FINANCIAL MATHEMATICS

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1 Stochastic Exponential

For a semimartingale X with $X_0 = 0$, the *stochastic exponential* of X, written $\mathcal{E}(X)$, is the (unique) semimartingale Z that is a solution of

$$Z_t = 1 + \int_0^t Z_{s-}\, dX_s.$$

The stochastic exponential is also known as the *Doléans-Dade exponential*. An important special case is when the semimartingale X is continuous. Then the stochastic exponential is given by

$$\mathcal{E}(X)_t = \exp\left\{X_t - \frac{1}{2}[X,X]_t\right\}.$$

The stochastic exponential is used to define a density process. In this context, consider also Girsanov's theorem. A central question that arises here is the following: Let M be a local martingale. When is $\mathcal{E}(M)$ a martingale? The only known general conditions that solve this problem are Kazamaki's criterion and Novikov's criterion. Moreover, these criteria apply only to local martingales with continuous paths. Novikov's criterion is a little less powerful than Kazamaki's, but it is much easier to check in practice. Since Novikov's criterion follows easily from Kazamaki's, we present both criteria here. Note that if M is a continuous local martingale, then $\mathcal{E}(M)$ is also a continuous local martingale. However, even if it is a uniformly integrable local martingale, it still need not be a martingale.

Lemma 1.1. *Let M be a continuous local martingale with $M_0 = 0$. Then $\mathcal{E}(M)$ is a supermartingale, hence $\mathbb{E}(\mathcal{E}(M)_t) \leq 1$ for all $t \geq 0$.*

Proof. Recall that $\mathcal{E}(M)_0 = 1$. Since M is a local martingale, $\mathcal{E}(M)$ is a nonnegative local martingale, whence a supermartingale.

Let T_n be a sequence of stopping times reducing $\mathcal{E}(M)$. Then $\mathbb{E}(\mathcal{E}(M)_{t\wedge T_n}) = 1$, and using Fatou's Lemma,

$$\mathbb{E}(\mathcal{E}(M)_t) = \mathbb{E}\left(\liminf_{n\to\infty} \mathcal{E}(M)_{t\wedge T_n}\right) \leq \liminf_{n\to\infty} \mathbb{E}(\mathcal{E}(M)_{t\wedge T_n}) = 1.$$

□

Theorem 1.2. *Let M be a continuous local martingale and T a bounded stopping time. Then*

$$\mathbb{E}\big[e^{\frac{1}{2}M_T}\big] \leq \mathbb{E}\big[e^{\frac{1}{2}[M,M]_T}\big]^{1/2}$$

Proof. Recall that

$$(\mathcal{E}(M)_T)^{1/2} = \left(e^{M_T - \frac{1}{2}[M,M]_T}\right)^{1/2} = e^{\frac{1}{2}M_T}\left(e^{-\frac{1}{2}[M,M]_T}\right)^{1/2},$$

which implies that

$$e^{\frac{1}{2}M_T} = (\mathcal{E}(M)_T)^{1/2}\left(e^{-\frac{1}{2}[M,M]_T}\right)^{1/2}$$

This, together with the Cauchy-Schwarz inequality and the fact that $\mathbb{E}(\mathcal{E}(M)_T) \leq 1$, gives the result. □

Lemma 1.3. *Let M be a continuous local martingale. Let $1 < p < \infty$, and set $\frac{1}{p} + \frac{1}{q} = 1$. Taking the supremum below over all bounded stopping times, assume that*

$$\sup_T \mathbb{E}\left(e^{\left(\frac{\sqrt{p}}{2\sqrt{p}-1}\right)M_T}\right) < \infty.$$

Then $\mathcal{E}(M)$ is an L^q-bounded martingale.

Proof. Let $1 < p < \infty$ and $r = \frac{\sqrt{p}+1}{\sqrt{p}-1}$. Then $s = \frac{\sqrt{p}+1}{2}$ and $\frac{1}{r} + \frac{1}{s} = 1$. Also, we note that $(q - \sqrt{\frac{q}{r}})s = \frac{\sqrt{p}}{2(\sqrt{p}-1)}$, which we use in the last equality of the proof. We have

$$\mathcal{E}(M)^q = e^{qM - \frac{q}{2}[M,M]_T} = e^{\sqrt{\frac{q}{r}}M - \frac{q}{2}[M,M]_T} e^{\left(q - \sqrt{\frac{q}{r}}\right)M}.$$

