar\theta > 0$, then the cost-efficient and the most-expensive payoffs are a.s. unique and are given by*

$$\underline{X}_T = G^{-1}(1 - F_{L_T}(L_T)) \quad \text{and} \quad \overline{X}_T = G^{-1}(F_{L_T}(L_T)). \tag{2.9}$$

The bounds in (a) *hold true with $F_{L_T}^{-1}(1-y)$ replaced by $F_{L_T}^{-1}(y)$.*

From this we directly obtain the following corollary where the notions increasing and decreasing are understood in the weak sense. The continuity assumption is only needed for the converse direction.

Corollary 2.1 (Characterization of cost-efficiency in Lévy models). *Let $(L_t)_{t\geq 0}$ be a Lévy process with continuous distribution F_{L_T} at maturity $T > 0$, and assume that a solution $\bar\theta$ of* (2.7) *exists.*

(a) *If $\bar\theta < 0$, a payoff $X_T \sim G$ is cost-efficient if and only if it is increasing in L_T.*
(b) *If $\bar\theta > 0$, a payoff $X_T \sim G$ is cost-efficient if and only if it is decreasing in L_T.*

For the most-expensive strategy, the reverse holds true.

Example 2.1.

(a) Applying Corollary 2.1 to the special payoff $X_T = S_T = S_0 e^{L_T}$ one obtains that buying one stock for S_0 at time $t = 0$ is a cost-efficient way to achieve a payoff with distribution $G = F_{S_T}$ at time T if and only if $\bar\theta < 0$.
(b) Assume again the setting of Corollary 2.1 and consider the payoff of a put option at time $T > 0$ with strike $K > 0$, i.e., $X_T^P = (K - S_T)_+ = (K - S_0 e^{L_T})_+$. X_T^P is a decreasing function of L_T and hence is cost-efficient if and only if $\bar\theta > 0$. For $\bar\theta < 0$, however, the classical put is the most-expensive way to realize a payoff with distribution $G = F_{X_T^P}$. Similarly, the payoff $X_T^C = (S_0 e^{L_T} - K)_+$ of a call option with strike K and maturity T is cost-efficient if $\bar\theta < 0$ and most-expensive if $\bar\theta > 0$.

The corollary also implies the inefficiency of path-dependent payoffs. Here we call a payoff X_T *path-dependent* if X_T does not solely depend on the asset price S_T at maturity time T (or equivalently on L_T), but at least on one more value S_t, resp. L_t, with $0 < t < T$. Consequently, a path-dependent payoff never is an increasing or decreasing function of L_T alone, and therefore, cannot be cost-efficient either. The only exception is the case $\bar\theta = 0$ which implies $Z_t^{\bar\theta} \equiv 1$ for all $0 \leq t \leq T$, thus $P = Q^0$ already is a risk-neutral measure. As is immediately obvious from the definition, the possible price range $[c(\underline{X}_T), c(\overline{X}_T)]$ of any payoff $X_T \sim G$ then shrinks to a singleton or, in other words, for $\bar\theta = 0$ every payoff X_T already is cost-efficient and cannot be improved further. This yields the following generalized version of Bernard *et al.* (2014, Corollary 3) in the Lévy market setting. For a related result, see also Cox & Leland (2000).

Corollary 2.2 (Inefficiency of path-dependent payoffs). *If $(L_t)_{t\geq 0}$ is a Lévy process with continuous distribution F_{L_T} at maturity $T > 0$ and a solution $\bar\theta$ of* (2.7) *exists, then path-dependent payoffs are not cost-efficient unless $\bar\theta = 0$.*

Remark 2.2. In some settings, path-dependent payoffs X_T can be improved by conditioning on S_T resp. L_T. Vanduffel *et al.* (2008) and Vanduffel *et al.* (2009)

proved that risk-averse investors with fixed investment horizon will always prefer the payoff $X'_T = E[X_T \,|\, S_T]$ to X_T in a Lévy market model where the real-world and risk-neutral measures P and Q are related by an Esscher transformation. More generally, path-dependent payoffs are suboptimal for risk-averse investors in any setting where the state price density is a function of S_T, see Kassberger & Liebmann (2011). Observe that the improved payoff X'_T is no longer path-dependent due to the conditioning on S_T, hence it fits into the present framework and may be enhanced further by applying Proposition 2.1. In Vanduffel *et al.* (2012), this approach is applied to Dollar cost averaging which is shown to be outperformed by a static strategy of investing in a suitable portfolio of path-independent options. Some general comparison results for prices of path-dependent options like Asian or lookback options are given in Bergenthum & Rüschendorf (2008).

The above results indicate that the sign of the risk-neutral Esscher parameter $\bar{\theta}$ in Lévy markets plays an essential role for the construction of cost-efficient strategies. Our next result states that in a bullish Lévy market scenario, i.e., a market where the expected return $E[S_t/S_0] = E[e^{L_t}]$ is greater than the risk-free return e^{rt}, the Esscher parameter $\bar{\theta}$ solving Eq. (2.7) must be negative to shrink the rate of return to the risk-free rate r. Similarly, in bearish markets where the expected return $E[e^{L_t}]$ is smaller than e^{rt} for all $t > 0$, we must have $\bar{\theta} > 0$ to adjust the rate of return accordingly. Observe that this line of argumentation requires $E[S_t] = S_0 E[e^{L_t}] = S_0 E[e^{L_1}]^t < \infty$ which is, of course, a quite natural condition.

Proposition 2.2 (Characterization of bullish and bearish markets). *Assume that the risk-neutral Esscher parameter $\bar{\theta}$ exists and $E[e^{L_1}] < \infty$. Then the market is bullish if and only if $\bar{\theta} < 0$, and it is bearish if and only if $\bar{\theta} > 0$.*

While the sign of the risk-neutral Esscher parameter $\bar{\theta}$ characterizes the market behaviour, the size of $|\bar{\theta}|$ reflects the magnitude of the drift of the price process and thus can be regarded as a measure for the strength of the market trend. Our next result implies that the efficiency loss $c(X_T) - c(\underline{X}_T)$ associated with the strategy X_T is increasing in the absolute value of the Esscher parameter $|\bar{\theta}|$. To clarify this, let us first define the function

$$l(\theta, \eta) = e^{-rT} E_\theta[X_T - \underline{X}_T],$$

where $\underline{X}_T$ is defined by Eqs. (2.8) or (2.9) as before, dependent on the sign of θ, and observe that the Esscher parameter $\bar{\theta} = \bar{\theta}(\eta)$ is a function of the parameters $\eta = (\eta_1, \ldots, \eta_k)$ of the driving Lévy process (see Eqs. (2.7), (2.12), and (2.13)). Then $l(\bar{\theta}(\eta), \eta) = c(X_T) - c(\underline{X}_T)$ is the efficiency loss of the strategy X_T.

Proposition 2.3. *Suppose that $E_\theta\big[(X_T - \underline{X}_T)^2\big] < \infty$, then $\frac{\partial l(\theta,\eta)}{\partial \theta} < 0$ for $\theta < 0$ and $\frac{\partial l(\theta,\eta)}{\partial \theta} > 0$ for $\theta > 0$.*

Hence, if the risk-neutral Esscher parameter $\bar{\theta}$ exists, then $E_{\bar{\theta}}\big[(X_T - \underline{X}_T)^2\big] < \infty$ ensures that the signs of $\frac{\partial l(\bar{\theta},\eta)}{\partial \theta}$ and $\bar{\theta}$ coincide, or, in other words, that the efficiency loss $l(\bar{\theta}, \eta)$ is increasing in $|\bar{\theta}|$. Note that this only allows to analyze the influence of the absolute value $|\bar{\theta}|$ on the efficiency loss for fixed η. To determine the influence of the parameter η_i on the latter, one has to consider

$$\frac{\mathrm{d}l(\bar{\theta},\eta)}{\mathrm{d}\eta_i} = \frac{\partial l(\bar{\theta},\eta)}{\partial \theta}\frac{\partial \bar{\theta}}{\partial \eta_i} + \frac{\partial l(\bar{\theta},\eta)}{\partial \eta_i} \tag{2.10}$$

(recall that $\bar{\theta} = \bar{\theta}(\eta)$). The derivative $\frac{\partial \bar{\theta}}{\partial \eta_i}$ is typically simple to calculate. As a consequence we obtain that e.g. the conditions $\frac{\partial l(\bar{\theta},\eta)}{\partial \eta_i} > 0$ and $\frac{\partial \bar{\theta}}{\partial \eta_i} > 0$ imply that for $\bar{\theta} > 0$ the efficiency loss $l(\bar{\theta}, \eta)$ is increasing in η_i. The functional dependence of

$l(\bar{\theta}, \eta)$ on the Lévy parameters η, when $\bar{\theta}$ is fixed, is illustrated in Figure 1 in the upcoming subsection for the case of the normal inverse Gaussian model.

2.3. **Models for the Lévy process.** In the last two decades more and more researchers started to use jump-diffusions and, more generally, Lévy processes as a valuable and flexible tool to model asset price processes as well as the term structure of interest rates. These typically provide a much better fit to real market data because the inherent jumps allow for a more realistic modeling and quantification of the risk of large price movements within short time intervals which are often severly underestimated in a pure diffusion framework. A comprehensive overview on the most prominent Lévy processes that already have been applied to financial modeling can be found in the books of Schoutens (2003) and Cont & Tankov (2004), for jump-diffusion models we also refer to Kou (2002). In the following we concentrate on the normal inverse Gaussian (NIG) and variance Gamma (VG) Lévy processes. We include the Brownian motion as a benchmark model here which allows us to compare the prices of cost-efficient strategies within the NIG and VG models with those that can be achieved in the classical Black–Scholes framework.

2.3.1. *Normal inverse Gaussian model.* The normal inverse Gaussian process was first applied to finance in Barndoff-Nielsen (1995), Barndorff-Nielsen (1998). Its generating distributions can be obtained as a normal mean-variance mixture with an inverse Gaussian mixing distribution. More specifically, if $X \sim NIG(\alpha, \beta, \delta, \mu)$, then the random variable X can be represented as follows:

$$X \stackrel{d}{=} \mu + \beta Z + \sqrt{Z}\, W, \tag{2.11}$$

where $\mu \in \mathbb{R}$, $W \sim N(0,1)$, and $Z \sim IG\big(\delta, \sqrt{\alpha^2 - \beta^2}\big)$ is an inverse Gaussian distributed random variable with $\delta > 0$ and $0 \le |\beta| < \alpha$ that is independent of W. This representation also entails that the infinite divisibility of the mixing inverse Gaussian distribution transfers to the NIG mixture distribution, thus there exists a Lévy process $(L_t)_{t \ge 0}$ with $\mathcal{L}(L_1) = NIG(\alpha, \beta, \delta, \mu)$. The parameter $\overline{\theta}$ of the risk-neutral Esscher measure $Q^{\overline{\theta}}$, i.e., the solution of (2.7) (if it exists) is given by

$$\bar{\theta}_{NIG} = -\frac{1}{2} - \beta + \frac{r-\mu}{\delta}\sqrt{\frac{\alpha^2}{1 + (\frac{r-\mu}{\delta})^2} - \frac{1}{4}}. \tag{2.12}$$

Note that $(L_t)_{t \ge 0}$ remains a NIG Lévy process under every Esscher measure Q^{θ}, but with parameter β replaced by $\beta + \theta$. Further properties and a derivation of these results are given in Appendix A.

Figure 1 illustrates the local behaviour of $l(\bar{\theta}, \eta)$ around the actually estimated Lévy parameters $\hat{\eta}$ for a put option on Allianz in the NIG model, i.e., the graphs show the efficiency loss $l(\bar{\theta}, \eta)$ for a fixed Esscher parameter $\bar{\theta} \equiv \bar{\theta}(\hat{\eta})$ when we vary individual parameters η_i of the Lévy model. The dotted lines always refer to the actually estimated parameter values $\hat{\eta}_i$. Let us consider, for example, the parameter $\eta_2 = \beta$. From Eq. (2.12) we directly obtain $\frac{\partial \bar{\theta}_{NIG}}{\partial \beta} = -1$. Moreover, the risk-neutral Esscher parameter $\bar{\theta}_{NIG}$ is negative in case of the Allianz stock (confer Table 1 in the next section), so it follows from Proposition 2.3 that $\frac{\partial l(\bar{\theta}_{NIG}(\hat{\eta}), \eta)}{\partial \bar{\theta}_{NIG}} < 0$. Figure 1 then indicates that the partial derivative $\frac{\partial l(\bar{\theta}_{NIG}(\hat{\eta}), \eta)}{\partial \beta}$ should be positive and thus we can conclude from Eq. (2.10) that the efficiency loss should exhibit an increasing behaviour in the skewness parameter β.

2.3.2. *Variance Gamma model.* The class of variance Gamma distributions was introduced in Madan & Seneta (1990) and Madan & Milne (1991) as a more realistic model for stock return distributions. Similar to NIG distributions, a variance

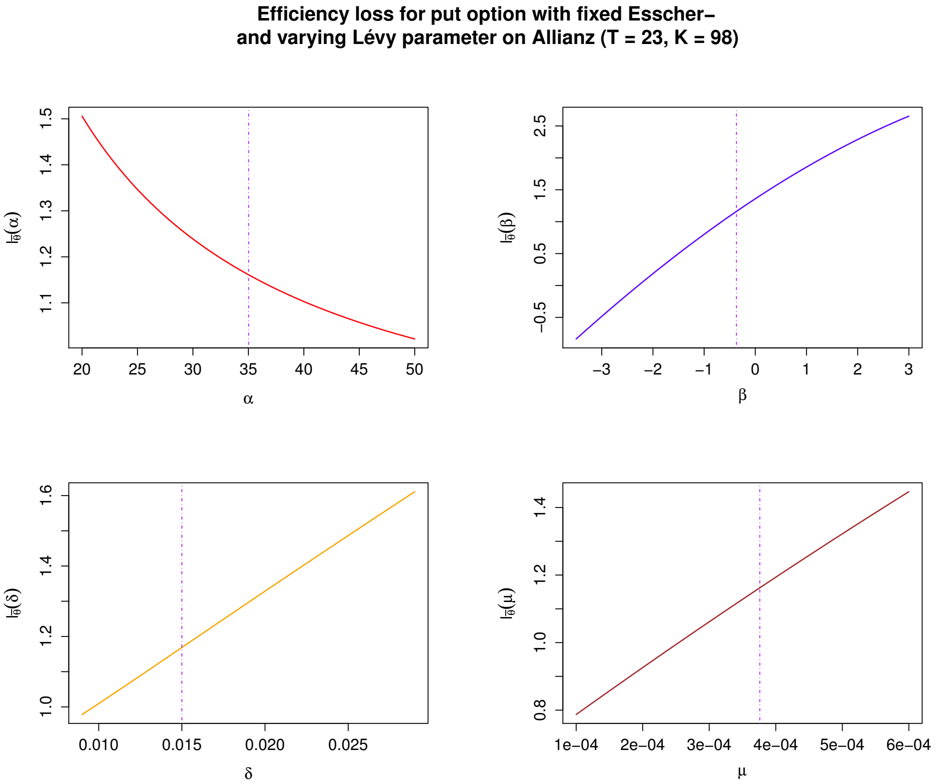

Figure 1. Dependence of $l(\bar{\theta}, \eta)$ on the Lévy parameters for a put option on Allianz in the NIG model when the Esscher parameter $\bar{\theta} = \bar{\theta}(\hat{\eta})$ is fixed. The dotted lines mark the actually estimated Lévy parameters, i.e., the components of $\hat{\eta}$ (see Table 1).

Gamma distributed random variable $X \sim VG(\lambda, \alpha, \beta, \mu)$ can be represented as a normal mean-variance mixture as in Eq. (2.11), but in this case the mixing variable $Z \sim G\big(\lambda, \frac{\alpha^2-\beta^2}{2}\big)$ is Gamma distributed with shape parameter $\lambda > 0$ and scale parameter $\frac{\alpha^2-\beta^2}{2}$ where $0 \leq |\beta| < \alpha$. Again, the infinite divisibility of $G\big(\lambda, \frac{\alpha^2-\beta^2}{2}\big)$ transfers to $VG(\lambda, \alpha, \beta, \mu)$.

Lemma 2.1 (b) and Eq. (A.6) in the appendix imply that the condition $2\alpha > 1$ is sufficient to guarantee a unique solution $\overline{\theta}$ of Eq. (2.7) in the VG case which is given by

$$\bar{\theta}_{VG} = \begin{cases} -\frac{1}{2} - \beta, & r = \mu, \\ -\frac{1}{1-e^{-\frac{r-\mu}{\lambda}}} - \beta + \operatorname{sign}(r-\mu)\sqrt{\frac{e^{-\frac{r-\mu}{\lambda}}}{\big(1-e^{-\frac{r-\mu}{\lambda}}\big)^2} + \alpha^2}, & r \neq \mu. \end{cases} \tag{2.13}$$

Under Q^{θ} the process $(L_t)_{t\geq 0}$ remains a VG Lévy process with parameter $\beta + \theta$ instead of β. For details see Appendix A.

