Existence of solutions of martingale problems using duality

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August 2020

Duality and martingale problems

- **Martingale problem:** $G_X : \mathcal{D}_X \subseteq \mathcal{C}_b(E_X) \to \mathcal{B}(E_X)$;
  $X = (X_t)_{t\geq 0}$ solves the G_X-martingale problem if

  $$\Big(f(X_t) - \int_0^t G_X f(X_s) ds\Big)_{t\geq 0}$$

  is a martingale for all $f \in \mathcal{D}_X$.

  $\to$ Uniqueness: by uniqueness of one-dimensional distributions

  $\to$ Existence: by approximation techniques
- **Duality:** X and Y are dual with respect to some H if

  $$\mathbb{E}_x[H(X_t, y)] = \mathbb{E}_y[H(x, Y_t)], \qquad t, x, y. \qquad (*)$$

  $\to$ If Y exists and $\Pi_X := \{H(., y) : y \in E_Y\}$ is separating, this shows uniqueness of one-dimensional distributions for X.
- **Here:** Use duality to show existence of G_X-martingale problem

  $\to$ Origin of the idea: Evans (1997), Dynkin (1993).

Duality and martingale problems

$$\mathbb{E}_x[H(X_t, y)] = \mathbb{E}_y[H(x, Y_t)] \qquad (*)$$

- If G_X is the generator for X and G_Y is the generator for Y, and

  $$G_X H(., y)(x) = G_Y H(x, .)(y),$$

  then, (on a probability space where X and Y are independent),

  $$\begin{aligned}\frac{d}{ds}\mathbb{E}_{(x,y)}[H(X_s, Y_{t-s})] &= \mathbb{E}_{(x,y)}[G_X H(., Y_{t-s})(X_s) - G_Y H(X_s, .)(Y_{t-s})] \\ &= 0,\end{aligned}$$

  and integrating gives $(*)$.

Existence and duality

- **Theorem:** (Depperschmidt, Greven, P., 2020) Let H be such that Π_X is separating, G_X be given and Y a Markov process which solves the G_Y-martingale problem.
  Assume that for all x, y, t, there exists $\mu_t(x, .)$ such that (some measurability condition holds and)

$$\mathbb{E}_y[H(x, Y_t)] = \int \mu_t(x, dx')H(x', y). \qquad (\diamond)$$

  Then, the G_X martingale problem is well-posed, its solution X satisfies $X_t \sim \mu_t(x, .)$ and

$$\mathbb{E}_x[H(X_t, y)] = \mathbb{E}_y[H(x, Y_t)] \qquad (*)$$

  holds.
  If Π_X is convergence determining and Y is Feller, then X is Feller as well.

- Proof: Chapman-Kolmogoroff for X follow from Y being Markov.

Example: Fleming-Viot process

- For $x \in \mathcal{P}(I)$ and $y \in E_Y := \bigcup_n \mathcal{C}_b(I^n)$,

$$H(x,y) = \int x^{\otimes}(d\underline{u})y(\underline{u}) = \langle x^{\otimes}, y\rangle$$

- Let $Y = (Y_t)_{t\geq 0}$ be E_Y-valued, with generator, for $y \in \mathcal{C}_b(I^n)$,

$$G_Y H(x,y) = \sum_{i\neq j} H(x, y\circ\theta_{ij}) - H(x,y)$$

with

$$\theta_{ij}(u_1, u_2, ...) = (u_1, ..., u_{j-1}, u_i, u_{j+1}, ...).$$

Example: Fleming-Viot process

- Is there some $X_t \sim \mu_t(x,.)$ satisfying $(\diamond)$?
- Fix x. The map $y \mapsto \mathbb{E}_y[\langle x^{\otimes}, Y_t\rangle]$ is a linear form, which can by continuity extended to a linear form on the set $\mathcal{C}_b(I^{\mathbb{N}})$.
- By the Riesz-Markov Theorem, there is $\mu \in \mathcal{P}(I^{\mathbb{N}})$ such that

$$\int \mu(d\underline{u})y(\underline{u}) = \mathbb{E}_y[\langle x^{\otimes}, Y_t\rangle].$$

- Since μ is invariant under coordinate permutations, by deFinetti's Theorem, there is an $\mathcal{P}(I)$-valued random variable X_t such that

$$\mathbb{E}[\langle X_t^{\otimes}, y\rangle] = \int \mu(d\underline{u})y(\underline{u}).$$

So, $(\diamond)$ holds and well-posedness of the martingale problem for

$$G_X\langle x^{\otimes}, y\rangle = \sum_{i\neq j}\langle(\theta_{ij})_*)x^{\otimes}, y\rangle - \langle x^{\otimes} y\rangle.$$

The solution is called the Fleming-Viot process.

Extensions, conclusion and outlook

- $G_X = G_X^1 + G_X^2$ allows for existence by duality using Trotters theorem;
- Use of Riesz-Markov Theorem only works on compact spaces; compactification might be required;
- Application: Tree-valued Fleming-Viot process with recombination
- The manuscript *Duality and the well-posedness of a martingale problem* at `https://arxiv.org/abs/1904.01564` should be updated soon.