Now, we apply Hölder's inequality for a stopping time S:

$$\begin{aligned}\mathbb{E}(\mathcal{E}(M)^q) &\leq \mathbb{E}\left(e^{\sqrt{qr}M_S - \frac{qr}{2}[M,M]_S}\right)^{1/r} \mathbb{E}\left(e^{s(q-\sqrt{\frac{q}{r}})M_S}\right)^{1/s} \\ &\leq \mathbb{E}\left(e^{\sqrt{qr}M_S - \frac{qr}{2}[M,M]_S}\right)^{1/r} \mathbb{E}\left(e^{\frac{\sqrt{p}}{2(\sqrt{p}-1)}M_S}\right)^{1/s}.\end{aligned}$$

Recalling that $\mathbb{E}\left(e^{rM_S}\right) \leq 1$, we have the result. □

Theorem 1.4 (Kazamaki's Criterion)**.** *Let M be a continuous local martingale. Suppose*

$$\sup_\tau \mathbb{E}\left\{e^{\frac{1}{2}M_\tau}\right\} < \infty,$$

where the supremum is taken over all bounded stopping times. Then $\mathcal{E}(M)$ is a uniformly integrable martingale.

Proof. Let $0 < a < 1$, and $p > 1$ be such that $\frac{\sqrt{p}}{\sqrt{p}-1} < 1/a$. Our hypothesis combined with the preceding lemma implies that $\mathcal{E}(aM)$ is an L^q bounded martingale, where $\frac{1}{p} + \frac{1}{q} = 1$, which in turn implies it is a uniformly integrable martingale. However,

$$\begin{aligned}\mathcal{E}(aM) &= e^{aM - \frac{a^2}{2}[M,M]} = e^{a^2 M - \frac{a^2}{2}[M,M]} e^{a(1-a)M} \\ &= \mathcal{E}(aM)^{a^2} e^{a(1-a)M}.\end{aligned}$$

Using Hölder's inequality with exponents a^{-2} and $(1-a^2)^{-1}$ yields (where the 1 on the left side comes from the uniform integrability):

$$\begin{aligned}1 = \mathbb{E}\{E(aM)_\infty\} &\le \mathbb{E}(\mathcal{E}(M)_\infty)^{a^2} \left(\mathbb{E}((e^{a(1-a)M_\infty})^{1/(1-a^2)})\right)^{1-a^2} \\ &= \mathbb{E}(\mathcal{E}(M)_\infty)^{a^2} \left(\mathbb{E}\{e^{\frac{1}{2}M_\infty}\}\right)^{2a(1-a)}\end{aligned}$$

Now let $a \to 1$; the second term on the right-hand side of the inequality converges to 1 since $2a(1-a) \to 0$. Thus,

$$1 \le \mathbb{E}(\mathcal{E}(M)_\infty),$$

and since it is always true that $1 \ge \mathbb{E}(E(M)_\infty)$, we are done. □

Theorem 1.5 (Novikov's Criterion)**.** *Let M be a continuous local martingale, and suppose that*

$$\mathbb{E}\left(e^{\frac{1}{2}[M,M]_\infty}\right) < \infty.$$

Then $\mathcal{E}(M)$ is a uniformly integrable martingale.

Proof. We have

$$\mathbb{E}\left(\mathcal{E}(M)_T\right) \le \left(\mathbb{E}\left(e^{\frac{1}{2}[M,M]_T}\right)\right)^{\frac{1}{2}},$$

where T is a stopping time.

Now, applying Kazamaki's criterion, we conclude that $\mathcal{E}(M)$ is a uniformly integrable martingale. □

2 Lévy's Theorem

Theorem 2.1 (Lévy's Theorem)**.** *A stochastic process $X = (X_t)_{t\ge 0}$ is a standard Brownian motion if and only if it is a continuous local martingale with $[X,X]_t = t$.*

Proof. We have already observed that a Brownian motion B is a continuous local martingale and that $[B,B]_t = t$. Thus, it remains to show sufficiency.

Fix $u \in \mathbb{R}$ and set $F(x,t) = \exp\{iux + \frac{1}{2}u^2 t\}$. Let $Z_t = F(X_t, t) = \exp\{iuX_t + \frac{1}{2}u^2 t\}$. Since $F \in C^2$, we can apply Itô's formula to obtain

$$Z_t = 1 + iu\int_0^t Z_s\, dX_s + \frac{u^2}{2}\int_0^t Z_s\, ds - \frac{u^2}{2}\int_0^t Z_s\, d[X,X]_s.$$

Using $[X,X]_t = t$, this simplifies to

$$Z_t = 1 + iu\int_0^t Z_s\, dX_s,$$

which is the exponential equation. Since X is a continuous local martingale, we now have that Z is also one (complex-valued, of course) by the martingale preservation property.