Remark 2.3. In many papers dealing with VG distributions, especially the ones of Madan and coauthors, a different parametrization $VG(\sigma, \nu, \theta, \tilde{\mu})$ is used (the VG parameter θ here should not be confused with the Esscher parameter). This is related to ours as follows:

$$\sigma^2 = \frac{2\lambda}{\alpha^2 - \beta^2}, \quad \nu = \frac{1}{\lambda}, \quad \theta = \beta\sigma^2 = \frac{2\beta\lambda}{\alpha^2-\beta^2}, \quad \tilde{\mu} = \mu.$$

2.3.3. *Samuelson model.* The classical benchmark model, which also is the basis of the Black–Scholes theory, is to assume that the stock price process $(S_0 e^{L_t})_{t\geq 0}$ follows a geometric Brownian motion. In this case, the driving Lévy process is given by

$$L_t = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t, \quad t \geq 0,$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion under the physical measure P, μ is the drift and σ the volatility parameter. Here we have $\mathcal{L}(L_t) = N\big((\mu - \frac{\sigma^2}{2})t, \sigma^2 t\big)$, and the mgf of L_1 equals

$$M_{N(\mu - \frac{\sigma^2}{2}, \sigma^2)}(u) = e^{u(\mu - \frac{\sigma^2}{2}) + \frac{u^2\sigma^2}{2}}.$$

Apparently, it is defined for all $u \in \mathbb{R}$ and tends to infinity if $u \to \pm\infty$, thus Lemma 2.1 (b) assures that a unique solution $\bar{\theta}$ of Eq. (2.7) exists which can easily be computed as $\bar{\theta}_N = \frac{r-\mu}{\sigma^2}$.

Remark 2.4. Since this model is complete and thus the risk-neutral measure Q is unique, the Esscher density process $(Z_t^{\bar{\theta}_N})_{t\geq 0}$ here must coincide with the state price density process $(Z_t)_{t\geq 0}$ obtained from Girsanov's theorem. This indeed is the case, as the following equation shows:

$$Z_t = \frac{e^{\frac{r-\mu}{\sigma} B_t}}{e^{\frac{(r-\mu)^2}{2\sigma^2} t}} = \frac{e^{\frac{r-\mu}{\sigma} B_t}}{E[e^{\frac{r-\mu}{\sigma} B_t}]} = \frac{e^{\frac{r-\mu}{\sigma^2}(L_t - t(\mu - \frac{\sigma^2}{2}))}}{E\big[e^{\frac{r-\mu}{\sigma^2}(L_t - t(\mu - \frac{\sigma^2}{2}))}\big]} = \frac{e^{\bar{\theta}_N L_t}}{M_{L_t}(\bar{\theta}_N)} = Z_t^{\bar{\theta}_N}.$$

3. Applications

In this section, we apply the results obtained so far to some common payoff distributions. More specifically, we consider European put and call options, forwards as well as spread trading strategies. Moreover, we provide some numerical results for the Lévy market settings discussed in Section 2.3. These calculations are based on estimated parameters from German stock price data for Allianz and Volkswagen from May 28, 2010, to September 28, 2012, which are shown in Figure 2. The estimated parameters from the daily log-returns of Allianz and Volkswagen are given in Table 1 below. The interest rate used to calculate $\bar{\theta}$ is $r = 4.2027 \cdot 10^{-6}$ which corresponds to the continuously compounded 1-Month-Euribor rate of October 1, 2012.

3.1. Put options ($\bar{\theta} < 0$).

Consider a long put option with strike $K > 0$ and maturity $T > 0$ whose payoff is $X_T^P = (K - S_T)_+ = (K - S_0 e^{L_T})_+$. As already remarked in Example 2.1 (b), X_T^P is monotonically decreasing in L_T, therefore the put option is inefficient if $\bar{\theta} < 0$ due to Corollary 2.1. The distribution function $G_P = F_{X_T^P}$ of the put payoff can easily be shown to equal

$$G_P(x) = P\big(X_T^P \leq x\big) = \begin{cases} 1, & \text{if } x \geq K, \\ 1 - F_{L_T}\big(\ln\big(\frac{K-x}{S_0}\big)\big), & \text{if } 0 \leq x < K, \\ 0, & \text{if } x < 0. \end{cases}$$

Its inverse is given by

$$G_P^{-1}(y) = \big(K - S_0 e^{F_{L_T}^{-1}(1-y)}\big)_+, \quad y \in (0,1), \tag{3.1}$$

which follows from solving the equation

$$1 - F_{L_T}\left(\ln\left(\frac{K-x}{S_0}\right)\right) = y$$

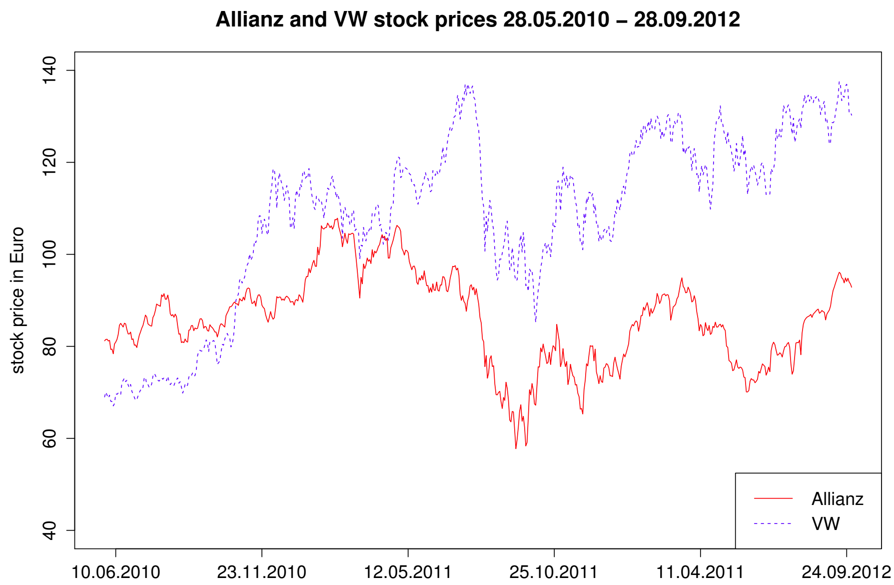

Figure 2. Daily closing prices of Allianz and Volkswagen used for parameter estimation.

	λ	α	β	δ	μ	$\bar{\theta}$
Allianz						
NIG	-0.5	35.020	-0.369	0.015	0.000376	-1.0127
VG	1.031	72.011	0.552	0.0	$1.941 \cdot 10^{-8}$	-1.0412
Normal		$\mu = 4.2757 \cdot 10^{-4}$, $\sigma = 0.0203$				-1.0314
Volkswagen						
NIG	-0.5	48.859	-0.842	0.0231	0.001451	-2.7087
VG	1.602	82.948	-2.165	0.0	0.00206	-2.7395
Normal		$\mu = 0.00129$, $\sigma = 0.0216$				-2.7447

Table 1. Estimated parameters from daily log-returns of Allianz and Volkswagen for the NIG-, the VG-, and the Samuelson model.

for x and noting that x must be non-negative since the range of X_T^P is $[0, K]$. Applying Proposition 2.1, for $\bar{\theta} < 0$ the cost-efficient payoff that generates the same distribution G_P as the long put is

$$\underline{X}_T^P = G_P^{-1}(F_{L_T}(L_T)) = \left(K - S_0 e^{F_{L_T}^{-1}(1 - F_{L_T}(L_T))}\right)_+. \tag{3.2}$$

Figure 3 displays the payoff X_T^P of a long put option on one Allianz stock with strike $K = 98$ and maturity $T = 23$ days, and its cost-efficient counterparts $\underline{X}_T^P$ for the three Lévy models under consideration. Although the latter payoff profiles look quite similar, a closer look reveals that the optimal payoff is model-dependent and varies slightly between the different models, in particular for large S_T.

Remark 3.1. Observe that the distribution function G_P and its inverse G_P^{-1} depend on the time to maturity. If the present time t is greater than zero, one has to replace T by $T - t$ and S_0 by S_t. This also implies that, in contrast to the vanilla put, the efficient put payoff is not static but a time-varying function. This is quite a natural feature which results from the fact that the efficient payoff has to take

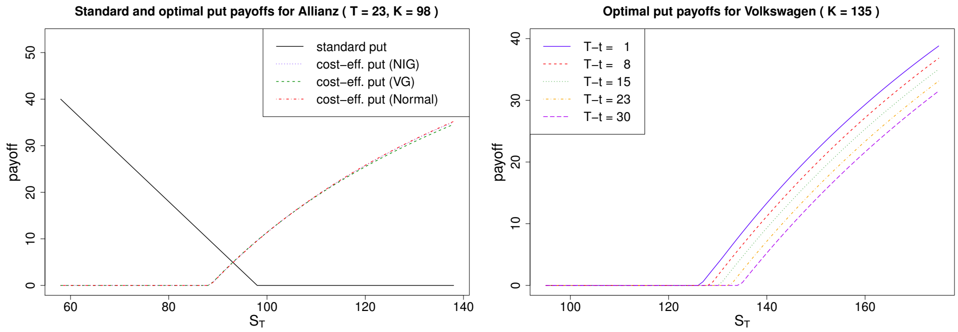

Figure 3. *Left:* Payoff functions of a classical put and its cost-efficient counterparts for Allianz. The initial stock price is $S_0 = 93.42$, the closing price of Allianz at October 1, 2012. *Right:* Cost-efficient put payoffs for different times to maturity within the VG model for Volkswagen. The initial stock price is always assumed to be $S_t = 130.55$.

the price movements of the underlying into account to ensure that its payoff distribution always coincides with that of its classical counterpart. The variation of the cost-efficient put payoff subject to different times to maturity is also illustrated in Figure 3 above.

However, one has to be aware that if an investor buys an efficient put, its payoff profile is fixed at the purchase date and will not be altered afterwards. Once bought or sold, the payoff distribution of a cost-efficient contract only equals that of its classical counterpart at the (initial) trading date, but no longer in the remaining time to maturity. To calculate the price $c(\underline{X}^P_{T,t})$ of an efficient put with a payoff fixed at time 0 at some later point in time $t > 0$, one has to resort to the fact that $S_T = S_0 e^{L_T} \overset{d}{=} S_0 e^{L_t + L_{T-t}} = S_t e^{L_{T-t}}$ and thus replace $L_T = \ln(S_T/S_0)$ in (3.2) by $\ln(S_t e^{L_{T-t}}/S_0)$, that is,

$$\underline{X}^P_{T,t} = \big(K - S_0 e^{F^{-1}_{L_T}(1-F_{L_T}(\ln(S_t e^{L_{T-t}}/S_0)))}\big)_+ . \tag{3.3}$$

The time t-price of an efficient put fixed at 0 then is $c(\underline{X}^P_{T,t}) = e^{-r(T-t)} E[Z^{\bar{\theta}}_{T-t} \underline{X}^P_{T,t}]$. We will use these facts later in Section 4 to derive the hedge deltas for efficient puts and calls. Observe that in contrast to the above notations, $c(\underline{X}^P_{T-t})$ and $\underline{X}^P_{T-t}$ denote the price resp. payoff of an efficient put with maturity T that is initiated at time t. For vanilla puts, this distinction is not necessary because $X^P_{T,t} = X^P_{T-t}$ and $c(X^P_{T,t}) = c(X^P_{T-t})$.

Recall that Figure 3 corresponds to a bullish market situation where $\bar{\theta} < 0$ (see Table 1) such that the classical put with payoff X^P_T is the most-expensive way to realize the payoff distribution G_P. However, if the market behaviour should suddenly switch at time t_s from bullish to bearish, that is, if the risk-neutral Esscher parameter $\bar{\theta}$ derived from market data should change its sign during the lifetime of the contract, then the roles of the payoffs are reversed: $X^P_{T-t_s}$ becomes cost-efficient, and the previously efficient payoffs $\underline{X}^P_{T,t_s}$ resp. $\underline{X}^P_{T-t_s}$ are most-expensive from that "switching time" t_s onwards. In other words, an initially optimal strategy may turn into the worst case if the market scenario significantly changes in between. This suggests that the present definition and construction of cost-efficient strategies might be extended to a more dynamic version that allows to accordingly react to reverse market movements. We do not exploit this idea further here, but leave it to future research.

Since the payoff function $X^{-P}_T = -(K - S_0 e^{L_T})_+$ of a short put with strike K and maturity T is monotonically increasing in L_T, a short put is cost-efficient if

$\bar{\theta} < 0$ and most-expensive if $\bar{\theta} > 0$. Analogously to the calculations for the long put, the inverse of the distribution G_{-P} of a short put is given by

$$G_{-P}^{-1}(y) = \big(S_0 e^{F_{L_T}^{-1}(y)} - K\big)_-, \quad y \in (0,1), \tag{3.4}$$

where $(x-y)_- := -(y-x)_+$. Applying Proposition 2.1, the cost-efficient payoff that generates the same distribution G_{-P} as the short put option for $\bar{\theta} > 0$ thus is

$$\underline{X}_T^{-P} = G_{-P}^{-1}(1 - F_{L_T}(L_T)) = \big(S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))} - K\big)_-. \tag{3.5}$$

It can easily be seen that

$$\underline{X}_T^P = -\overline{X}_T^{-P} \text{ and } c\big(\underline{X}_T^P\big) = -c\big(\overline{X}_T^{-P}\big)$$

as investors simply take opposite positions. The one who is long has to pay for entering the put while the one who is short receives an upfront payment. Thus, for the long position it is optimal to minimize the cost of the put while the investor who is short wants to maximize the initial cash inflow.

In Table 2 below, we compare the costs of a long put on Allianz and Volkswagen with their cost-efficient counterparts for the Lévy models discussed in Section 2.3. All computations are based on the estimated parameters given in Table 1. The initial stock prices S_0 of Allianz resp. Volkswagen are the closing prices on October 1, 2012, and the time to maturity is chosen to be $T = 23$ trading days, meaning that the put options mature on November 1, 2012. According to Proposition 2.1 and Eq. (3.1), the cost of the efficient put can be calculated by

$$c\big(\underline{X}_T^P\big) = E\big[e^{-rT} Z_T^{\bar{\theta}} \underline{X}_T^P\big] = \frac{1}{M_{dist}(\bar{\theta})} \int_0^1 e^{\bar{\theta} F_{dist}^{-1}(1-y)} \big(K - S_0 e^{F_{dist}^{-1}(y)}\big)_+ \, dy$$

where *dist* is either $NIG(\alpha, \beta, \delta T, \mu T)$, $VG(\lambda T, \alpha, \beta, \mu T)$, or $N((\mu - \frac{\sigma^2}{2})T, \sigma^2 T)$.

Using Eqs. (2.7) and (A.5), the cost $c(X_T^P)$ of the vanilla put in the NIG model is given by

$$\begin{aligned} c(X_T^P) &= E_{\bar{\theta}}\big[e^{-rT}(K - S_T)_+\big] \\ &= e^{-rT} \int_{-\infty}^{\ln(K/S_0)} (K - S_0 e^x) Z_T^{\bar{\theta}} \, d_{NIG(\alpha,\beta,\delta T,\mu T)}(x) \, dx \\ &= K e^{-rT} F_{NIG(\alpha,\beta+\bar{\theta},\delta T,\mu T)}\big(\ln\big(\tfrac{K}{S_0}\big)\big) - S_0 F_{NIG(\alpha,\beta+\bar{\theta}+1,\delta T,\mu T)}\big(\ln\big(\tfrac{K}{S_0}\big)\big), \end{aligned} \tag{3.6}$$

and for the VG model one analogously obtains

$$c(X_T^P) = K e^{-rT} F_{VG(\lambda T,\alpha,\beta+\bar{\theta},\mu T)}\big(\ln\big(\tfrac{K}{S_0}\big)\big) - S_0 F_{VG(\lambda T,\alpha,\beta+\bar{\theta}+1,\mu T)}\big(\ln\big(\tfrac{K}{S_0}\big)\big).$$

	$c(X_T^P)$	$c(\underline{X}_T^P)$	Efficiency loss in %
Allianz			
NIG	6.4495	5.2825	18.09
VG	6.3681	5.2270	17.92
Normal	6.4324	5.2683	18.10
Volkswagen			
NIG	8.0064	4.0871	48.95
VG	7.9765	4.0603	49.10
Normal	7.9909	4.0749	49.01

Table 2. Comparison of the cost of a long put option on Allianz and Volkswagen, resp., and the corresponding cost-efficient payoffs in different Lévy models. Initial stock price, strike, and time to maturity are $S_0 = 93.42$, $K = 98$, $T = 23$ for Allianz and $S_0 = 130.55$, $K = 135$, $T = 23$ for Volkswagen. The other parameters needed for the calculations are taken from Table 1.

In the Samuelson model, $c(X_T^P)$ is given by the well-known Black–Scholes put price formula.

The results show that the savings from choosing the cost-efficient strategies can be quite substantial: For Allianz, the cost of the efficient put is less than 83% of the price of the plain vanilla put, and in case of Volkswagen the vanilla put is almost twice as expensive as the efficient put. The great differences in the efficiency losses of the Allianz and Volkswagen puts may seem somewhat surprising at first glance because the stock price to strike ratio $\frac{S_0}{K}$ is roughly the same in both cases (0.953 for Allianz and 0.967 for Volkswagen), but is suggested by Proposition 2.3 if we suppose that a change in $\bar{\theta}$ has a much greater impact on the efficiency loss than a modification of the Lévy parameters η. Since the payoffs X_T^P and $\underline{X}_T^P$ are both bounded by the strike K (see (3.2)), the condition $E_{\bar{\theta}}\big[(X_T^P - \underline{X}_T^P)^2\big] < \infty$ here is trivially fulfilled, so Proposition 2.3 assures that the efficiency loss is increasing in $|\bar{\theta}|$ if we can neglect the influence of η here. As can be seen from Table 1, the value of $|\bar{\theta}|$ for Volkswagen is more than 2.5 times as large as that of Allianz, and this is also reflected in the magnitude of the efficiency losses in Table 2. However, for each stock itself the efficiency losses obtained under the different Lévy models are of almost the same size and thus seem to be widely model-independent.

3.2. **Call options ($\bar{\theta} > 0$).** Consider a long call option with strike $K > 0$, maturity $T > 0$, and payoff $X_T^C = (S_T - K)_+ = (S_0 e^{L_T} - K)_+$. As already pointed out before in Example 2.1 (b), X_T^C is monotonically increasing in L_T, hence the long call option is not cost-efficient if $\bar{\theta} > 0$. Its distribution function $G_C = F_{X_T^C}$ can easily be derived as

$$G_C(x) = P\big(X_T^C \le x\big) = \begin{cases} 0, & \text{if } x < 0, \\ F_{L_T}\big(\ln\big(\frac{K+x}{S_0}\big)\big), & \text{if } x \ge 0. \end{cases} \tag{3.7}$$

The corresponding inverse is given by

$$G_C^{-1}(y) = \big(S_0 e^{F_{L_T}^{-1}(y)} - K\big)_+, \quad y \in (0,1). \tag{3.8}$$

Applying Proposition 2.1 again, for $\bar{\theta} > 0$ the cost-efficient payoff that generates the same distribution G_C as the long call option is given by

$$\underline{X}_T^C = G_C^{-1}(1 - F_{L_T}(L_T)) = \Big(S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))} - K\Big)_+. \tag{3.9}$$

Similarly, one can show that the short call is inefficient for $\bar{\theta} < 0$ as its payoff function $X_T^{-C} = -(S_0 e^{L_T} - K)_+$ is monotonically decreasing in L_T. The distribution function G_{-C} of the short call payoff is

$$G_{-C}(x) = P\big(X_T^{-C} \le x\big) = \begin{cases} 1, & \text{if } x \ge 0, \\ 1 - F_{L_T}\big(\ln\big(\frac{K-x}{S_0}\big)\big), & \text{if } x < 0, \end{cases}$$

and for its inverse one obtains

$$G_{-C}^{-1}(y) = -\big(S_0 e^{F_{L_T}^{-1}(1-y)} - K\big)_+, \quad y \in (0,1).$$

Thus, the cost-efficient strategy for a short call in the case $\bar{\theta} < 0$ is

$$\underline{X}_T^{-C} = G_{-C}^{-1}(F_{L_T}(L_T)) = -\big(S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))} - K\big)_+. \tag{3.10}$$

From the preceding equations we obtain that for $\bar{\theta} > 0$,

$$\underline{X}_T^C = -\overline{X}_T^{-C} \text{ and } c\big(\underline{X}_T^C\big) = -c\big(\overline{X}_T^{-C}\big), \tag{3.11}$$

under the general assumptions of Proposition 2.1. If $\bar{\theta} < 0$, analogous relations hold.

3.3. **Forwards.** The payoff function $X_T^{-F} = K - S_T = K - S_0 e^{L_T}$ of a short forward with delivery price K is strictly decreasing in L_T and thus inefficient if $\bar{\theta} < 0$. The corresponding distribution function G_{-F} is given by

$$G_{-F}(x) = P(K - S_0 e^{L_T} \leq x) = 1 - F_{L_T}\left(\ln\left(\frac{K-x}{S_0}\right)\right)$$

and has the inverse

$$G_{-F}^{-1}(y) = K - S_0 e^{F_{L_T}^{-1}(1-y)}, \quad y \in (0,1).$$

By Proposition 2.1, the cost-efficient strategy for a short forward in case $\bar{\theta} < 0$ is

$$\underline{X}_T^{-F} = G_{-F}^{-1}(F_{L_T}(L_T)) = K - S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))}. \tag{3.12}$$

Remark 3.2. Observe that the payoff $\underline{X}_T^{-F}$ and hence the cost $c\big(\underline{X}_T^{-F}\big)$ of the efficient short forward depend on the distribution of L_T and hence on the specific Lévy model one has chosen. In contrast to this, simple no-arbitrage arguments show that the cost $c\big(X_T^{-F}\big)$ of the standard short forward (if the underlying provides no income during the lifetime of the contract) is given by $c\big(X_T^{-F}\big) = Ke^{-rT} - S_0$, and thus, is obviously model-independent.

Recall that the payoff of a short forward equals the payoff of the sum of a long put and a short call with the same strike K and maturity T, i.e.,

$$X_T^{-F} = K - S_T = (K - S_T)_+ - (S_T - K)_+.$$

This decomposition suggests that the cost-efficient strategy of a short forward may alternatively be derived as the combination of the cost-efficient strategies for a long put and a short call which are both inefficient if $\bar{\theta} < 0$. Indeed, from Eqs. (3.2) and (3.10) we have

$$\begin{aligned}\underline{X}_T^{-F} &= \big(K - S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))}\big)_+ - \big(S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))} - K\big)_+ \\ &= \underline{X}_T^{P} + \underline{X}_T^{-C}.\end{aligned} \tag{3.13}$$

Analogously, a long forward is inefficient if $\bar{\theta} > 0$. Its payoff X_T^F corresponds to the sum of the payoffs of a long call and a short put

$$X_T^F = S_T - K = (S_T - K)_+ - (K - S_T)_+. \tag{3.14}$$

Here, one also obtains that the payoff of a cost-efficient long forward

$$\underline{X}_T^F = G_F^{-1}(1 - F_{L_T}(L_T)) = S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))} - K \tag{3.15}$$

corresponds to the sum of an efficient long call and an efficient short put.

The decomposition of the cost-efficient strategy of a forward into the sum of cost-efficient strategies for a call and a put suggests that there exists some cost-efficient analogue to the put-call parity which says that the prices of a long call and a long put can be derived from each other by just adding resp. subtracting the corresponding forward price. However, this classical result essentially relies on the fact that the payoffs and costs of short positions are just the negative payoffs resp. costs of the corresponding long positions, but this relationship does not apply to cost-efficient payoffs anymore. Thus, in general the price of a cost-efficient long put cannot be obtained as the difference of the prices of an efficient long call and an efficient long forward. The terms in Eq. (3.13) cannot be simply rearranged because altering between long and short positions here means to switch between cost-efficient and most-expensive payoffs, so the correct put-call parity within this framework is given by the following proposition.

Proposition 3.1 (Cost-efficient put-call parity). *The cost-efficient long forward payoff* $\underline{X}_T^F$ *allows the decomposition* $\underline{X}_T^F = \underline{X}_T^C + \underline{X}_T^{-P}$ *which implies* $\underline{X}_T^C = \underline{X}_T^F + \overline{X}_T^P$*, and thus,* $c(\underline{X}_T^C) = c(\underline{X}_T^F) + c(\overline{X}_T^P)$.

For the price of a cost-efficient long put one analogously obtains

$$c(\underline{X}_T^P) = c(\underline{X}_T^{-F}) + c(\overline{X}_T^C). \tag{3.16}$$

Remember that for $\bar{\theta} < 0$ we have $X_T^C = \underline{X}_T^C$, $X_T^F = \underline{X}_T^F$, and $X_T^P = \overline{X}_T^P$, whereas for $\bar{\theta} > 0$ the standard payoffs $X_T^P = \underline{X}_T^P$, $X_T^{-F} = \underline{X}_T^{-F}$, and $X_T^C = \overline{X}_T^C$ are already cost-efficient, resp. most-expensive.

3.4. **Spread trading strategies.** A bull spread is a combination of a long call C_1 with strike $K_1 > 0$ and a short call $-C_2$ with strike $K_2 > K_1$. The payoff is given by

$$X_T^{bull} = \left(S_0 e^{L_T} - K_1\right)_+ - \left(S_0 e^{L_T} - K_2\right)_+$$

and thus is increasing in L_T. Hence, the bull spread is not cost-efficient if $\bar{\theta} > 0$. Its distribution function is

$$G_{bull}(x) = \begin{cases} 0, & \text{if } x < 0, \\ F_{L_T}\left(\ln\left(\frac{K_1 + x}{S_0}\right)\right), & \text{if } 0 \le x < K_2 - K_1, \\ 1, & \text{if } x \ge K_2 - K_1, \end{cases}$$

and the corresponding inverse can be represented by

$$G_{bull}^{-1}(y) = \left(S_0 e^{F_{L_T}^{-1}(y)} - K_1\right)_+ - \left(S_0 e^{F_{L_T}^{-1}(y)} - K_2\right)_+.$$

Note that for $x < K_2 - K_1$, the distribution function $G_{bull}(x)$ coincides with that of the long call C_1, therefore it is not surprising that the first summand of the inverse $G_{bull}^{-1}(y)$ is equal to $G_{C_1}^{-1}(y)$. The second summand here is necessary to ensure that the quantile function takes only values in the range $[0, K_2 - K_1]$ of X_T^{bull}. If $\bar{\theta} > 0$, the cost-efficient strategy corresponding to such a bull spread then is

$$\underline{X}_T^{bull} = G_{bull}^{-1}(1 - F_{L_T}(L_T)) = \underline{X}_T^{C_1} - \underline{X}_T^{C_2} = \underline{X}_T^{C_1} + \overline{X}_T^{-C_2} \tag{3.17}$$

where the last equalities follow from Eqs. (3.9), (3.10), and (3.11). Hence, the efficient bull spread payoff $\underline{X}_T^{bull}$ is equivalent to a long position in an efficient call C_1 and a short position in a most-expensive call C_2.

A bear spread is a combination of a short put with strike $K_1 > 0$ and a long put with strike $K_2 > K_1$. Its payoff thus is

$$X_T^{bear} = \left(K_2 - S_0 e^{L_T}\right)_+ - \left(K_1 - S_0 e^{L_T}\right)_+$$

which is decreasing in L_T and thus inefficient if $\bar{\theta} < 0$. Similarly to the bull spread, we derive the cost-efficient payoff of the bear spread for $\bar{\theta} < 0$ as

$$\underline{X}_T^{bear} = \underline{X}_T^{P_2} - \underline{X}_T^{P_1} = \underline{X}_T^{P_2} + \overline{X}_T^{-P_1}, \tag{3.18}$$

which corresponds to the sum of an efficient long put P_2 with strike K_2 and a most-expensive short put $-P_1$ with strike K_1.

From the above examples, one may have the impression that the cost-efficient strategy for any combination of long and short puts or calls can easily be obtained by just replacing the long positions by their cost-efficient and the short positions by their most-expensive counterparts. However, this is not true in general as the following counterexample shows: Consider a butterfly spread which is the combination of two long calls C_3 and C_1 with strikes $K_3 > K_1 > 0$, and two short calls

$-C_2$ with strike $K_2 = \frac{K_1+K_3}{2}$. The payoff X_T^{bfly} of a butterfly spread is thus given by

$$X_T^{bfly} = \big(S_0 e^{L_T} - K_1\big)_+ + \big(S_0 e^{L_T} - K_3\big)_+ - 2\big(S_0 e^{L_T} - K_2\big)_+,$$

and the corresponding distribution function G_{bfly} can be shown to equal

$$G_{bfly}(x) = \begin{cases} 0, & \text{if } x < 0, \\ F_{L_T}\big(\ln\big(\frac{K_1+x}{S_0}\big)\big) + 1 - F_{L_T}\big(\ln\big(\frac{K_3-x}{S_0}\big)\big), & \text{if } 0 \le x < \frac{K_3-K_1}{2}, \\ 1, & \text{if } x \ge \frac{K_3-K_1}{2}. \end{cases}$$

The distribution function has a more complex form because the payoff X_T^{bfly} is not monotonic in L_T, and it can easily be checked that the inverse G_{bfly}^{-1} does not admit a representation in form of a sum of $G_{C_1}^{-1}$, $G_{C_3}^{-1}$, and $G_{-C_2}^{-1}$. Therefore, the relation $\underline{X}_T^{bfly} = \underline{X}_T^{C_1} + \underline{X}_T^{C_3} + 2\overline{X}_T^{-C_2}$ cannot be valid either.

For several further options like self-quanto calls and puts as well as straddles explicit or semi-explicit formulas can be derived analogously as in the preceding sections.

4. Delta hedging of cost-efficient strategies in Lévy models

In the previous section, we provided a semi-explicit formula for the cost of a cost-efficient strategy which is valuable for many financial applications since it can be easily evaluated numerically. For practitioners, however, this formula might still be unsatisfying unless an explicit hedging strategy for the cost-efficient payoff exists. In this section, we want to deal with the hedging problem and first provide some formulas of possible hedging strategies for efficient puts and calls which we then apply to hedge the efficient puts on Allianz and Volkswagen discussed in Section 3.1. We also develop alternative hedging strategies for vanilla puts and calls based on a series of efficient puts resp. calls with decreasing times to maturity and compare them to the classical ones. In the following we focus on deriving formulas for delta hedges, i.e., the derivative of the cost of a strategy with respect to the underlying. If the underlying asset is traded sufficiently liquid in the market, delta hedging probably is one of the simplest, but nevertheless fairly effective ways to cover a risky position and is therefore widely used in practice.

4.1. **Delta hedging of efficient puts and calls.** Consider the payoffs $X_T^P = (K - S_T)_+$ and $X_T^C = (S_T - K)_+$ of a put resp. call option with strike $K > 0$ and maturity $T > 0$. We already saw in Example 2.1 (b) and Sections 3.1 and 3.2 that the payoff X_T^P becomes inefficient for $\bar\theta < 0$ while the payoff X_T^C is inefficient if $\bar\theta > 0$. As was pointed out in Remark 3.1, the payoff profiles of the corresponding cost-efficient counterparts are fixed at their issuance dates and cannot be altered afterwards. Using Eq. (3.3), the writer of an efficient put initiated at time 0 thus has to hedge the price

$$\begin{aligned} c(\underline{X}_{T,t}^P) &= e^{-r(T-t)} E[Z_{T-t}^{\bar\theta}\, \underline{X}_{T,t}^P] \\ &= \frac{1}{M_{L_{T-t}}(\bar\theta)} \int_{-\infty}^{+\infty} e^{\bar\theta y - r(T-t)} \Big(K - S_0 e^{F_{L_T}^{-1}(1 - F_{L_T}(\ln(S_t e^y/S_0)))}\Big)_+ F_{L_{T-t}}(dy) \\ &= \frac{e^{-r(T-t)}}{M_{L_{T-t}}(\bar\theta)} \int_0^1 e^{\bar\theta F_{L_{T-t}}^{-1}(z)} \Big(K - S_0 e^{F_{L_T}^{-1}\big(1 - F_{L_T}\big(\ln\big(\frac{S_t e^{F_{L_{T-t}}^{-1}(z)}}{S_0}\big)\big)\big)}\Big)_+ dz \quad (4.1) \end{aligned}$$

at time t. In case of an efficient call, the price to be hedged at time t can analogously be derived as

$$c(\underline{X}^C_{T,t}) = \frac{e^{-r(T-t)}}{M_{L_{T-t}}(\bar\theta)} \int_0^1 e^{\bar\theta F^{-1}_{L_{T-t}}(z)} \Big(S_0 e^{F^{-1}_{L_T}\big(1-F_{L_T}\big(\ln\big(\frac{S_t e^{F^{-1}_{L_{T-t}}(z)}}{S_0}\big)\big)\big)} - K \Big)_+ dz \quad (4.2)$$

The following theorem provides explicit formulas for the derivatives of $c(\underline{X}^P_{T,t})$ and $c(\underline{X}^C_{T,t})$ with respect to the price S_t of the underlying at time t.

Theorem 4.1 (Deltas for cost-efficient puts and calls). *Let $(L_t)_{t\geq 0}$ be a Lévy process with a continuous and strictly increasing distribution F_{L_t} for all $t \in [0,T]$, and assume that a solution $\bar\theta$ of (2.7) exists. For each $t \in [0,T)$ we have:*

(a) *If $\bar\theta < 0$, the delta $\underline{\Delta}^P_t$ of the cost-efficient long put $\underline{X}^P_{T,t}$ initiated at time 0 is*

$$\begin{aligned}\underline{\Delta}^P_t &= \frac{S_0}{S_t}\frac{e^{-r(T-t)}}{M_{L_{T-t}}(\bar\theta)} \\ &\quad \cdot \int_{y^P_{T,t}}^{+\infty} e^{\bar\theta y + F^{-1}_{L_T}\big(1-F_{L_T}\big(\ln\big(\frac{S_t e^y}{S_0}\big)\big)\big)} \frac{d_{L_T}\big(\ln\big(\frac{S_t e^y}{S_0}\big)\big)}{d_{L_T}\big(1-F_{L_T}\big(\ln\big(\frac{S_t e^y}{S_0}\big)\big)\big)} F_{L_{T-t}}(dy),\end{aligned} \quad (4.3)$$

where $d_{L_T}(y)$ is the density of F_{L_T}, and $y^P_{T,t} = \ln\big(\frac{S_0}{S_t}\big) + F^{-1}_{L_T}\big(1-F_{L_T}\big(\ln\big(\frac{K}{S_0}\big)\big)\big)$.