Moreover, stopping Z at a fixed time τ_0, Z_{τ_0}, we have that Z_{τ_0} is bounded and hence a martingale. It then follows for $0 \leq s < t$ that

$$\mathbb{E}\{\exp\{iu(X_t - X_s)\} \mid \mathcal{F}_s\} = \exp\left\{-\frac{1}{2}u^2(t-s)\right\}.$$

Since this holds for any $u \in \mathbb{R}$, we conclude that $X_t - X_s$ is independent of $\mathcal{F}_s$ and that it is normally distributed with mean zero and variance $t - s$. Therefore, X is a Brownian motion. □

Theorem 2.2 (Lévy's Theorem: Multi-dimensional Version)**.** *Let $X = (X^1, \ldots, X^n)$ be continuous local martingales such that*

$$[X^i, X^j]_t = \begin{cases} t, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Then X is a standard n-dimensional Brownian motion.

3 Arbitrage and Duality in Nondominated Discrete-time Models

3.1 Existence of optimal superhedging strategies

In this section, we obtain the existence of optimal superhedging strategies via an elementary closedness property with respect to pointwise convergence. For the remainder of this section, $\mathcal{P}$ is any nonempty collection of probability measures on a general measurable space $(\Omega, \mathcal{F})$ with filtration $\{\mathcal{F}_t\}_{t \in \{0,1,\ldots,T\}}$, and $S = (S_0, S_1, \ldots, S_T)$ is any collection of $\mathcal{F}$-measurable, $\mathbb{R}^d$-valued random variables S_t.

Denote by $\mathcal{H}$ of trading strategies, i.e., the set of all predictable processes. For $H \in \mathcal{H}$, the wealth process $H \bullet S$ is defined by the discrete stochastic integral, and the condition NA($\mathcal{P}$) says that $H \bullet S_T \geq 0$ $\mathcal{P}$-q.s. implies $H \bullet S_T = 0$ $\mathcal{P}$-q.s.

We write $\mathcal{L}^0_+$ for the set of all nonnegative random variables. The following result states that the cone $\mathcal{C}$ of all claims which can be superreplicated from initial capital $x = 0$ is closed under pointwise convergence.

Theorem 3.1. *Let $\mathcal{C} := \{H \bullet S_T \mid H \in \mathcal{H}\} - \mathcal{L}^0_+$. If NA($\mathcal{P}$) holds, then $\mathcal{C}$ is closed under $\mathcal{P}$-q.s. convergence; that is, if $\{W^n\}_{n \geq 1} \subseteq \mathcal{C}$ and W is a random variable such that $W^n \to W$ $\mathcal{P}$-q.s., then $W \in \mathcal{C}$.*

Proof. Let

$$W^n = H^n \bullet S_T - K^n$$

be a sequence in $\mathcal{C}$ which converges $\mathcal{P}$-q.s. to a random variable W; we need to show that $W = H \bullet S_T - K$ for some $H \in \mathcal{H}$ and $K \in \mathcal{L}^0_+$. We shall use an induction over the number of periods in the market. The claim is trivial when there are zero periods. Hence, we show the passage from $T-1$ to T periods; more precisely, we shall assume that the claim is proved for any market with dates $\{1, 2, \ldots, T\}$, and we deduce the case with dates $\{0, 1, \ldots, T\}$.

For any real matrix M, let index(M) be the number of rows in M which vanish identically. Now let $\mathbb{H}_1$ be the random $(d \times \infty)$-matrix whose columns are given by the vectors $H_1^1, H_1^2, \dots$. Then index$(\mathbb{H}_1)$ is a random variable with values in $\{0, 1, \dots, d\}$. If index$(\mathbb{H}_1) = d$ $\mathcal{P}$-q.s., we have $H_1^n = 0$ for all n, so that setting $H_1 = 0$, we conclude immediately by the induction assumption. For the general case, we use another induction over $i = d, d-1, \dots, 0$; namely, we assume that the result is proved whenever index$(\mathbb{H}_1) \geq i$ $\mathcal{P}$-q.s., and we show how to pass to $i - 1$.

Indeed, assume that index$(\mathbb{H}_1) \geq i - 1 \in \{0, \dots, d-1\}$; we shall construct H separately on finitely many sets forming a partition of Ω. Consider first the set

$$\Omega_1 := \big\{\liminf \big|H_1^n\big| < \infty\big\} \in \mathcal{F}_0.$$

We can find $\mathcal{F}_0$-measurable random indices n_k such that on Ω_1, $H_1^{n_k}$ converges pointwise to a (finite) $\mathcal{F}_0$-measurable random vector H_1. As the sequence

$$\tilde{W}^k := W^{n_k} - H_1^{n_k} \Delta S_1 = \sum_{t=2}^{T} H_t^{n_k} \Delta S_t - K^{n_k}$$

converges to $W - H_1 \Delta S_1 =: \tilde{W}$ $\mathcal{P}$-q.s. on Ω_1, we can now apply the induction assumption to obtain $H_2, \dots, H_T$ and $K \geq 0$ such that

$$\tilde{W} = \sum_{t=2}^{T} H_t \Delta S_t - K$$

and therefore $W = H \bullet S_T - K$ on Ω_1. It remains to construct H on

$$\Omega_2 := \Omega_1^c = \big\{\liminf \big|H_1^n\big| = +\infty\big\}.$$

Let

$$G_1^n := \frac{H_1^n}{1 + |H_1^n|}.$$

As $|G_1^n| \leq 1$, there exist $\mathcal{F}_0$-measurable random indices n_k such that $G_1^{n_k}$ converges pointwise to an $\mathcal{F}_0$-measurable random vector G_1, and clearly $|G_1| = 1$ on Ω_2. Moreover, on Ω_2, we have $W^{n_k}/(1 + |H_1^{n_k}|) \to 0$ and hence $-G_1 \Delta S_1$ is the $\mathcal{P}$-q.s. limit of

$$\sum_{t=2}^{T} \frac{H_t^{n_k}}{1 + |H_1^{n_k}|} \Delta S_t - \frac{K^{n_k}}{1 + |H_1^{n_k}|}.$$

By the induction assumption, it follows that there exist $\tilde{H}_2, \dots, \tilde{H}_T$ such that $\sum_{t=2}^{T} \tilde{H}_t \Delta S_t \geq -G_1 \Delta S_1$ on $\Omega_2 \in \mathcal{F}_0$. Therefore,

$$G_1 \Delta S_1 + \sum_{t=2}^{T} \tilde{H}_t \Delta S_t = 0 \qquad \text{on } \Omega_2, \tag{1}$$

since otherwise the trading strategy $(G_1, \tilde{H}_2, \dots, \tilde{H}_T)\mathbf{1}_{\Omega_2}$ would violate NA$(\mathcal{P})$. As $|G_1| = 1$ on Ω_2, we have that for every $\omega \in \Omega_2$, at least one component $G_1^j(\omega)$ of $G_1(\omega)$ is nonzero. Therefore,

$$\begin{aligned}
\Lambda_1 &:= \Omega_2 \cap \big\{G_1^1 \neq 0\big\}, \\
\Lambda_j &:= \big(\Omega_2 \cap \big\{G_1^j \neq 0\big\}\big) \setminus (\Lambda_1 \cup \dots \cup \Lambda_{j-1}), \qquad j = 2, \dots, d
\end{aligned}$$

defines an $\mathcal{F}_0$-measurable partition of Ω_2. We then consider the vectors

$$\bar{H}_t^n := H_t^n - \sum_{j=1}^d \mathbf{1}_{\Lambda_j} \frac{H_1^{n,j}}{G_1^j}(G_1 \mathbf{1}_{\{t=1\}} + \tilde{H}_t \mathbf{1}_{\{t \geq 2\}}), \qquad t = 1, \dots, T.$$

Note that $\bar{H}^n \bullet S_T = H^n \bullet S_T$ by (1). Hence, we still have $W = \lim \bar{H}^n \bullet S_T - K^n$. However, on Ω_2, the resulting matrix $\bar{\mathbb{H}}_1$ now has $\mathrm{index}(\bar{\mathbb{H}}_1) \geq i$ since we have created an additional vanishing row: the jth component of $\bar{H}_1^n$ vanishes on Λ_j by construction, while the jth row of $\mathbb{H}_1$ cannot vanish on $\Lambda_j \subseteq \{G_1^j \neq 0\}$ by the definition of G_1. We can now apply the induction hypothesis for indices greater or equal to i to obtain H on Ω_2. Recalling that $\Omega = \Omega_1 \cup \Omega_2$, we have shown that there exist $H \in \mathcal{H}$ and $K \geq 0$ such that $W = H \bullet S_T - K$. □

Theorem 3.2. *Let* $\mathrm{NA}(\mathcal{P})$ *hold, and let* f *be a random variable. Then*

$$\pi(f) := \inf\{x \in \mathbb{R} \mid \exists H \in \mathcal{H} \text{ such that } x + H \bullet S_T \geq f \ \mathcal{P}\text{-q.s.}\} > -\infty,$$

and there exists $H \in \mathcal{H}$ *such that* $\pi(f) + H \bullet S_T \geq f$ $\mathcal{P}$*-q.s.*

Proof. The claim is trivial if $\pi(f) = \infty$. Suppose that $\pi(f) = -\infty$. Then, for all $n \geq 1$, there exists $H^n \in \mathcal{H}$ such that $-n + H^n \bullet S_T \geq f$ $\mathcal{P}$-q.s. and hence

$$H^n \bullet S_T \geq f + n \geq (f + n) \wedge 1 \qquad \mathcal{P}\text{-q.s.}$$

That is, $W^n := (f + n) \wedge 1 \in \mathcal{C}$ for all $n \geq 1$. Now Theorem 3.1 yields that $1 = \lim W^n$ is in $\mathcal{C}$, which clearly contradicts $\mathrm{NA}(\mathcal{P})$.