(b) *If $\bar\theta > 0$ and $c(\underline{X}^C_{T,t}) < \infty$, the delta $\underline{\Delta}^C_t$ of the cost-efficient long call $\underline{X}^C_{T,t}$ initiated at time 0 is*

$$\begin{aligned}\underline{\Delta}^C_t &= -\frac{S_0}{S_t}\frac{e^{-r(T-t)}}{M_{L_{T-t}}(\bar\theta)} \\ &\quad \cdot \int_{-\infty}^{y^C_{T,t}} e^{\bar\theta y + F^{-1}_{L_T}\big(1-F_{L_T}\big(\ln\big(\frac{S_t e^y}{S_0}\big)\big)\big)} \frac{d_{L_T}\big(\ln\big(\frac{S_t e^y}{S_0}\big)\big)}{d_{L_T}\big(1-F_{L_T}\big(\ln\big(\frac{S_t e^y}{S_0}\big)\big)\big)} F_{L_{T-t}}(dy),\end{aligned} \quad (4.4)$$

where $y^C_{T,t} = y^P_{T,t}$.

Eqs. (4.3) and (4.4) of the previous theorem especially entail that the deltas of efficient puts and calls just have opposite signs compared to their classical counterparts. This is in line with the intuition because Figure 3 has shown that the payoff of an efficient put is reversed to that of a vanilla put and bears some similarities to the payoff of a vanilla call, and a similar observation can be made when comparing the payoffs of standard and efficient calls. In Section 4.3, we shall demonstrate that delta hedging of cost-efficient puts can be efficiently applied in practice and that the obtained hedge errors are usually not greater, but often even smaller than those of the corresponding vanilla puts.

4.2. **Delta hedging of vanilla puts and calls using cost-efficient strategies.** In the previous section, we considered just one efficient call resp. put whose payoff profile was fixed at the initial date $t = 0$. However, one can also obtain alternative hedging strategies for vanilla calls and puts by making use of the time-varying payoff functions discussed in Section 3.1. To see how and why this works, recall that a cost-efficient put that is initiated at time t and has time to maturity $T-t$ is given by $\underline{X}^P_{T-t} = \big(K - S_t e^{F^{-1}_{L_{T-t}}(1-F_{L_{T-t}}(L_{T-t}))}\big)_+$. Using the fact that $L_0 = 0$ almost surely for every Lévy process L, one easily obtains that $\underline{X}^P_{T-t} \to (K - S_T)_+$ for $t \to T$. Alternatively, this also can be deduced from the fact that the asset price S_T is a known constant at maturity, therefore the payoff-distribution of the standard put at time T is the degenerate distribution (unit mass) located at $(K - S_T)_+$. Since by definition the corresponding efficient put must have the same payoff-distribution, its payoff profile at time T must coincide with the latter. Analogous conclusions also hold, of course, for cost-efficient and vanilla calls. This implies that the prices of efficient puts and calls converge to those of their classical counterparts if the time to maturity goes to zero, that is, $c(X^P_{T-t}) - c(\underline{X}^P_{T-t}) \to 0$ if $t \to T$ (see Figures 7 and 8). Therefore, a delta hedge which reproduces the evolution of the efficient put prices $(c(\underline{X}^P_{T-t}))_{0\le t\le T}$ can be regarded as an alternative way to hedge the final put payoff $(K - S_T)_+$ at maturity.

Remark 4.1. The general risk-neutral pricing rule implies that the sequence $(e^{-rt}c(X^P_{T-t}))_{0\le t\le T}$ of the discounted vanilla put prices is a Q-martingale. Because $c(\underline{X}^P_{T-t}) < c(X^P_{T-t})$ for all $t < T$ and $c(\underline{X}^P_{T-T}) = c(X^P_{T-T})$, the sequence $(e^{-rt}c(\underline{X}^P_{T-t}))_{0\le t\le T}$ cannot be a Q-martingale, and hence every trading strategy which replicates the values $(c(\underline{X}^P_{T-t}))_{0\le t\le T}$ of the efficient put prices cannot be self-financing either. This, of course, also applies to our alternative hedging strategy. However, almost every trading strategy which is self-financing in the theory of continuous-time trading will loose this property in practice, because there only discrete-time trading is possible which inevitably leads to hedging errors. Reducing these surely is the predominant task from a hedger's point of view. As we shall see in Section 4.3, the alternative delta hedging strategies have the potential to outperform the classical ones in this context. Therefore, they might be interesting for practitioners although they lack the theoretical appealing feature of being self-financing.

The following theorem gives explicit representations of the alternative deltas for vanilla calls and puts. To better distinguish them from the classical hedge deltas Δ^C_t, Δ^P_t, we denote them by $\Delta^{roC}_t, \Delta^{roP}_t$ here. This notation should also reflect the similarity of these delta hedges to a rolling-over strategy where an investor buys at each day t the cost-efficient put $\underline{X}^P_{T-t}$ or call $\underline{X}^C_{T-t}$ initiated at that day and sells it on the following day to buy the actual efficient put $\underline{X}^P_{T-t-1}$ resp. call $\underline{X}^C_{T-t-1}$ instead.

Theorem 4.2 ("Rollover"-deltas for vanilla puts and calls). *Let $(L_t)_{t\ge 0}$ be a Lévy process with a continuous and strictly increasing distribution F_{L_t} for all $t \in [0, T]$, and assume that a solution $\bar\theta$ of (2.7) exists. Then for each $t \in [0, T)$ we have:*

(a) *If $\bar\theta < 0$, the alternative delta Δ^{roP}_t of the long vanilla put X^P_T at time t is*

$$\Delta^{roP}_t = -\frac{1}{M_{L_{T-t}}(\bar\theta)} \int_0^{F_{L_{T-t}}\left(\ln\left(\frac{K}{S_t}\right)\right)} e^{\bar\theta F^{-1}_{L_{T-t}}(1-y) + F^{-1}_{L_{T-t}}(y) - r(T-t)} dy. \qquad (4.5)$$

(b) *If $\bar\theta > 0$, the alternative delta Δ^{roC}_t of the long vanilla call X^C_T at time t is*

$$\Delta^{roC}_t = \frac{1}{M_{L_{T-t}}(\bar\theta)} \int_0^{1-F_{L_{T-t}}\left(\ln\left(\frac{K}{S_t}\right)\right)} e^{\bar\theta F^{-1}_{L_{T-t}}(y) + F^{-1}_{L_{T-t}}(1-y) - r(T-t)} dy. \qquad (4.6)$$

Eqs. (4.5) and (4.6) especially imply that the alternative deltas $\Delta^{roP}_t, \Delta^{roC}_t$ for the vanilla puts and calls have the same sign as their classical counterparts Δ^P_t, Δ^C_t, which is in line with the intuition. We now compare the magnitudes of Δ^{roP}_t and Δ^P_t resp. Δ^{roC}_t and Δ^C_t and show that the absolute values of the rollover-deltas are always smaller for calls and in many cases also for puts.

Theorem 4.3 (Comparison of deltas). *Let $(L_t)_{t\ge 0}$ be a Lévy process with a continuous and strictly increasing distribution F_{L_t} for all $t \in [0, T]$, and assume that a solution $\bar\theta$ of (2.7) exists.*

(a) *For vanilla calls, we have the following relations for each $t \in [0, T)$:*
If $\bar\theta > 0$, then $0 \le \Delta^{roC}_t \le \Delta^C_t$. For $\bar\theta < 0$ we have $\Delta^{roC}_t = \Delta^C_t$.

(b) *In the put case, we have* $\Delta_t^{roP} = \Delta_t^P$ *for* $\bar{\theta} > 0$ *and each* $t \in [0, T)$.
If $\bar{\theta} < 0$ *and* $F_{L_{T-t}}\big(\ln\big(\frac{K}{S_t}\big)\big) \leq q^*$ *where* $q^* \in (0.5, 1]$ *is the unique positive root of the function* $D_P : [0, 1] \to \mathbb{R}$,

$$D_P(q) = \frac{1}{M_{L_{T-t}}(\bar{\theta})} \int_0^q e^{\bar{\theta} F^{-1}_{L_{T-t}}(y) + F^{-1}_{L_{T-t}}(y)} - e^{\bar{\theta} F^{-1}_{L_{T-t}}(1-y) + F^{-1}_{L_{T-t}}(y)}\, dy,$$

then $\Delta_t^P \leq \Delta_t^{roP} \leq 0$.

Remark 4.2. Observe again that the condition in Theorem 4.3 (b) is time-dependent, thus even if $\Delta_0^P \leq \Delta_0^{roP}$ holds at the initial time $t = 0$, this does not necessarily mean that $\Delta_t^P \leq \Delta_t^{roP}$ for all $t \in (0, T)$. However, in practical examples this inequality typically holds throughout the lifetime of the contract as the subsequent examples show. The fact that the rollover-deltas are smaller than their classical counterparts also implies that the corresponding hedge portfolios react less sensitive to changes in value of the underlying and thus may provide more robust hedging strategies for vanilla puts and calls.

4.3. **Applications to real market data.** In the following, we illustrate the theoretical findings by some practical examples for the put case which continue the calculations in Section 3.1. We first consider the price evolution $(c(X^P_{T-t}))_{0\leq t\leq T}$ of a vanilla put and a cost-efficient put $c(\underline{X}^P_{T,t})_{0\leq t\leq T}$ on the Allianz and the Volkswagen stock which are assumed to be issued on October 1, 2012, and to mature on November 1, 2012. Figure 4 shows the prices of the Allianz stock and the corresponding puts with strike $K = 98$ within the aforementioned time period, as well as the values of the deltas $(\Delta_t^P)_{0\leq t\leq T}$ resp. $(\underline{\Delta}_t^P)_{0\leq t\leq T}$ associated to both puts. Here, all calculations are based on the NIG model; the NIG parameters for Allianz can be found in Table 1. The deltas $\underline{\Delta}_t^P$ of the efficient put were calculated using Eq. (4.3) from Theorem 4.1, and an explicit formula for their counterparts Δ_t^P of the vanilla put in the NIG model can be easily derived from Eq. (3.6): Observing that

$$\begin{aligned} d_{NIG(\alpha,\beta+\bar{\theta}+1,\delta(T-t),\mu(T-t))}(x) &\underset{(A.5)}{=} \frac{e^{(\bar{\theta}+1)x}}{M_{L_{T-t}}(\bar{\theta}+1)}\, d_{NIG(\alpha,\beta,\delta(T-t),\mu(T-t))}(x) \\ &\underset{(2.7)}{=} \frac{e^x}{e^{r(T-t)}} \frac{e^{\bar{\theta}x}}{M_{L_{T-t}}(\bar{\theta})}\, d_{NIG(\alpha,\beta,\delta(T-t),\mu(T-t))}(x) \\ &\underset{(A.5)}{=} \frac{e^x}{e^{r(T-t)}}\, d_{NIG(\alpha,\beta+\bar{\theta},\delta(T-t),\mu(T-t))}(x), \end{aligned}$$

we here obtain

$$\begin{aligned} \Delta_t^P = \frac{\partial c(X^P_{T-t})}{\partial S_t} &= -\frac{Ke^{-r(T-t)}}{S_t}\, d_{NIG(\alpha,\beta+\bar{\theta},\delta(T-t),\mu(T-t))}\big(\ln\big(\tfrac{K}{S_t}\big)\big) \\ &\quad - F_{NIG(\alpha,\beta+\bar{\theta}+1,\delta(T-t),\mu(T-t))}\big(\ln\big(\tfrac{K}{S_t}\big)\big) \\ &\quad + d_{NIG(\alpha,\beta+\bar{\theta}+1,\delta(T-t),\mu(T-t))}\big(\ln\big(\tfrac{K}{S_t}\big)\big) \\ &= -\frac{Ke^{-r(T-t)}}{S_t}\, d_{NIG(\alpha,\beta+\bar{\theta},\delta(T-t),\mu(T-t))}\big(\ln\big(\tfrac{K}{S_t}\big)\big) \\ &\quad - F_{NIG(\alpha,\beta+\bar{\theta}+1,\delta(T-t),\mu(T-t))}\big(\ln\big(\tfrac{K}{S_t}\big)\big) \\ &\quad + \frac{Ke^{-r(T-t)}}{S_t}\, d_{NIG(\alpha,\beta+\bar{\theta},\delta(T-t),\mu(T-t))}\big(\ln\big(\tfrac{K}{S_t}\big)\big) \\ &= -F_{NIG(\alpha,\beta+\bar{\theta}+1,\delta(T-t),\mu(T-t))}\big(\ln\big(\tfrac{K}{S_t}\big)\big). \end{aligned}$$

As is obvious from Figure 4, the price of the cost-efficient put evolves almost exactly in the opposite way as that of the vanilla put. This reflects the fact that the payoff

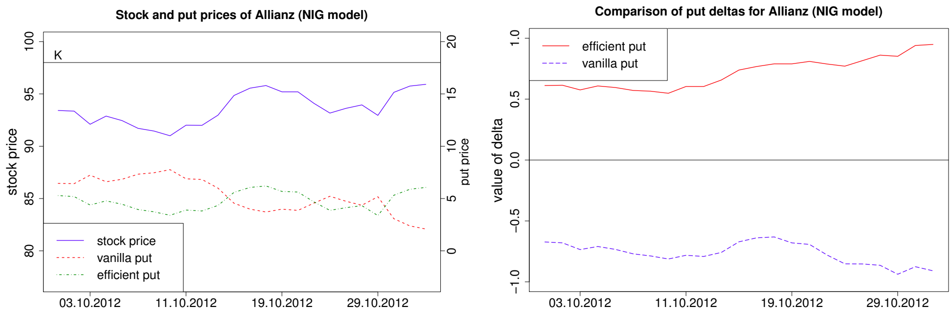

Figure 4. *Left:* Stock price of Allianz from October 1, 2012, to November 1, 2012, and the prices $c(X^P_{T-t})$, $c(\underline{X}^P_{T,t})$ of the associated vanilla resp. efficient put. *Right:* Comparison of the deltas Δ^P_t and $\underline{\Delta}^P_t$ of the vanilla and the efficient put on Allianz.

profiles of both puts are, in some sense, reversed to each other (see Figure 3); the efficient put roughly behaves like a vanilla call. However, the efficient put ends in the money although the price of the Allianz stock remains below the strike price at maturity because its payoff function already takes positive values for some $S_T < K$. The opposite behaviour of the efficient and the vanilla put is also mirrored in the values of the associated deltas. Because the values of the deltas at maturity are not relevant for hedging purposes anymore, Figure 4 only shows the deltas up to one day to maturity, that is, from October 1, 2012, to October 31, 2012. The results obtained for the other two Lévy models (normal and VG) look quite similar and therefore are not plotted here separately. This is also in line with our previous estimations and calculations. Since the risk-neutral Esscher parameter roughly was of the same size for all three models (see Table 1) and also the put prices and efficiency losses in Table 2 were almost identical, one should not expect greater differences here.

Figure 5 below shows the evolution of the prices of the Volkswagen stock and the cost-efficient and vanilla puts on it with strike $K = 135$ as well as the corresponding deltas. Again, the results do not differ much between all three Lévy models under consideration, thus we only show the plots for the VG case. The delta of the vanilla put in this model can be derived analogously as above to be

$$\Delta^P_t = \frac{\partial c(X^P_{T-t})}{\partial S_t} = -F_{VG(\lambda(T-t),\alpha,\beta+\bar{\theta}+1,\mu(T-t))}\big(\ln\big(\tfrac{K}{S_t}\big)\big).$$

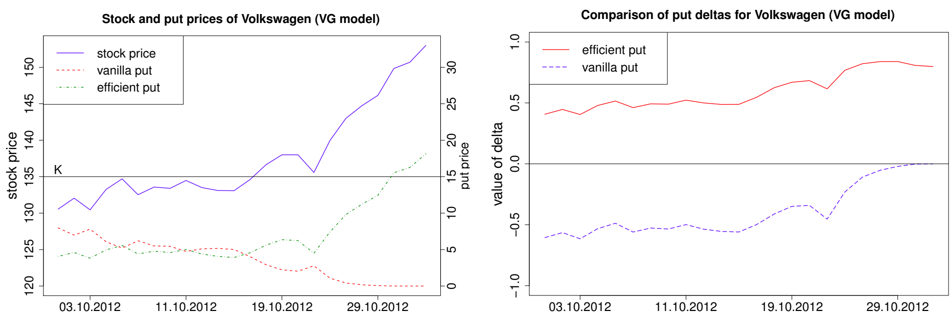

Figure 5. *Left:* Stock price of Volkswagen from October 1, 2012, to November 1, 2012, and the prices $c(X^P_{T-t})$, $c(\underline{X}^P_{T,t})$ of the associated vanilla resp. efficient put. *Right:* Comparison of the deltas $\Delta^P_t, \underline{\Delta}^P_t$ of the vanilla and the efficient put on Volkswagen.

Note that in this example we have $S_T > K$, therefore the vanilla put expires worthless, and the corresponding delta converges to zero, whereas the efficient put ends deep in the money.

However, computing the put deltas is only one side of the coin, market participants will surely be more interested in how well the hedging strategies based on them work in practice. The NIG and VG models are incomplete from the very beginning, so one cannot expect perfect hedging there, but also the Samuelson model is only complete in theory. Since in reality just discrete hedging is feasible, one will encounter hedge errors within this framework, too. The magnitude of these errors is, of course, relevant for practical applications. Therefore, we also calculate and compare the hedge errors that occur in delta hedging of the vanilla and efficient puts on Allianz and Volkswagen considered before.