On the other hand, if $\pi(f)$ is finite, then $W^n := f - \pi(f) - 1/n \in \mathcal{C}$ for all $n \geq 1$ and thus $f - \pi(f) = \lim W^n \in \mathcal{C}$ by Theorem 3.1, which yields the existence of H. □

3.2 The one-period case

In this section, we prove the First Fundamental Theorem and the Superhedging Theorem in the one-period case; these results could also be generalized to the multi-period case. We consider an arbitrary measurable space $(\Omega, \mathcal{F})$ with a filtration $(\mathcal{F}_0, \mathcal{F}_1)$, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and a nonempty convex set $\mathcal{P} \subseteq \mathfrak{P}(\Omega)$. The stock price process is given by a deterministic vector $S_0 \in \mathbb{R}^d$ and an $\mathcal{F}_1$-measurable, $\mathbb{R}^d$-valued random vector S_1. We write ΔS for $S_1 - S_0$ and note that Q is a martingale measure simply if $E_Q[\Delta S] = 0$; with the convention $\infty - \infty := -\infty$. Moreover, we have $\mathcal{H} = \mathbb{R}^d$. We write $Q \lll \mathcal{P}$ if there is a $P \in \mathcal{P}$ such that $Q \ll P$. We define $\mathcal{Q} = \{Q \in \mathfrak{P}(\Omega) \mid Q \lll \mathcal{P}, E_Q[\Delta S] = 0\}$.

3.2.1 First fundamental theorem

Theorem 3.3. *The following are equivalent:*

(i) $\mathrm{NA}(\mathcal{P})$ *holds;*

(ii) for all $P \in \mathcal{P}$ *there exists* $Q \in \mathcal{Q}$ *such that* $P \ll Q$.

Proof. Exercise 10.1. □

In the following, we will use a separation theorem in $\mathbb{R}^k$. First, the following definitions: For any two convex subsets C_1 and C_2 of $\mathbb{R}^k$, a hyperplane H is said to separate C_1 and C_2 if C_1 is contained in one of the halfspaces corresponding to H and C_2 is contained in the other halfspace. A hyperplane separates C_1 and C_2 properly if both are not wholly contained in the hyperplane. For $\Gamma \subseteq \mathbb{R}^k$ we define the affine hull $\operatorname{aff}(\Gamma)$ by

$$\operatorname{aff}(\Gamma) \coloneqq \{x \in \mathbb{R}^k \mid x = \sum_{i=1}^{n} \alpha_i x_i, n \in \mathbb{N}, x_i \in \gamma, \sum_{i=1}^{n} \alpha_i = 1\}.$$

The relative interior $\operatorname{ri}(\Gamma)$ of Γ is defined by

$$\operatorname{ri}(\Gamma) \coloneqq \{x \in \Gamma \mid \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \cap \operatorname{aff}(\Gamma) \subset \gamma\}.$$

If Γ is convex then its relative interior is given by

$$\operatorname{ri}(\Gamma) = \{x \in \Gamma \mid \forall y \in \Gamma \setminus \{x\}\ \exists z \in \Gamma, \lambda \in (0,1) \text{ s.t. } x = \lambda y + (1-\lambda) z\}.$$

Theorem 3.4. *Let C_1 and C_2 be non-empty convex subsets of $\mathbb{R}^k$. There exists a hyperplane properly separating C_1 and C_2 if and only if $\operatorname{ri} C_1 \cap \operatorname{ri} C_2 = \emptyset$.*

Lemma 3.5 (Fundamental lemma)**.** *Let $\mathrm{NA}(\mathcal{P})$ hold, and let f be a random variable. Then*

$$0 \in \operatorname{ri}\big\{E_R[\Delta S] \mid R \in \mathfrak{P}(\Omega), R \lll \mathcal{P}, E_R\big[|\Delta S| + |f|\big] < \infty\big\} \subseteq \mathbb{R}^d.$$

Similarly, given $P \in \mathcal{P}$, we also have

$$0 \in \operatorname{ri}\big\{E_R[\Delta S] \mid R \in \mathfrak{P}(\Omega), P \ll R \lll \mathcal{P}, E_R\big[|\Delta S| + |f|\big] < \infty\big\} \subseteq \mathbb{R}^d.$$