We assume that the hedge portfolios are rebalanced daily, hence the portfolio weights δ_t (amount of stock at time t) and b_t (amount of money on the savings account at t) just have to be calculated at the discrete times $t = 0, 1, \ldots, T-1$. For the vanilla puts $\delta_t = \Delta_t^P$, and in case of the efficient puts we have $\delta_t = \underline{\Delta}_t^P$. Depending on the put type under consideration, we analogously set $c_t = c(X_{T-t}^P)$ or $c_t = c(\underline{X}_{T,t}^P)$, respectively. At the initial time $t = 0$, the hedge portfolio is set up with the weights δ_0 and $b_0 = -\delta_0 S_0 + c_0$ since the writer of the put obtains c_0 from the buyer, shorts $|\delta_0|$ stocks and deposits all incomes on his savings account. At time $t > 0$, the value of the portfolio *before* rebalancing is $\delta_{t-1} S_t + e^r b_{t-1}$, and we define the corresponding hedge error by

$$e_t := c_t - \delta_{t-1} S_t - e^r b_{t-1},$$

so positive hedge errors mean losses and negative gains. At the end of the trading day, the new weights δ_t and $b_t = c_t - \delta_t S_t$ are chosen to ensure that the value of the portfolio again exactly coincides with the present put price. Using the above definition of e_t, we can alternatively represent b_t in the form

$$b_t = e_t + e^r b_{t-1} + S_t(\delta_{t-1} - \delta_t).$$

This means that the hedge error is nothing but the amount of money one has to additionally inject in or withdraw from the savings account after adapting the stock position to make the value of the hedge portfolio congruent with the current put price.

Remark 4.3. In general, the size of the hedge error also depends on the rebalancing frequency and the continuity properties of the payoff function. Our empirical results below show that for standard and efficient puts a daily rebalancing of the portfolio already is sufficient to get a fairly precise approximation to the current option prices. A thorough theoretical analysis of the behaviour of hedge errors resulting from delta and quadratic hedging strategies in exponential Lévy models can be found in Broden & Tankov (2011).

The upper graphs of Figure 6 display the hedge errors obtained from delta hedging of the different puts on Allianz and Volkswagen. At the beginning, the hedge errors of the efficient and the vanilla puts behave fairly similarly, but with time passing the distinctions increase. This might again be explained by the different shapes of the payoff profiles and the different signs of the corresponding deltas which lead to more pronounced differences in the hedge errors as the time to maturity becomes smaller. The sums $\sum_{t=0}^{22} |e_t|$ of the absolute hedge errors for Allianz are 1.296 (efficient put) and 1.798 (vanilla put), for Volkswagen we obtain 1.794 (efficient put) resp. 2.252 (vanilla put). This indicates that cost-efficient options can be hedged at least as efficiently as standard options. However, since the prices of vanilla and efficient puts can differ significantly over time, one should not only

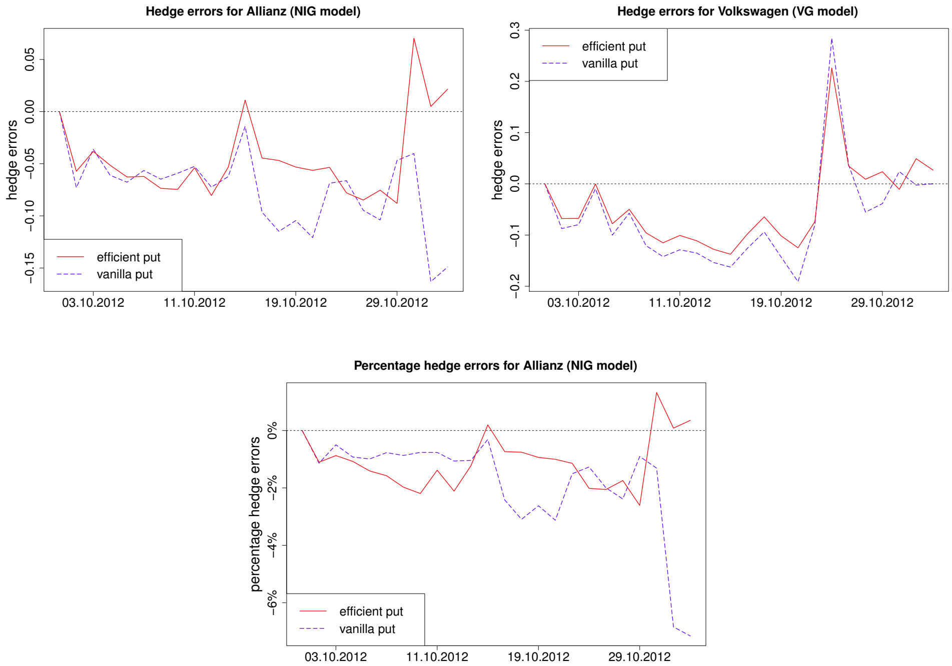

Figure 6. *Top:* Delta hedge errors of the efficient and vanilla puts on Allianz with strike $K = 98$ and Volkswagen with strike $K = 135$ maturing on November 1, 2012. *Bottom:* Percentage hedge errors of the efficient and vanilla puts on Allianz.

look at the absolute hedge errors to confirm this assertion, but also take the relative or percentage hedge errors $\tilde{e}_t := \frac{e_t}{c_t}$ into account. The values of $\tilde{e}_t$ for the Allianz puts are shown in the lower graph of Figure 6 above. For the efficient put, we obtain $\sum_{t=0}^{22} |\tilde{e}_t| = 0.299$, and the corresponding value for the vanilla put is 0.438. Analogous computations for the Volkswagen puts would not make much sense here because there the vanilla put ends up deep out of the money, therefore the $\tilde{e}_t$ would tend to infinity as $t \to T$.

In the last part of this section, we want to compare the alternative hedging strategy for vanilla puts based on the rollover-deltas Δ_t^{roP} with its classical counterpart and investigate if it can really provide an efficient and more robust way to hedge the final put payoff $(K - S_T)_+$ as was expected from Theorem 4.3. For this purpose, we again consider the vanilla puts on Allianz and Volkswagen with

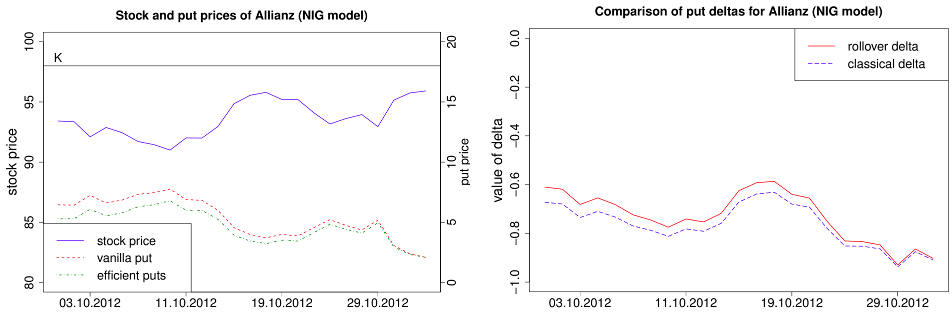

Figure 7. *Left:* Stock price of Allianz from October 1, 2012, to November 1, 2012, and the prices $c(X_{T-t}^P)$, $c(\underline{X}_{T-t}^P)$ of the associated vanilla resp. efficient puts. *Right:* Comparison of the deltas Δ_t^P, Δ_t^{roP} of the vanilla put on Allianz shown on the left.

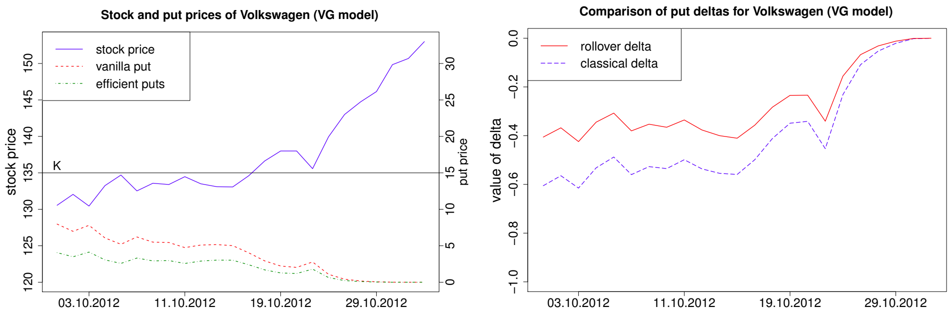

Figure 8. *Left:* Stock price of Volkswagen from October 1, 2012, to November 1, 2012, and the prices $c(X^P_{T-t})$, $c(\underline{X}^P_{T-t})$ of the associated vanilla resp. efficient puts. *Right:* Comparison of the deltas Δ^P_t, Δ^{roP}_t of the vanilla put on Volkswagen shown on the left.

the same strikes and maturity as before, but now contrast the corresponding price processes $(c(X^P_{T-t}))_{0\le t\le T}$ with the series $(c(\underline{X}^P_{T-t}))_{0\le t\le T}$ of prices of efficient puts which are newly initiated at each day t. Figures 7 and 8 show the stock and put price processes for Allianz in the NIG model and for Volkswagen in the VG model, respectively, as well as a graphical comparison of the associated classical put deltas Δ^P_t and rollover-deltas Δ^{roP}_t. The condition of Theorem 4.3 (b) is obviously fulfilled for all $0 \le t \le T$, the absolute values of the rollover-deltas are always smaller than those of the classical deltas for both stocks.

This indicates that the hedging strategies based on the rollover-deltas may indeed allow for a less expensive way to replicate the final put payoff. The advantage of lower hedging costs might be annihilated by larger hedging errors though. Therefore one also has, of course, to take these into account before coming to a definite conclusion. Using some of the notations from above, we define the hedge error for the alternative hedging strategy by

$$e_t := c(\underline{X}^P_{T-t}) - \Delta^{roP}_t S_t - e^r b_{t-1}.$$

Observe that we do not use the time-t-price $c(X^P_{T-t})$ of the vanilla put in the above definition although we want to hedge its final payoff. Since the rollover-deltas Δ^{roP}_t are intended to replicate the prices $c(\underline{X}^P_{T-t})$, and $c(\underline{X}^P_{T-t}) < c(X^P_{T-t})$ for all $0 \le t < T$ because $\bar{\theta} < 0$ here, a comparison of the value of the hedge portfolio at time t with $c(X^P_{T-t})$ would lead to a systematic overestimation of the hedge error. Moreover, we only consider options of European type here. Therefore it is more important to look at the hedge error at maturity which tells us how precise the hedging strategies can reproduce the final obligation of the writer of the option. At time T, however, we have $c(\underline{X}^P_{T-T}) = c(X^P_{T-T}) = (K - S_T)_+$ as pointed out before, so there the hedge error is defined without ambiguity.

So let us finally take a look at the hedge errors obtained from the two delta-hedging strategies for the vanilla puts on Allianz and Volkswagen which are visualized in Figure 9. For Allianz, the hedge errors e_T at maturity are -0.149 for the classical delta hedge and -0.085 for the alternative rollover-delta hedge, and the sum $\sum_{t=0}^{22} |e_t|$ of the absolute hedge errors is 1.789 for the classical and 0.802 for the rollover hedge. The final hedge errors e_T for the Volkswagen put are zero for both hedging strategies (which is not so surprising because the vanilla put expires worthless here), and the sums of the absolute hedge errors are 2.252 for the classical and 0.983 for the rollover hedge. This shows that the latter can yield at least comparable and often even more accurate results than the classical delta hedging strategy. In case of the Allianz put, the classical delta hedge tends to superhedge the option, that is, the value of the hedge portfolio is always greater than the option

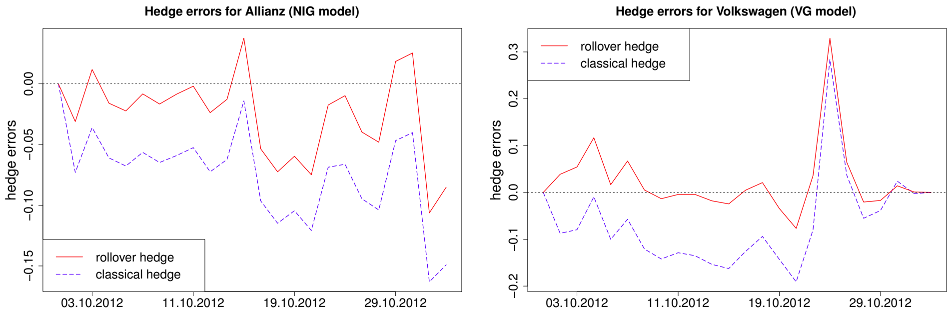

Figure 9. *Left:* Delta hedge errors of the vanilla put on Allianz with strike $K = 98$ maturing on November 1, 2012, in the NIG model. *Right:* Delta hedge errors of the vanilla put on Volkswagen with strike $K = 135$ maturing on November 1, 2012, in the VG model.

price. The rollover hedge does the same on most days, but produces smaller absolute hedge errors. In view of Theorem 4.3, we suppose that analogous assertions will also hold for calls and probably also for more complex options.

5. Conclusion

We applied the concept of cost-efficiency to general Lévy market models where the risk-neutral measure is obtained by an Esscher transform. Explicit criteria for cost-efficiency were derived and applied to various financial derivatives, and we established a cost-efficient version of the put-call parity. Furthermore, we proved that the magnitude of the efficiency loss increases if the market trend, resp. the drift of the underlying, becomes more pronounced. Numerical examples of cost-efficient puts were presented which provide evidence that the savings from switching to cost-efficient strategies can be quite substantial. We found that the efficiency losses obtained under different Lévy models were of almost the same magnitude, and thus, seem to be widely model-independent.

Further, we derived explicit formulas for the Greek delta of cost-efficient puts and calls and developed alternative hedging strategies for vanilla puts and calls based on rollover trading strategies involving their cost-efficient counterparts. We also proved that the absolute values of the corresponding rollover-deltas are smaller than the classical hedge deltas. This suggests that the alternative delta hedging strategies can often be more accurate and lead to smaller hedge errors. In a practical application using German stock price data we demonstrated that the computation of all the deltas is numerically tractable. The cost-efficient puts can be hedged as accurately as their vanilla counterparts, and the alternative hedging strategies for vanilla puts indeed have the potential to outperform the classical ones. This indicates that cost-efficient strategies provide a more advantageous way to achieve and hedge a final payoff, and thus, may be an appropriate tool to increase market efficiency.

Appendix A. Derivation of the risk-neutral Esscher parameters

Normal inverse Gaussian model. We first provide some properties of NIG distributions as introduced in (2.11). The Lebesgue density $d_{NIG(\alpha,\beta,\delta,\mu)}$ can be obtained by calculating

$$\begin{aligned} d_{NIG(\alpha,\beta,\delta,\mu)}(x) &= \int_0^\infty d_{N(\mu+\beta y)}(x)\, d_{IG(\delta,\sqrt{\alpha^2-\beta^2})}(y)\, dy \\ &= n(\alpha,\beta,\delta)\, \frac{K_1\big(\alpha\sqrt{\delta^2+(x-\mu)^2}\big)}{\sqrt{\delta^2+(x-\mu)^2}}\, e^{\beta(x-\mu)}, \end{aligned}$$

where $K_1(x)$ is the modified Bessel function of third kind with index 1, and the norming constant $n(\alpha,\beta,\delta)$ is given by

$$n(\alpha,\beta,\delta)=\frac{\alpha\delta}{\pi}\,e^{\delta\sqrt{\alpha^2-\beta^2}}.$$

The corresponding mgf $M_{NIG(\alpha,\beta,\delta,\mu)}$ then can easily be derived observing that

$$\begin{aligned}M_{NIG(\alpha,\beta,\delta,\mu)}(u) &= \int_{-\infty}^{\infty} e^{ux} d_{NIG(\alpha,\beta,\delta,\mu)}(x)\,dx\\ &= \int_{-\infty}^{\infty} e^{u\mu}\frac{n(\alpha,\beta,\delta)}{n(\alpha,\beta+u,\delta)}\,d_{NIG(\alpha,\beta+u,\delta,\mu)}(x)\,dx\\ &= e^{u\mu}\frac{n(\alpha,\beta,\delta)}{n(\alpha,\beta+u,\delta)} = e^{u\mu+\delta\sqrt{\alpha^2-\beta^2}-\delta\sqrt{\alpha^2-(\beta+u)^2}} \qquad \text{(A.1)}\end{aligned}$$

which obviously is defined for all $u\in(-\alpha-\beta,\alpha-\beta)$. Hence, Assumption 2.1 is fulfilled if $\alpha-\beta-(-\alpha-\beta)=2\alpha>1$. However, we have

$$\lim_{u\to\pm\alpha-\beta} M_{NIG(\alpha,\beta,\delta,\mu)}(u)=e^{(\pm\alpha-\beta)\mu+\delta\sqrt{\alpha^2-\beta^2}},$$

that is, the mgf tends to a finite limit at the boundaries of this interval. According to Lemma 2.1, it thus may not always be possible to find a solution $\bar\theta$ of Eq. (2.7). If it exists, it is given by

$$\bar\theta_{NIG}=-\frac12-\beta+\frac{r-\mu}{\delta}\sqrt{\frac{\alpha^2}{1+(\frac{r-\mu}{\delta})^2}-\frac14}. \qquad \text{(A.2)}$$

For the derivation of this expression note that the defining Eq. (2.7) for the risk-neutral Esscher parameter in the NIG model becomes

$$e^r=\frac{M_{NIG(\alpha,\beta,\delta,\mu)}(\bar\theta_{NIG}+1)}{M_{NIG(\alpha,\beta,\delta,\mu)}(\bar\theta_{NIG})}=e^{\mu-\delta\sqrt{\alpha^2-(\beta+\bar\theta_{NIG}+1)^2}+\delta\sqrt{\alpha^2-(\beta+\bar\theta_{NIG})^2}},$$

or equivalently,

$$\frac{r-\mu}{\delta}=\sqrt{\alpha^2-(\beta+\bar\theta_{NIG})^2}-\sqrt{\alpha^2-(\beta+\bar\theta_{NIG}+1)^2}. \qquad \text{(A.3)}$$

Under Assumption 2.1, which is equivalent to $2\alpha>1$ as seen above, Lemma 2.1 states that there can exist at most one solution $\bar\theta_{NIG}$ to (A.3) which obviously must also fulfill the additional constraints $|\beta+\bar\theta_{NIG}|<\alpha$ and $|\beta+\bar\theta_{NIG}+1|<\alpha$.