Proof. We show only the second claim; the first one can be obtained by omitting the lower bound P in the subsequent argument. We fix P and f; moreover, we set $\mathcal{I}_k := \{1, \dots, d\}^k$ for $k = 1, \dots, d$ and

$$\Theta := \big\{R \in \mathfrak{P}(\Omega) \mid P \ll R \lll \mathcal{P}, E_R\big[|\Delta S| + |f|\big] < \infty\big\}.$$

Note that $\Theta \neq \emptyset$ by a standard construction of an equivalent measure $R \sim P$ under which ΔS and f is integrable. Given $I \in \mathcal{I}_k$, we denote

$$\Gamma_I := \big\{E_R\big[\big(\Delta S^i\big)_{i \in I}\big] \mid R \in \Theta\big\} \subseteq \mathbb{R}^k;$$

then our claim is that $0 \in \operatorname{ri}\Gamma_I$ for $I = (1, \dots, d) \in \mathcal{I}_d$. It is convenient to show more generally that $0 \in \operatorname{ri}\Gamma_I$ for all $I \in \mathcal{I}_k$ and all $k = 1, \dots, d$. We proceed by induction.

Consider first $k = 1$ and $I \in \mathcal{I}_k$; then I consists of a single number $i \in \{1, \dots, d\}$. If $\Gamma_I = \{0\}$, the result holds trivially, so we suppose that there exists $R \in \Theta$ such that $E_R[\Delta S^i] \neq 0$. We focus on the case $E_R[\Delta S^i] > 0$; the reverse case is similar. Then, $\mathrm{NA}(\mathcal{P})$ implies that $A := \{\Delta S^i < 0\}$ satisfies $R_1(A) > 0$ for some $R_1 \in \mathcal{P}$. By replacing R_1 with $R_2 := (R_1 + P)/2$ we also have that $R_2 \gg P$, and finally we replace R_2 with an equivalent probability R_3 such that $E_{R_3}[|\Delta S| + |f|] < \infty$; as a result, we have found $R_3 \in \Theta$ satisfying $E_{R_3}[\mathbf{1}_A \Delta S^i] < 0$. But then $R' \sim R_3 \gg P$ defined by

$$\frac{dR'}{dR_3} = \frac{\mathbf{1}_A + \varepsilon}{E_{R_3}[\mathbf{1}_A + \varepsilon]} \tag{2}$$

satisfies $R' \in \Theta$ and $E_{R'}[\Delta S^i] < 0$ for $\varepsilon > 0$ chosen small enough. Now set

$$R_\lambda := \lambda R + (1-\lambda)R' \in \Theta$$

for each $\lambda \in (0,1)$; then

$$0 \in \big\{E_{R_\lambda}\big[\Delta S^i\big] \mid \lambda \in (0,1)\big\} \subseteq \operatorname{ri}\big\{E_R\big[\Delta S^i\big] \mid R \in \Theta\big\},$$

which was the claim for $k=1$.

Let $1 < k \le d$ be such that $0 \in \operatorname{ri}\Gamma_I$ for all $I \in \mathcal{I}_{k-1}$; we show hat $0 \in \operatorname{ri}\Gamma_I$ for all $I \in \mathcal{I}_k$. Suppose that there exists $I = (i_1, \dots, i_k) \in \mathcal{I}_k$ such that $0 \notin \operatorname{ri}\Gamma_I$. Then, the convex set Γ_I can be separated from the origin, see Theorem 3.4; that is, we can find $y = (y^1, \dots, y^k) \in \mathbb{R}^k$ such that $|y| = 1$ and

$$0 \le \inf\left\{E_R\left[\sum_{j=1}^k y^j \Delta S^{i_j}\right] \mid R \in \Theta\right\}.$$

Using an argument similar to that which precedes (2) (Exercise 10.2), this implies that $\sum_{j=1}^k y^j \Delta S^{i_j} \ge 0$ $\mathcal{P}$-q.s., and thus $\sum_{j=1}^k y^j \Delta S^{i_j} = 0$ $\mathcal{P}$-q.s. by $\mathrm{NA}(\mathcal{P})$. As $|y|=1$, there exists $1 \le l \le k$ such that $y^l \neq 0$, and we obtain that

$$\Delta S^l = -\sum_{j=1}^k \delta_{j \neq l}\big(y^j/y^l\big)\Delta S^{i_j} \qquad \mathcal{P}\text{-q.s.}$$

Using the definition of the relative interior, the assumption that $0 \notin \operatorname{ri}\Gamma_I$ then implies that $0 \notin \operatorname{ri}\Gamma_{I'}$, where $I' \in \mathcal{I}_{k-1}$ is the vector obtained from I by deleting the lth entry. This contradicts our induction hypothesis. ☐

3.2.2 Superhedging theorem

We can now establish the Superhedging Theorem in the one-period case.

Theorem 3.6. *Let $\mathrm{NA}(\mathcal{P})$ hold, and let f be a random variable. Then*

$$\sup_{Q \in \mathcal{Q}} E_Q[f] = \pi(f) := \inf\big\{x \in \mathbb{R} \mid \exists H \in \mathbb{R}^d \text{ such that } x + H\Delta S \ge f \ \mathcal{P}\text{-q.s.}\big\}. \tag{3}$$

Moreover, $\pi(f) > -\infty$, and there exists $H \in \mathbb{R}^d$ such that $\pi(f) + H\Delta S \ge f$ $\mathcal{P}$-q.s.

The last statement is a consequence of Theorem 3.2. For the proof of (3), the inequality "$\le$" is left as an exercise (Exercise 10.4); For the inequality "$\ge$" we need to find $Q_n \in \mathcal{Q}$ such that $E_{Q_n}[f] \to \pi(f)$. Our construction proceeds in two steps. In the subsequent lemma, we find "approximate" martingale measures R_n such that $E_{R_n}[f] \to \pi(f)$; in its proof, it is important to relax the martingale property as this allows us to use arbitrary measure changes. In the second step, we replace R_n by true martingale measures, on the strength of the Fundamental Lemma: it implies that if R is any probability with $E_R[\Delta S]$ close to the origin, then there exists a perturbation of R which is a martingale measure.

Lemma 3.7. *Let $\mathrm{NA}(\mathcal{P})$ hold, and let f be a random variable with $\pi(f) = 0$. There exist probabilities $R_n \lll \mathcal{P}$, $n \ge 1$ such that*

$$E_{R_n}[\Delta S] \to 0 \quad \text{and} \quad E_{R_n}[f] \to 0.$$

Proof. The set
$$\Theta := \big\{R \in \mathfrak{P}(\Omega) \mid R \lll \mathcal{P}, E_R\big[|\Delta S| + |f|\big] < \infty\big\}$$
is nonempty. Introduce the set
$$\Gamma := \big\{E_R\big[(\Delta S, f)\big] \mid P \in \Theta\big\} \subseteq \mathbb{R}^{d+1};$$
then our claim is equivalent to $0 \in \overline{\Gamma}$, where $\overline{\Gamma}$ denotes the closure of Γ in $\mathbb{R}^{d+1}$. Suppose for contradiction that $0 \notin \overline{\Gamma}$, and note that Γ is convex because $\mathcal{P}$ is convex. Thus, $\overline{\Gamma}$ can be separated strictly from the origin; that is, there exist $(y, z) \in \mathbb{R}^d \times \mathbb{R}$ with $|(y, z)| = 1$ and $\alpha > 0$ such that
$$0 < \alpha = \inf_{R \in \Theta} E_R[y\Delta S + zf].$$
Using again a similar argument as before (2), this implies that
$$0 < \alpha \leq y\Delta S + zf \qquad \mathcal{P}\text{-q.s.} \tag{4}$$
Suppose that $z < 0$; then this yields that
$$f \leq \big|z^{-1}\big|y\Delta S - \big|z^{-1}\big|\alpha \qquad \mathcal{P}\text{-q.s.},$$
which implies that $\pi(f) \leq -|z^{-1}|\alpha < 0$ and thus contradicts the assumption that $\pi(f) = 0$. Hence, we must have $0 \leq z \leq 1$. But as $\pi(zf) = z\pi(f) = 0 < \alpha/2$, there exists $H \in \mathbb{R}^d$ such that $\alpha/2 + H\Delta S \geq zf$ $\mathcal{P}$-q.s., and then (4) yields
$$0 < \alpha/2 \leq (y + H)\Delta S \qquad \mathcal{P}\text{-q.s.},$$
which contradicts $\mathrm{NA}(\mathcal{P})$. This completes the proof. □

Lemma 3.8. *Let* $\mathrm{NA}(\mathcal{P})$ *hold, let* f *be a random variable and let* $R \in \mathfrak{P}(\Omega)$ *be such that* $R \lll \mathcal{P}$ *and* $E_R[|\Delta S| + |f|] < \infty$. *Then there exists* $Q \in \mathcal{Q}$ *such that* $E_Q[|f|] < \infty$ *and*
$$\big|E_Q[f] - E_R[f]\big| \leq c\big(1 + \big|E_R[f]\big|\big)\big|E_R[\Delta S]\big|,$$
where $c > 0$ *is a constant independent of* R *and* Q.