Case 1: $r=\mu$
Here we have that

$$0=\sqrt{\alpha^2-(\beta+\bar\theta_{NIG})^2}-\sqrt{\alpha^2-(\beta+\bar\theta_{NIG}+1)^2}$$

and hence $(\beta+\bar\theta_{NIG})^2=(\beta+\bar\theta_{NIG}+1)^2$, which obviously is fulfilled iff

$$\bar\theta_{NIG}=-\frac12-\beta.$$

This is a proper solution since by Assumption 2.1

$$\alpha>\frac12=|\beta+\bar\theta_{NIG}|=|\beta+\bar\theta_{NIG}+1|.$$

Case 2: $r\neq\mu$
To simplify notations, we set $r^*:=\frac{r-\mu}{\delta}$ and $\beta^*:=\beta+\bar\theta_{NIG}$ in the following. Using these abbreviations, Eq. (A.3) can be rewritten as

$$\sqrt{\alpha^2-\beta^{*2}-(2\beta^*+1)}=\sqrt{\alpha^2-\beta^{*2}}-r^*.$$

Squaring this equation and isolating the term $\sqrt{\alpha^2 - \beta^{*2}}$ yields

$$\sqrt{\alpha^2 - \beta^{*2}} = \frac{(1 + r^{*2}) + 2\beta^*}{2r^*}.$$

Squaring again and reorganizing terms we finally obtain the following quadratic equation for β^*:

$$\beta^{*2} + \beta^* + \frac{1 + r^{*2}}{4} - \frac{\alpha^2 r^{*2}}{1 + r^{*2}} = 0.$$

The solutions to this quadratic equation are given by

$$\beta^* = \beta + \bar{\theta}_{NIG} = -\frac{1}{2} \pm r^* \sqrt{\frac{\alpha^2}{1 + r^{*2}} - \frac{1}{4}} \Longrightarrow \bar{\theta}_{NIG} = -\frac{1}{2} - \beta \pm r^* \sqrt{\frac{\alpha^2}{1 + r^{*2}} - \frac{1}{4}}.$$

Note that the above solutions only exist if $2\alpha > \sqrt{1 + r^{*2}}$ which is more restrictive than Assumption 2.1. From Eq. (A.3) we conclude that for $r^* > 0$ we must have $(\beta + \bar{\theta}_{NIG})^2 < (\beta + \bar{\theta}_{NIG} + 1)^2$, which is equivalent to $-\frac{1}{2} - \beta < \bar{\theta}_{NIG}$. If $r^* < 0$, we analogously arrive at the constraint $-\frac{1}{2} - \beta > \bar{\theta}_{NIG}$. Comparing this with the above solutions of the quadratic equation for β^*, we finally see that the only possible solution for the risk-neutral Esscher parameter is

$$\bar{\theta}_{NIG} = -\frac{1}{2} - \beta + \frac{r - \mu}{\delta} \sqrt{\frac{\alpha^2}{1 + (\frac{r-\mu}{\delta})^2} - \frac{1}{4}}. \tag{A.4}$$

However, observe that this is a possible, but not a definitive solution! One additionally has to check if the obtained $\bar{\theta}_{NIG}$ really solves the initial Eq. (A.3). There exist sets of NIG parameters which fulfill all necessary constraints, however, the value $\bar{\theta}_{NIG}$ calculated according to (A.4) is not a valid solution of (A.3). Take, for example, $(\alpha, \beta, \delta, \mu) = (1, -0.1, 0.05, 0)$ and $r = 0.06$, then we have $2 = 2\alpha > \sqrt{1 + r^{*2}} = 1.56205$, and calculating $\bar{\theta}_{NIG}$ according to (A.4) yields $\bar{\theta}_{NIG} = 0.07975404$. Clearly, this $\bar{\theta}_{NIG}$ also fulfills the additional constraints $|\beta + \bar{\theta}_{NIG}| < \alpha$ and $|\beta + \bar{\theta}_{NIG} + 1| < \alpha$, but inserting this value and the other parameters into Eq. (A.3) one sees that the latter is violated.

Note that the characteristic function ϕ_{NIG} of an NIG distribution can be obtained via the relation $\phi_{NIG}(u) = M_{NIG}(iu)$. Since for every Lévy process it holds that $\phi_{L_t}(u) = \phi_{L_1}(u)^t$, one immediately obtains from (A.1) that $\phi_{L_t}(u) = \phi_{NIG(\alpha,\beta,\delta,\mu)}(u)^t = \phi_{NIG(\alpha,\beta,\delta t,\mu t)}(u)$, hence for an NIG Lévy process $(L_t)_{t\geq 0}$ we have $\mathcal{L}(L_t) = NIG(\alpha, \beta, \delta t, \mu t)$ for all $t > 0$. Similar arguments as used above in (A.1) then show that for any Esscher transform the density $d^\theta_{L_t}$ of L_t under the measure Q^θ is

$$\begin{aligned} d^\theta_{L_t}(x) &= \frac{e^{\theta x}}{M_{NIG(\alpha,\beta,\delta t,\mu t)}(\theta)}\, d_{NIG(\alpha,\beta,\delta t,\mu t)}(x) \\ &= \frac{n(\alpha, \beta + \theta, \delta t)}{n(\alpha, \beta, \delta t)}\, e^{\theta(x - \mu t)} d_{NIG(\alpha,\beta,\delta t,\mu t)}(x) = d_{NIG(\alpha,\beta+\theta,\delta t,\mu t)}(x), \end{aligned} \tag{A.5}$$

that is, $(L_t)_{t\geq 0}$ remains an NIG Lévy process under every Esscher transform Q^θ, but with different parameter $\beta \rightsquigarrow \beta + \theta$.

Variance Gamma model. Again, we first summarize some properties of the VG model as introduced in Section 2.3. The corresponding Lebesgue density $d_{VG(\lambda,\alpha,\beta,\mu)}$ is given by

$$\begin{aligned} d_{VG(\lambda,\alpha,\beta,\mu)}(x) &= \int_0^\infty d_{N(\mu+\beta y)}(x)\, d_{G(\lambda,(\alpha^2-\beta^2)/2)}(y)\, dy \\ &= m(\lambda, \alpha, \beta)\, |x - \mu|^{\lambda - \frac{1}{2}}\, K_\lambda(\alpha|x - \mu|)\, e^{\beta(x-\mu)} \end{aligned}$$

with the norming constant

$$m(\lambda,\alpha,\beta)=\frac{(\alpha^2-\beta^2)^\lambda}{\sqrt{\pi}(2\alpha)^{\lambda-\frac{1}{2}}\Gamma(\lambda)}.$$

With the same reasoning as in (A.1) one obtains the mgf

$$M_{VG(\lambda,\alpha,\beta,\mu)}(u)=e^{u\mu}\frac{m(\lambda,\alpha,\beta)}{m(\lambda,\alpha,\beta+u)}=e^{u\mu}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+u)^2}\right)^\lambda \tag{A.6}$$

which again is defined for all $u\in(-\alpha-\beta,\alpha-\beta)$. Observe that here we have $\lim_{u\to\pm\alpha-\beta}M_{VG(\lambda,\alpha,\beta,\mu)}(u)=\infty$, such that due to by Lemma 2.1 (b) the condition $2\alpha>1$ is sufficient to guarantee a unique solution $\bar\theta$ of Eq. (2.7) in the VG model. The defining equation for the risk-neutral Esscher parameter here becomes

$$e^r=\frac{M_{VG(\lambda,\alpha,\beta,\mu)}(\bar\theta_{VG}+1)}{M_{VG(\lambda,\alpha,\beta,\mu)}(\bar\theta_{VG})}=e^\mu\left(\frac{\alpha^2-(\beta+\bar\theta_{VG})^2}{\alpha^2-(\beta+\bar\theta_{VG}+1)^2}\right)^\lambda,$$

or equivalently,

$$e^{\frac{r-\mu}{\lambda}}=\frac{\alpha^2-(\beta+\bar\theta_{VG})^2}{\alpha^2-(\beta+\bar\theta_{VG}+1)^2}. \tag{A.7}$$

Case 1: $r=\mu$
In this case, (A.7) becomes

$$\alpha^2-(\beta+\bar\theta_{VG})^2=\alpha^2-(\beta+\bar\theta_{VG}+1)^2$$

which apparently is solved by $\bar\theta_{VG}=-\frac{1}{2}-\beta$. This is a proper solution as can be seen analogously as in the NIG model.

Case 2: $r\neq\mu$
To simplify the notation and formulas in the derivation of $\bar\theta_{VG}$, we set, similarly as before, $r^*:=\frac{r-\mu}{\lambda}$ and $\beta^*=\beta+\bar\theta_{VG}$. Multiplying both sides of (A.7) with $\alpha^2-(\beta^*+1)^2$ yields

$$e^{r^*}\left(\alpha^2-(\beta^*+1)^2\right)=\alpha^2-\beta^{*2}.$$

Expanding the expressions and rearranging terms we obtain

$$\beta^{*2}+\frac{2}{1-e^{-r^*}}\beta^*+\left(\frac{1}{1-e^{-r^*}}-\alpha^2\right)=0.$$

The solutions of this quadratic equation are given by

$$\beta^*=\beta+\bar\theta_{VG}=-\frac{1}{1-e^{-r^*}}\pm\sqrt{\frac{e^{-r^*}}{(1-e^{-r^*})^2}+\alpha^2}.$$

Thus, the possible risk-neutral Esscher parameters are

$$\bar\theta_{VG}=-\frac{1}{1-e^{-r^*}}-\beta\pm\sqrt{\frac{e^{-r^*}}{(1-e^{-r^*})^2}+\alpha^2}. \tag{A.8}$$

Observe that the mgf M_{VG} is only defined on the interval $(-\alpha-\beta,\alpha-\beta)$, therefore $\bar\theta_{VG}\in(-\alpha-\beta,\alpha-\beta-1)$ must hold. Further, note that we always have $\sqrt{\frac{e^{-r^*}}{(1-e^{-r^*})^2}+\alpha^2}>\alpha$. Now suppose that $r>\mu$, then $e^{-r^*}<1$, or equivalently, $-\frac{1}{1-e^{-r^*}}<0$. Thus,

$$-\frac{1}{1-e^{-r^*}}-\beta-\sqrt{\frac{e^{-r^*}}{(1-e^{-r^*})^2}+\alpha^2}<-\beta-\alpha.$$

Hence, this solution of (A.7) does not lie within $(-\alpha-\beta, \alpha-\beta-1)$, so the unique solution in the case $r > \mu$ is given by

$$\bar{\theta}_{VG} = -\frac{1}{1-e^{-\frac{r-\mu}{\lambda}}} - \beta + \sqrt{\frac{e^{-\frac{r-\mu}{\lambda}}}{\left(1-e^{-\frac{r-\mu}{\lambda}}\right)^2} + \alpha^2}. \tag{A.9}$$

If on the other hand $r < \mu$, then we have $e^{-r*} > 1$, resp. $-\frac{1}{1-e^{-r*}} > 0$, so the above solution (A.9) lies outside $(-\alpha-\beta, \alpha-\beta-1)$ because

$$-\frac{1}{1-e^{-r^*}} - \beta + \sqrt{\frac{e^{-r^*}}{(1-e^{-r^*})^2} + \alpha^2} > -\beta + \alpha.$$

Consequently, the unique solution in the case $r < \mu$ is given by

$$\bar{\theta}_{VG} = -\frac{1}{1-e^{-\frac{r-\mu}{\lambda}}} - \beta - \sqrt{\frac{e^{-\frac{r-\mu}{\lambda}}}{\left(1-e^{-\frac{r-\mu}{\lambda}}\right)^2} + \alpha^2}. \tag{A.10}$$

Again, we have $\phi_{VG}(u) = M_{VG}(iu)$ and conclude from (A.6) that $\phi_{L_t}(u) = \phi_{VG(\lambda,\alpha,\beta,\mu)}(u)^t = \phi_{VG(\lambda t,\alpha,\beta,\mu t)}(u)$, i.e., for a VG Lévy process $(L_t)_{t\geq 0}$ it holds that $\mathcal{L}(L_t) = VG(\lambda t, \alpha, \beta, \mu t)$ for all $t > 0$. Similarly as in the NIG case, one can also show that every Esscher transform of the real-world measure P only affects the parameter β and $(L_t)_{t\geq 0}$ remains a VG Lévy process under Q^θ, but with different parameter $\beta + \theta$.

Appendix B. Proofs

Proof of Proposition 2.1: Let X_T be a payoff with distribution G, denote the distribution function of $Z_T^{\bar{\theta}} = \frac{e^{\bar{\theta} L_T}}{M_{L_T}(\bar{\theta})}$ by $F_{Z_T^{\bar{\theta}}}$ and observe that the representation of $Z_T^{\bar{\theta}}$ implies that the continuity of F_{L_T} transfers to $F_{Z_T^{\bar{\theta}}}$. From Theorem 2.1 we already know that the cost-efficient payoff in general is given by $\underline{X}_T = G^{-1}(1 - F_{Z_T^{\bar{\theta}}}(Z_T^{\bar{\theta}}))$. If now $\bar{\theta} < 0$, then

$$1 - F_{Z_T^{\bar{\theta}}}(x) = 1 - P\big(Z_T^{\bar{\theta}} \leq x\big) = F_{L_T}\left(\frac{1}{\bar{\theta}} \ln\big(x M_{L_T}(\bar{\theta})\big)\right),$$

from which $\underline{X}_T = G^{-1}(F_{L_T}(L_T))$ follows immediately. In a similar way one obtains $\overline{X}_T = G^{-1}(1 - F_{L_T}(L_T))$, and the representations for the case $\bar{\theta} > 0$ can be proven analogously. The formulas for the price bounds are easily obtained by observing that the above equation implies $F^{-1}_{Z_T^{\bar{\theta}}}(x) = \frac{e^{\bar{\theta} F^{-1}_{L_T}(1-x)}}{M_{L_T}(\bar{\theta})}$ which just has to be inserted into the general Eqs. (2.5) of Theorem 2.1.

To show the a.s. uniqueness of the strategies, suppose that again $\bar{\theta} < 0$ and $\underline{X}'_T \sim G$ is another cost-efficient strategy with payoff-distribution G. Then $(\underline{X}'_T, Z_T^{\bar{\theta}})$ must also be countermonotonic by Theorem 2.1, hence $\underline{X}'_T = h(Z_T^{\bar{\theta}})$ a.s. for some decreasing function h. On the other hand, we know that $\underline{X}_T = G^{-1}(1 - F_{Z_T^{\bar{\theta}}}(Z_T^{\bar{\theta}}))$. But since both $\underline{X}_T$ and $\underline{X}'_T$ have the same distribution function G, it must hold that $h(z) = G^{-1}(1 - F_{Z_T^{\bar{\theta}}}(z))$ for almost all $z \in \mathbb{R}$ and hence $\underline{X}_T = \underline{X}'_T$ a.s. The proofs of uniqueness in the other cases follow analogously. □

Proof of Corollary 2.1: Suppose that $\bar{\theta} < 0$ and $\underline{X}_T$ is a cost-efficient payoff with distribution function G. On the one hand, due to Proposition 2.1, it holds that $\underline{X}_T = G^{-1}(F_{L_T}(L_T))$ almost surely, such that $\underline{X}_T$ is increasing in L_T, i.e., $X_T = v(L_T)$ for some measurable, increasing function v. On the other hand, if a payoff X_T is increasing in L_T, then the representation $Z_T^{\bar{\theta}} = \frac{e^{\bar{\theta} L_T}}{M_{L_T}(\bar{\theta})}$, together with the assumption $\bar{\theta} < 0$, implies that $X_T = h(Z_T)$ where $h(z) = v\big(\bar{\theta}^{-1}\ln\big(zM_{L_T}(\bar{\theta})\big)\big)$ is decreasing. Therefore, X_T and Z_T are countermonotonic and hence X_T is cost-efficient due to Theorem 2.1 (b). The second statement as well as the reverse for the most-expensive strategies can be shown analogously. □

Proof of Proposition 2.2: Assume $\bar{\theta} < 0$ first. Then $f(x) := e^{\bar{\theta}x}$ and $g(x) := e^x$ are decreasing resp. increasing functions on $\mathbb{R}$. Hence, $f(L_1)$ and $g(L_1)$ are countermonotonic random variables and thus $\mathrm{Cov}(f(L_1), g(L_1)) \leq 0$, (see Lehman (1966, Lemma 3)). Therefore, we obtain

$$0 \geq \mathrm{Cov}(e^{\bar{\theta}L_1}, e^{L_1}) = M_{L_1}(\bar{\theta}+1) - M_{L_1}(\bar{\theta})M_{L_1}(1),$$

and equality holds true if and only if $e^{\bar{\theta}L_1}$ and e^{L_1} are independent, which is not true here because L_1 is nondegenerate by Assumption 2.1. Further observe that the assumptions made in the Proposition ensure the finiteness of all expressions on the right hand side. Together with (2.7), the latter inequality thus implies

$$E[e^{L_t}] = M_{L_1}(1)^t > \left(\frac{M_{L_1}(\bar{\theta}+1)}{M_{L_1}(\bar{\theta})}\right)^t = e^{rt}$$

for all $t > 0$, which corresponds to a bullish market.

To show the converse, assume now a bullish market scenario where the expected return $E[e^{L_t}]$ is greater than e^{rt} for all $t > 0$, or equivalently,

$$M_{L_1}(1) > e^r = \frac{M_{L_1}(\bar{\theta}+1)}{M_{L_1}(\bar{\theta})},$$

since $\bar{\theta}$ is the unique solution of Eq. (2.7). From latter inequality we infer that $M_{L_1}(\bar{\theta}+1) < M_{L_1}(\bar{\theta})M_{L_1}(1)$ and hence $\mathrm{Cov}(e^{\bar{\theta}L_1}, e^{L_1}) < 0$. From this we directly conclude that the risk-neutral Esscher parameter $\bar{\theta}$ must be strictly negative because otherwise $f(x) = e^{\bar{\theta}x}$ and $g(x) = e^x$ are both non-decreasing functions on $\mathbb{R}$, implying that $\mathrm{Cov}(f(L_1), g(L_1))$ cannot be negative if L_1 is nondegenerate. The statement for the bearish market can be shown analogously. □

Proof of Proposition 2.3: By Definition 2.2, we have $\theta \in (a, b)$ where $a < 0 < b$, so there exists a sufficiently small $\epsilon > 0$ such that also $\theta - \epsilon, \theta + \epsilon \in (a, b)$. Thus, we have

$$E_\theta[e^{\pm\epsilon L_T}] = \frac{E[e^{(\theta\pm\epsilon)L_T}]}{M_{L_T}(\theta)} = \frac{M_{L_T}(\theta\pm\epsilon)}{M_{L_T}(\theta)} < \infty$$

and conclude that L_T also has a mgf $M^\theta_{L_T}(u)$ under the Esscher measure Q^θ which is well-defined and finite at least on the open interval $(-\epsilon, \epsilon)$. In particular, this implies $E_\theta[L_T^2] < \infty$ and hence $E_\theta[|L_T|] < \infty$.

Thus, we can differentiate the Esscher density $Z^\theta_T = \frac{e^{\theta L_T}}{M_{L_T}(\theta)}$ with respect to θ and obtain

$$\begin{aligned}\frac{\partial Z^\theta_T}{\partial\theta} &= \frac{L_T e^{\theta L_T} M_{L_T}(\theta) - e^{\theta L_T} M'_{L_T}(\theta)}{M_{L_T}(\theta)^2} \\ &= Z^\theta_T L_T - \frac{e^{\theta L_T} E[L_T e^{\theta L_T}]}{M_{L_T}(\theta)^2} = Z^\theta_T (L_T - E_\theta[L_T])\end{aligned}$$

where in the second equality we used that $M'_{L_T}(\theta) = E\big[\frac{\partial}{\partial\theta}e^{\theta L_T}\big]$. The interchange between differentiation and integration here is justified because $E\big[\big|\frac{\partial}{\partial\theta}e^{\theta L_T}\big|\big] = E[|L_T|e^{\theta L_T}] = M_{L_T}(\theta)E_\theta[|L_T|] < \infty$ as shown above. Further, observe that X_T does not depend on θ, and by Proposition 2.1, neither does $\underline{X}_T$, such that we have

$\frac{\partial Z_T^\theta(X_T-\underline{X}_T)}{\partial\theta} = \frac{\partial Z_T^\theta}{\partial\theta}(X_T-\underline{X}_T)$ and

$$\begin{aligned} E\left[\left|\frac{\partial Z_T^\theta}{\partial\theta}(X_T-\underline{X}_T)\right|\right] &= E\big[\big|Z_T^\theta(L_T-E_\theta[L_T])(X_T-\underline{X}_T)\big|\big] \\ &\le E\big[Z_T^\theta(|L_T|+E_\theta[|L_T|])\,|X_T-\underline{X}_T|\big] \\ &= E_\theta\big[|L_T|\,|X_T-\underline{X}_T|\big]+E_\theta\big[|L_T|\big]E_\theta\big[|X_T-\underline{X}_T|\big]<\infty \end{aligned}$$

because $E_\theta[L_T^2]<\infty$ and, by assumption, also $E_\theta\big[(X_T-\underline{X}_T)^2\big]<\infty$. This again allows to interchange differentiation and integration in the following calculation which yields, similarly as above,

$$\begin{aligned} \frac{\partial l(\theta,\eta)}{\partial\theta} &= e^{-rT}\frac{\partial E\big[Z_T^\theta(X_T-\underline{X}_T)\big]}{\partial\theta} = e^{-rT}E\left[\frac{\partial Z_T^\theta}{\partial\theta}\,(X_T-\underline{X}_T)\right] \\ &= e^{-rT}E\big[Z_T^\theta(L_T-E_\theta[L_T])(X_T-\underline{X}_T)\big] \\ &= e^{-rT}\Big(E\big[Z_T^\theta L_T(X_T-\underline{X}_T)\big]-E_\theta\big[L_T\big]E\big[Z_T^\theta(X_T-\underline{X}_T)\big]\Big) \\ &= e^{-rT}\mathrm{Cov}_\theta(L_T,X_T-\underline{X}_T). \end{aligned}$$

Hence, $l(\theta,\eta)$ is increasing in θ iff $\mathrm{Cov}_\theta(L_T,X_T)\ge\mathrm{Cov}_\theta(L_T,\underline{X}_T)$. The latter inequality is fulfilled for $\theta>0$, because $\underline{X}_T$ then is defined by Eq. (2.9) and hence is a decreasing function of L_T, implying that $(L_T,\underline{X}_T)$ is a countermonotonic pair of random variables, and thus, has the smallest covariance among all pairs of random variables possessing the same marginal distributions. Analogously, one obtains that $l(\theta,\eta)$ is decreasing for $\theta<0$, because in this case $\underline{X}_T$ is defined by Eq. (2.8) and thus is an increasing function of L_T. Therefore, $(L_T,\underline{X}_T)$ is comonotonic and hence $\mathrm{Cov}_\theta(L_T,X_T)\le\mathrm{Cov}_\theta(L_T,\underline{X}_T)$. □

Proof of Theorem 4.1: (a) Fix $t\in[0,T)$ and recall that by Eq. (4.1) the time-t-price of the efficient put initiated at time 0 is given by

$$c(\underline{X}_{T,t}^P)=\frac{1}{M_{L_{T-t}}(\bar\theta)}\int_{-\infty}^{+\infty}e^{\bar\theta y-r(T-t)}\big(K-S_0e^{F_{L_T}^{-1}(1-F_{L_T}(\ln(S_te^y/S_0)))}\big)_+\,F_{L_{T-t}}(dy)$$

The above integrand $f(S_t,y):\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}_+$ with

$$f(S_t,y)=e^{\bar\theta y-r(T-t)}\big(K-S_0e^{F_{L_T}^{-1}(1-F_{L_T}(\ln(S_te^y/S_0)))}\big)_+$$

is bounded for all $S_t\ge0$ by $g(y)=Ke^{\bar\theta y}$ which is integrable with respect to $F_{L_{T-t}}$ because $\int_{-\infty}^{+\infty}g(y)F_{L_{T-t}}(dy)=KM_{L_{T-t}}(\bar\theta)<\infty$. Moreover, $f(S_t,y)$ is differentiable in S_t for all $y\in\mathbb{R}$ (apart from the point $S_t=S_0e^{F_{L_T}^{-1}(1-F_{L_T}(\ln(K/S_0)))-y}$, but since the left- and right-hand side derivatives are bounded, this can be neglected here), and the partial derivative is

$$\begin{aligned} \frac{\partial f(S_t,y)}{\partial S_t} &= \frac{S_0}{S_t}e^{\bar\theta y-r(T-t)+F_{L_T}^{-1}(1-F_{L_T}(\ln(S_te^y/S_0)))} \\ &\quad\cdot\frac{d_{L_T}\big(\ln\big(\frac{S_te^y}{S_0}\big)\big)}{d_{L_T}\big(1-F_{L_T}\big(\ln\big(\frac{S_te^y}{S_0}\big)\big)\big)}\,\mathbb{1}_{\big(\ln(\frac{S_0}{S_t})+F_{L_T}^{-1}(1-F_{L_T}(\ln(\frac{K}{S_0}))),\infty\big)}(y) \end{aligned}$$

where d_{L_T} denotes the Lebesgue density of F_{L_T} which exists and is strictly positive on $\mathbb{R}$ due to our assumptions on F_{L_T}. Since $\bar\theta<0$, $1-F_{L_T}\big(\ln\big(\frac{S_te^y}{S_0}\big)\big)\to0$ and $d_{L_T}\big(\ln\big(\frac{S_te^y}{S_0}\big)\big)\to0$ for $y\to\infty$ (the latter must hold for any probability density), we see that $\frac{\partial}{\partial S_t}f(S_t,y)\le\frac{S_0}{S_td_{L_t}(0)}$ for sufficiently large y. Therefore, the partial derivative is integrable with respect to $F_{L_{T-t}}$, so we can interchange differentiation and integration and obtain

$$\begin{aligned}\underline{\Delta}_t^P &= \frac{\partial c(\underline{X}_{T,t}^P)}{\partial S_t} = \frac{\partial}{\partial S_t}\frac{1}{M_{L_{T-t}}(\bar{\theta})}\int_{-\infty}^{+\infty} f(S_t,y)\,F_{L_{T-t}}(dy)\\ &= \frac{1}{M_{L_{T-t}}(\bar{\theta})}\int_{-\infty}^{+\infty}\frac{\partial f(S_t,y)}{\partial S_t}\,F_{L_{T-t}}(dy)\\ &= \frac{S_0}{S_t}\frac{e^{-r(T-t)}}{M_{L_{T-t}}(\bar{\theta})}\\ &\quad\cdot\int_{y_{T,t}^P}^{+\infty} e^{\bar{\theta}y+F_{L_T}^{-1}(1-F_{L_T}(\ln(S_te^y/S_0)))}\frac{d_{L_T}\big(\ln\big(\frac{S_te^y}{S_0}\big)\big)}{d_{L_T}\big(1-F_{L_T}\big(\ln\big(\frac{S_te^y}{S_0}\big)\big)\big)}\,F_{L_{T-t}}(dy)\end{aligned}$$

where $y_{T,t}^P = \ln\big(\frac{S_0}{S_t}\big) + F_{L_T}^{-1}\big(1-F_{L_T}\big(\ln\big(\frac{K}{S_0}\big)\big)\big)$.

(b) In case of an efficient call we have $c(\underline{X}_{T,t}^C) = \frac{1}{M_{L_{T-t}}(\bar{\theta})}\int_{-\infty}^{+\infty}\tilde{f}(S_t,y)\,F_{L_{T-t}}(dy)$ with

$$\tilde{f}(S_t,y) = e^{\bar{\theta}y-r(T-t)}\big(S_0e^{F_{L_T}^{-1}(1-F_{L_T}(\ln(S_te^y/S_0)))}-K\big)_+$$

and, by assumption, $\tilde{f}(S_t,y)$ is integrable w.r.t. $F_{L_{T-t}}$ for all $S_t \geq 0$. Moreover, $\tilde{f}(S_t,y)$ is differentiable in S_t (again, apart from $S_t = S_0e^{F_{L_T}^{-1}(1-F_{L_T}(\ln(K/S_0)))-y}$), and the partial derivative is

$$\begin{aligned}\frac{\partial\tilde{f}(S_t,y)}{\partial S_t} &= -\frac{S_0}{S_t}e^{\bar{\theta}y-r(T-t)+F_{L_T}^{-1}(1-F_{L_T}(\ln(S_te^y/S_0)))}\\ &\quad\cdot\frac{d_{L_T}\big(\ln\big(\frac{S_te^y}{S_0}\big)\big)}{d_{L_T}\big(1-F_{L_T}\big(\ln\big(\frac{S_te^y}{S_0}\big)\big)\big)}\,\mathbb{1}_{\big(-\infty,\ln(\frac{S_0}{S_t})+F_{L_T}^{-1}(1-F_{L_T}(\ln(\frac{K}{S_0})))\big)}(y).\end{aligned}$$

Similarly as above, we have $d_{L_T}\big(\ln\big(\frac{S_te^y}{S_0}\big)\big)\to 0$ and $d_{L_T}\big(1-F_{L_T}\big(\ln\big(\frac{S_te^y}{S_0}\big)\big)\big)\to d_{L_T}(1)$ for $y\to-\infty$, so the quotient of the densities remains bounded as $y\to-\infty$, thus the integrability of $\tilde{f}(S_t,y)$ with respect to $F_{L_{T-t}}$ readily transfers to $\frac{\partial\tilde{f}(S_t,y)}{\partial S_t}$. Hence, we can again interchange differentiation and integration and obtain

$$\begin{aligned}\underline{\Delta}_t^C &= \frac{\partial c(\underline{X}_{T,t}^C)}{\partial S_t} = \frac{1}{M_{L_{T-t}}(\bar{\theta})}\int_{-\infty}^{+\infty}\frac{\partial\tilde{f}(S_t,y)}{\partial S_t}\,F_{L_{T-t}}(dy)\\ &= -\frac{S_0}{S_t}\frac{e^{-r(T-t)}}{M_{L_{T-t}}(\bar{\theta})}\\ &\quad\cdot\int_{-\infty}^{y_{T,t}^C} e^{\bar{\theta}y+F_{L_T}^{-1}(1-F_{L_T}(\ln(S_te^y/S_0)))}\frac{d_{L_T}\big(\ln\big(\frac{S_te^y}{S_0}\big)\big)}{d_{L_T}\big(1-F_{L_T}\big(\ln\big(\frac{S_te^y}{S_0}\big)\big)\big)}\,F_{L_{T-t}}(dy)\end{aligned}$$

with $y_{T,t}^C = y_{T,t}^P$. □

Proof of Theorem 4.2: (a) Fix $t\in[0,T)$. The price $c(\underline{X}_{T-t}^P) = c(\underline{X}_{T-t,0}^P)$ of the cost-efficient long put with maturity T that is initiated at time t is obtained from Proposition 2.1 (a) and Eq. (3.1) by replacing T by $T-t$ as well as S_0 by S_t:

$$c(\underline{X}_{T-t}^P) = \frac{e^{-r(T-t)}}{M_{L_{T-t}}(\bar{\theta})}\int_0^1 e^{\bar{\theta}F_{L_{T-t}}^{-1}(z)}\Big(K-S_te^{F_{L_{T-t}}^{-1}(1-z)}\Big)_+\,dz$$

It can easily be seen that the above integrand $f(S_t,z):\mathbb{R}_+\times[0,1]\to\mathbb{R}_+$ with

$$f(S_t,z) = e^{\bar{\theta}F_{L_{T-t}}^{-1}(z)-r(T-t)}\big(K-S_te^{F_{L_{T-t}}^{-1}(1-z)}\big)_+$$

is integrable with respect to z for all $S_t \geq 0$ since it is bounded by the function $g(z) = K\, e^{\bar{\theta} F^{-1}_{L_{T-t}}(z) - r(T-t)}$. For the integrability of $g(z)$, observe that

$$\int_0^1 g(z)\, dz = K \int_0^1 e^{\bar{\theta} F^{-1}_{L_{T-t}}(z) - r(T-t)} dz$$

$$= K \int_{-\infty}^{+\infty} e^{\bar{\theta} x - r(T-t)} d_{L_{T-t}}(x)\, dx = K e^{-r(T-t)}\, M_{L_{T-t}}(\bar{\theta}) < \infty,$$

where $d_{L_{T-t}}$ denotes the density of $F_{L_{T-t}}$. Moreover, $f(S_t, z)$ is differentiable in S_t for all $z \in [0, 1]$ (apart from the point $S_t = K e^{F^{-1}_{L_{T-t}}(1-z)}$, but again the left- and right-hand side derivatives are bounded), and the partial derivative is

$$\frac{\partial}{\partial S_t} f(S_t, z) = e^{\bar{\theta} F^{-1}_{L_{T-t}}(z) - r(T-t)} \big(-e^{F^{-1}_{L_{T-t}}(1-z)}\big) \mathbb{1}_{[1 - F_{L_{T-t}}(\ln(K/S_t)), 1]}(z).$$

Its absolute value is bounded by the integrable function

$$\tilde{g}(z) = \frac{K}{S_t}\, e^{\bar{\theta} F^{-1}_{L_{T-t}}(z) - r(T-t)}.$$

Hence, we can interchange differentiation and integration and obtain

$$\Delta_t^{roP} = \frac{\partial}{\partial S_t} c(\underline{X}^P_{T-t}) = \frac{1}{M_{L_{T-t}}(\bar{\theta})} \int_0^1 \frac{\partial}{\partial S_t} f(S_t, z)\, dz$$

$$= -\frac{1}{M_{L_{T-t}}(\bar{\theta})} \int_{1 - F_{L_{T-t}}(\ln(K/S_t))}^1 e^{\bar{\theta} F^{-1}_{L_{T-t}}(z) + F^{-1}_{L_{T-t}}(1-z) - r(T-t)} dz$$

$$= -\frac{1}{M_{L_{T-t}}(\bar{\theta})} \int_0^{F_{L_{T-t}}(\ln(K/S_t))} e^{\bar{\theta} F^{-1}_{L_{T-t}}(1-y) + F^{-1}_{L_{T-t}}(y) - r(T-t)} dy.$$