Proof. Let $\Theta = \{R' \in \mathfrak{P}(\Omega) \mid R' \lll \mathcal{P}, E_{R'}[|\Delta S| + |f|] < \infty\}$ and
$$\Gamma = \big\{E_{R'}[\Delta S] \mid R' \in \Theta\big\}.$$
If $\Gamma = \{0\}$, then $R \in \Theta$ is itself a martingale measure, and we are done. So let us assume that the vector space $\operatorname{span}\Gamma \subseteq \mathbb{R}^d$ has dimension $k > 0$, and let $e_1, \dots, e_k$ be an orthonormal basis. Lemma 3.5 shows that $0 \in \operatorname{ri}\Gamma$; hence, we can find $P_i^\pm \in \Theta$ and $\alpha_i^\pm > 0$ such that
$$\alpha_i^\pm E_{P_i^\pm}[\Delta S] = \pm e_i, \qquad 1 \leq i \leq k.$$
Note also that $P_i^\pm, \alpha_i^\pm$ do not depend on R.

Let $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$ be such that $-E_R[\Delta S] = \sum_{i=1}^k \lambda_i e_i$. Then we have $|\lambda| = |E_R[\Delta S]|$ and
$$-E_R[\Delta S] = \int \Delta S \, d\mu \qquad \text{for } \mu := \sum_{i=1}^k \lambda_i^+ \alpha_i^+ P_i^+ + \lambda_i^- \alpha_i^- P_i^-,$$

where λ_i^+ and λ_i^- denote the positive and the negative part of λ_i. Define the probability Q by

$$Q = \frac{R+\mu}{1+\mu(\Omega)};$$

then $R \ll Q \lll \mathcal{P}$ and $E_Q[\Delta S] = 0$ by construction. Moreover,

$$\begin{aligned} \big|E_Q[f] - E_R[f]\big| &= \frac{1}{1+\mu(\Omega)}\int f\,d\mu - \frac{\mu(\Omega)}{1+\mu(\Omega)}E_R[f] \\ &\leq \int f\,d\mu + \mu(\Omega)E_R[f] \\ &\leq c|\lambda|\big(1+\big|E_R[f]\big|\big), \end{aligned}$$

where c is a constant depending only on $\alpha_i^{\pm}$ and $E_{P_i^{\pm}}[f]$. It remains to recall that $|\lambda| = |E_R[\Delta S]|$. □

Proof of Theorem 3.6. The last claim holds by Theorem 3.2, so $\pi(f) > -\infty$. Let us first assume that f is bounded from above; then $\pi(f) < \infty$, and by a translation we may even suppose that $\pi(f) = 0$. By Theorem 3.3, the set $\mathcal{Q}$ of martingale measures is nonempty; moreover, $E_Q[f] \leq \pi(f) = 0$ for all $Q \in \mathcal{Q}$ by Exercise 10.4. Thus, we only need to find a sequence $Q_n \in \mathcal{Q}$ such that $E_{Q_n}[f] \to 0$. Indeed, Lemma 3.7 yields a sequence $R_n \lll \mathcal{P}$ such that $E_{R_n}[\Delta S] \to 0$ and $E_{R_n}[f] \to 0$. Applying Lemma 3.8 to each R_n, we obtain a sequence $Q_n \in \mathcal{Q}$ such that $E_{Q_n}[|f|] < \infty$ and

$$\big|E_{Q_n}[f] - E_{R_n}[f]\big| \leq c\big(1+\big|E_{R_n}[f]\big|\big)\big|E_{R_n}[\Delta S]\big| \to 0;$$

as a result, we have $E_{Q_n}[f] \to 0$ as desired.

It remains to discuss the case where f is not bounded from above. By the previous argument, we have

$$\sup_{Q\in\mathcal{Q}} E_Q[f\wedge n] = \pi(f\wedge n), \qquad n \in \mathbb{N}; \tag{5}$$

we pass to the limit on both sides. Indeed, on the one hand, we have

$$\sup_{Q\in\mathcal{Q}} E_Q[f\wedge n] \nearrow \sup_{Q\in\mathcal{Q}} E_Q[f]$$

by the monotone convergence theorem (applied to all Q such that $E_Q[f^-] < \infty$). On the other hand, it also holds that

$$\pi(f\wedge n) \nearrow \pi(f),$$

because if $\alpha := \sup_n \pi(f\wedge n)$, then $(f\wedge n) - \alpha \in \mathcal{C}$ for all n and thus $f - \alpha \in \mathcal{C}$ by Theorem 3.1, and in particular $\pi(f) \leq \alpha$. □