(b) The price $c(\underline{X}^C_{T-t})$ of the cost-efficient call with maturity T that is initiated at time t can analogously derived as in part (a) from Proposition 2.1 b) and Eq. (3.8) to be

$$c(\underline{X}^C_{T-t}) = \frac{e^{-r(T-t)}}{M_{L_{T-t}}(\bar{\theta})} \int_0^1 e^{\bar{\theta} F^{-1}_{L_{T-t}}(z)} \Big(S_t e^{F^{-1}_{L_{T-t}}(1-z)} - K\Big)_+ dz.$$

Here we consider the function

$$\tilde{f}(S_t, z) : \mathbb{R}_+ \times [0, 1] \to \mathbb{R}_+, \qquad \tilde{f}(S_t, z) = e^{\bar{\theta} F^{-1}_{L_{T-t}}(z) - r(T-t)} \big(S_t e^{F^{-1}_{L_{T-t}}(1-z)} - K\big)_+$$

which is integrable with respect to z for all $S_t \geq 0$ because $c(\underline{X}^C_{T-t}) \leq c(X^C_{T-t}) \leq e^{-r(T-t)} E_{\bar{\theta}}[S_t e^{L_{T-t}}] = S_t$. Further, $\tilde{f}(S_t, z)$ is differentiable in S_t for all $z \in [0, 1]$ (apart from $S_t = K e^{-F^{-1}_{L_{T-t}}(1-z)}$ which again is negligible here), and we have

$$\frac{\partial}{\partial S_t} \tilde{f}(S_t, z) = e^{\bar{\theta} F^{-1}_{L_{T-t}}(z) - r(T-t)} \cdot e^{F^{-1}_{L_{T-t}}(1-z)}\, \mathbb{1}_{[0, 1 - F_{L_{T-t}}(\ln(K/S_t))]}(z) \geq 0.$$

Clearly, the integrability of $\tilde{f}(S_t, z)$ with respect to z readily transfers to $\frac{\partial}{\partial S_t} \tilde{f}(S_t, z)$, thus we can again interchange differentiation and integration and obtain

$$\Delta_t^{roC} = \frac{\partial}{\partial S_t} c(\underline{X}^C_{T-t}) = \frac{1}{M_{L_{T-t}}(\bar{\theta})} \int_0^1 \frac{\partial}{\partial S_t} \tilde{f}(S_t, z)\, dz$$

$$= \frac{1}{M_{L_{T-t}}(\bar{\theta})} \int_0^{1 - F_{L_{T-t}}(\ln(K/S_t))} e^{\bar{\theta} F^{-1}_{L_{T-t}}(z) + F^{-1}_{L_{T-t}}(1-z) - r(T-t)} dz.$$

□

Proof of Theorem 4.3: (a) Since the vanilla and the efficient call coincide for $\bar\theta < 0$, the equation $\Delta_t^{roC} = \Delta_t^C$ is immediately obvious, thus we only have to consider the case $\bar\theta > 0$. Fix an arbitrary $t \in [0,T)$. Because the vanilla call is most-expensive for $\bar\theta > 0$, we can combine Proposition 2.1 and Eq. (3.8) to represent its time-t-price by

$$c(X^C_{T-t}) = \frac{1}{M_{L_{T-t}}(\bar\theta)} \int_0^1 e^{\bar\theta F^{-1}_{L_{T-t}}(y) - r(T-t)} \big(S_t e^{F^{-1}_{L_{T-t}}(y)} - K\big)_+ \, dy$$

from which one can derive completely analogously as in the proof of Theorem 4.2 the following formula for the corresponding delta:

$$\begin{aligned}\Delta_t^C &= \frac{1}{M_{L_{T-t}}(\bar\theta)} \int_{F_{L_{T-t}}\left(\ln\left(\frac{K}{S_t}\right)\right)}^1 e^{\bar\theta F^{-1}_{L_{T-t}}(y) + F^{-1}_{L_{T-t}}(y) - r(T-t)} \, dy \\ &= \frac{1}{M_{L_{T-t}}(\bar\theta)} \int_0^{1-F_{L_{T-t}}\left(\ln\left(\frac{K}{S_t}\right)\right)} e^{\bar\theta F^{-1}_{L_{T-t}}(1-y) + F^{-1}_{L_{T-t}}(1-y) - r(T-t)} \, dy. \qquad \text{(B.1)}\end{aligned}$$

Because $\Delta_t^{roC} \geq 0$ by Theorem 4.2, the assertion of the theorem is proven if we can show that $\Delta_t^C - \Delta_t^{roC} \geq 0$. Comparing Eqs. (4.6) and (B.1), the latter inequality obviously is equivalent to the statement that the function $D_C : [0,1] \to \mathbb{R}$, defined by

$$D_C(q) = \frac{1}{M_{L_{T-t}}(\bar\theta)} \int_0^q e^{\bar\theta F^{-1}_{L_{T-t}}(1-y) + F^{-1}_{L_{T-t}}(1-y)} - e^{\bar\theta F^{-1}_{L_{T-t}}(y) + F^{-1}_{L_{T-t}}(1-y)} \, dy,$$

is nonnegative for all $q \in [0,1]$. We have $D_C(0) = 0$ and

$$\begin{aligned}D_C(1) &= \frac{1}{M_{L_{T-t}}(\bar\theta)} \int_0^1 e^{\bar\theta F^{-1}_{L_{T-t}}(1-y) + F^{-1}_{L_{T-t}}(1-y)} - e^{\bar\theta F^{-1}_{L_{T-t}}(y) + F^{-1}_{L_{T-t}}(1-y)} \, dy \\ &= \int_{-\infty}^{\infty} \frac{e^{\bar\theta z}}{M_{L_{T-t}}(\bar\theta)} e^z f_{L_{T-t}}(z) \, dz - \int_{-\infty}^{\infty} \frac{e^{\bar\theta z}}{M_{L_{T-t}}(\bar\theta)} e^{F^{-1}_{L_{T-t}}(1-F_{L_{T-t}}(z))} f_{L_{T-t}}(z) \, dz \\ &= E\left[Z^{\bar\theta}_{T-t} \frac{S_{T-t}}{S_0}\right] - E\left[Z^{\bar\theta}_{T-t} e^{F^{-1}_{L_{T-t}}(1-F_{L_{T-t}}(L_{T-t}))}\right] \geq 0\end{aligned}$$

because $\frac{S_{T-t}}{S_0} = e^{L_{T-t}} \overset{d}{=} e^{F^{-1}_{L_{T-t}}(1-F_{L_{T-t}}(L_{T-t}))}$, but $Z^{\bar\theta}_{T-t}$, $e^{L_{T-t}}$ are comonotonic and $Z^{\bar\theta}_{T-t}$, $e^{F^{-1}_{L_{T-t}}(1-F_{L_{T-t}}(L_{T-t}))}$ are countermonotonic for $\bar\theta > 0$. Moreover,

$$D_C'(q) = \frac{1}{M_{L_{T-t}}(\bar\theta)} \left[e^{\bar\theta F^{-1}_{L_{T-t}}(1-q) + F^{-1}_{L_{T-t}}(1-q)} - e^{\bar\theta F^{-1}_{L_{T-t}}(q) + F^{-1}_{L_{T-t}}(1-q)}\right]$$

from which we conclude

$$D_C'(q) = 0 \iff e^{\bar\theta F^{-1}_{L_{T-t}}(1-q)} = e^{\bar\theta F^{-1}_{L_{T-t}}(q)} \iff q = 0.5.$$

The assumptions on $F_{L_{T-t}}$ imply that $F^{-1}_{L_{T-t}}(q)$ is strictly increasing as well, so the above calculations further show that $D_C'(q) \geq 0$ for $q \leq 0.5$ and $D_C'(q) \leq 0$ for $q \geq 0.5$. Hence, the function $D_C(q)$ is increasing on $[0, 0.5]$ and decreasing on $[0.5, 1]$ with boundary values $D_C(0) = 0$ and $D_C(1) \geq 0$ which yields that $D_C(q) \geq 0$ for all $q \in [0,1]$, and thus $\Delta_t^C - \Delta_t^{roC} \geq 0$.

(b) For $\bar\theta > 0$, the equality $\Delta_t^{roP} = \Delta_t^P$ again follows from the fact that vanilla and efficient put coincide in this case, therefore we assume $\bar\theta < 0$ in the following. Then the vanilla put is most-expensive, and combining Proposition 2.1 and Eq. (3.1) allows to represent its time-t-price as

$$c(X^P_{T-t}) = \frac{1}{M_{L_{T-t}}(\bar\theta)} \int_0^1 e^{\bar\theta F^{-1}_{L_{T-t}}(1-y) - r(T-t)} \big(K - S_t e^{F^{-1}_{L_{T-t}}(1-y)}\big)_+ \, dy$$

from which the delta can be derived as

$$\Delta_t^P = -\frac{1}{M_{L_{T-t}}(\bar\theta)} \int_{1-F_{L_{T-t}}\left(\ln\left(\frac{K}{S_t}\right)\right)}^{1} e^{\bar\theta F^{-1}_{L_{T-t}}(1-y)+F^{-1}_{L_{T-t}}(1-y)-r(T-t)}\,dy$$
$$= -\frac{1}{M_{L_{T-t}}(\bar\theta)} \int_0^{F_{L_{T-t}}\left(\ln\left(\frac{K}{S_t}\right)\right)} e^{\bar\theta F^{-1}_{L_{T-t}}(y)+F^{-1}_{L_{T-t}}(y)-r(T-t)}\,dy. \quad \text{(B.2)}$$

Because $\Delta_t^{roP}, \Delta_t^P \le 0$, the assertion of the theorem is equivalent to $\Delta_t^{roP} - \Delta_t^P \ge 0$. Analogously as in the call case we see by comparing Eqs. (4.5) and (B.2) that for given values K and S_t we have $\Delta_t^{roP} - \Delta_t^P \ge 0$ if and only if $D_P\left[F_{L_{T-t}}\left(\ln\left(\frac{K}{S_t}\right)\right)\right] \ge 0$, where the function $D_P(q) : [0,1] \to \mathbb{R}$ is defined by

$$D_P(q) = \frac{1}{M_{L_{T-t}}(\bar\theta)} \int_0^q e^{\bar\theta F^{-1}_{L_{T-t}}(y)+F^{-1}_{L_{T-t}}(y)} - e^{\bar\theta F^{-1}_{L_{T-t}}(1-y)+F^{-1}_{L_{T-t}}(y)}\,dy.$$

We have $D_P(0) = 0$ and calculate, similarly as before, that

$$D_P(1) = E\left[Z^{\bar\theta}_{T-t}\frac{S_{T-t}}{S_0}\right] - E\left[Z^{\bar\theta}_{T-t}\, e^{F^{-1}_{L_{T-t}}(1-F_{L_{T-t}}(L_{T-t}))}\right] \le 0$$

since $Z^{\bar\theta}_{T-t}$, $e^{L_{T-t}}$ are countermonotonic for $\bar\theta < 0$ and $Z^{\bar\theta}_{T-t}$, $e^{F^{-1}_{L_{T-t}}(1-F_{L_{T-t}}(L_{T-t}))}$ are comonotonic. Further,

$$D'_P(q) = \frac{1}{M_{L_{T-t}}(\bar\theta)}\left[e^{\bar\theta F^{-1}_{L_{T-t}}(q)+F^{-1}_{L_{T-t}}(q)} - e^{\bar\theta F^{-1}_{L_{T-t}}(1-q)+F^{-1}_{L_{T-t}}(q)}\right]$$

and hence

$$D'_P(q) = 0 \iff e^{\bar\theta F^{-1}_{L_{T-t}}(1-q)} = e^{\bar\theta F^{-1}_{L_{T-t}}(q)} \iff q = 0.5.$$

Since $F^{-1}_{L_{T-t}}(q)$ is strictly increasing and $\bar\theta < 0$, we see that $D'_P(q) \ge 0$ for $q \le 0.5$ and $D'_P(q) \le 0$ for $q \ge 0.5$, consequently the function $D_P(q)$ has a positive maximum in $q = 0.5$ and is strictly decreasing on $(0.5, 1]$. The fact that $D_P(1) \le 0$ then implies the existence of a unique $q^* \in (0.5, 1]$ with $D_P(q^*) = 0$ and $D_P(q) \ge 0$ for all $q \in [0, q^*]$. This proves the assertion. □

References

O.E. Barndorff-Nielsen (1995), Normal inverse Gaussian processes and the modelling of stock returns, Technical Report 300, Department of Theoretical Statistics, University of Aarhus.

O.E. Barndorff-Nielsen (1998), Processes of normal inverse Gaussian type, *Finance and Stochastics* **2**(1), 41–68.

J. Bergenthum & L. Rüschendorf (2008), Comparison results for path-dependent options, *Statistics & Decisions* **26**(1), 53–72.

C. Bernard & P.P. Boyle (2010), Explicit representation of cost-efficient strategies, working paper, available at SSRN (`http://ssrn.com/abstract=1695543`).

C. Bernard, P.P. Boyle, & S. Vanduffel (2014), Explicit representation of cost-efficient strategies, forthcoming in *Finance* **35**(2).

M. Brodén & P. Tankov (2011), Tracking errors from discrete hedging in exponential Lévy models, *International Journal of Theoretical and Applied Finance* **14**(6), 803–837.

T. Chan (1999), Pricing contingent claims on stocks driven by Lévy processes, *Annals of Applied Probability* **9**, 504–528.

R. Cont & P. Tankov (2004), *Financial Modelling with Jump Processes*, first edition, Financial Mathematics Series, Boca Raton, FL: Chapman & Hall/CRC.

J.C. Cox & H.E. Leland (2000), On dynamic investment strategies, *Journal of Economic Dynamics & Control* **24** (11-12), 1859–1880.

R.-A. Dana (2005), A representation result for concave Schur concave functions, *Mathematical Finance* **15**(4), pp. 613–634.

P. Dybvig (1988), Distributional analysis of portfolio choice, *Journal of Business* **61**(3), 369–393.

P. Dybvig (1988), Inefficient dynamic portfolio strategies or how to throw away a million dollars in the stock market, *Review of Financial Studies* **1**(1), 67–88.

F. Esche & M. Schweizer (2005), Minimal entropy preserves the Lévy property: how and why, *Stochastic Processes and Their Applications* **115**(2), 299–327.

H. Föllmer & A. Schied (2004), *Stochastic Finance. An Introduction in Discrete Time*, second revised and extended edition, Berlin: de Gruyter.

H.U. Gerber & E.S.W. Shiu (1994), Option pricing by Esscher transforms, *Transactions of Society of Actuaries* **46**, 99–191.

T. Goll & L. Rüschendorf (2002), Minimal distance martingale measures and optimal portfolios consistent with observed market prices, In: *Stochastic Processes and Related Topics* (R. Buckdahn and H.J. Engelbert and M. Yor, eds.), Stochastics Monographs, Taylor & Francis, pp. 141–154.

F. Hubalek & C. Sgarra (2006), Esscher transforms and the minimal entropy martingale measure for exponential Lévy models, *Quantitative Finance* **6**(2), 125–145.

E. Jouini & H. Kallal (2001), Efficient trading strategies in the presence of market frictions, *Review of Financial Studies* **14**(2), 343–369.

S. Kassberger & T. Liebmann (2011), When are path-dependent payoffs suboptimal?, *Journal of Banking & Finance* **36**, 1304–1310.

U. Keller (1997), Realistic modelling of financial derivatives, PhD thesis, University of Freiburg.

S.G. Kou (2002), A jump-diffusion model for option pricing, *Management Science* **48**(8), 1086–1101.

E.L. Lehman (1966), Some concepts of dependence, *Annals of Mathematical Statistics* **37**(5), 1137–1153.

D.B. Madan & F. Milne (1991), Option pricing with V.G. martingale components, *Mathematical Finance* **1**(4), 39–55.

D.B. Madan & E. Seneta (1990), The Variance Gamma (V.G.) model for share market returns, *Journal of Business* **63**, 511–524.

Y. Miyahara (1999), Minimal entropy martingale measures of jump type price processes in incomplete asset markets, *Asia–Pacific Financial Markets* **6**, 97–113.

S. Raible (2000), Lévy processes in finance: theory, numerics, and empirical facts, PhD thesis, University of Freiburg.

L. Rüschendorf (2012), Risk bounds, worst case dependence and optimal claims and contracts, *Proceedings of the AFMATH Conference*, Brussels, 23–36.

L. Rüschendorf (2013), *Mathematical Risk Analysis. Dependence, Risk Bounds, Optimal Allocations and Portfolios* Berlin: Springer-Verlag.

W. Schoutens (2003), *Lévy Processes in Finance: Pricing Financial Derivatives*, Wiley Series in Probability and Statistics, Chichester: Wiley.

S. Vanduffel & X. Chen and J. Dhaene and M. Goovaerts and L. Henrard (2008), Optimal approximations for risk measures of sums of lognormals based on conditional expectations, *Journal of Computational and Applied Mathematics* **221**(1), 202–218.

S. Vanduffel, A. Chernih, M. Maj, & W. Schoutens (2009), A note on the suboptimality of path-dependent pay-offs in Lévy markets, *Applied Mathematical Finance* **16**(4), 315–330.

S. Vanduffel, A. Ahcan, L. Henrard, & M. Maj (2012), An explicit option-based strategy that outperforms Dollar cost averaging, *International Journal of Theoretical and Applied Finance* **15**(2) (`http://www.worldscientific.com/doi/abs/10.1142/S0219024912500136`).