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Übungsblatt 11

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Definition 1 (T-forward measure). Let $(B_t)_{t\le T}$, $B_t = e^{\int_0^t r_s ds}$ be the bank account/numeraire in a financial market. If $\mathbb{Q}$ is a risk-neutral measure, then the forward measure $\mathbb{Q}^T$ on $\mathscr{F}_T$ is defined via the Radon Nikodym density process Z with respect to $\mathbb{Q}$ given by

$$Z_t = \frac{P_t(T)}{P_0(T)B_t}.$$

Aufgabe 1 (4 Points)**.** Let $F_t(T,S)$ be the simple forward rate for $[T,S]$ prevailing at t, which is given by

$$F_t(T,S) = \frac{1}{S-T}\left(\frac{P_t(T)}{P_t(S)} - 1\right), \quad t \in [0,T].$$

Show that $(F_t(T,S))_{t\in[0,T]}$ is a martingale with respect to some forward measure Q^U; that is

$$F_t(T,S) = E_{Q^U}[F_T(T,S)\,|\,\mathscr{F}_t] \quad \text{for all } t \in [0,T].$$

What is U?

Hint. Use the identity

$$P_t(T) = B_t\, E_Q\left[\frac{1}{B_T}\,\middle|\,\mathscr{F}_t\right], \quad t \in [0,T].$$

Aufgabe 2 (8 Points)**.** Consider a HJM model

$$f_t(T) = f_0(T) + \int_0^t \alpha_s(T)ds + \int_0^t \sigma_s(T)dW'_s$$

under an equivalent local martingale measure $Q \sim P$ with a driving $\mathbb{R}^m$-valued standard Q-Wiener process W'. We fix two dates T and S.

1. Show that the density process of Q^S relative to Q^T is given by

$$\frac{P_0(T)}{P_0(S)}\frac{P(S)}{P(T)}.$$

2. Show that the density process of Q^S relative to Q^T also equals

$$\mathcal{E}\big((\Sigma(S) - \Sigma(T)) \cdot W^T\big),$$

where $\Sigma_t(T) = -\int_t^T \sigma_t(s)ds$ and $W^T = W' - \Sigma(T)\cdot\lambda$ for $t \in [0,T]$ (here λ denotes the Lebesgue measure).

3. Now, we suppose $T < S$ and fix a strike price $K > 0$. We consider the price of a European call option on the S-bond at time $t = 0$, which is given by

$$\pi = E_Q\left[\frac{(P_T(S) - K)^+}{B_T}\right].$$

Show that

$$\pi = P_0(S)Q^S\Big(\ln\Big(\frac{P_T(T)}{P_T(S)}\Big) \le \ln\Big(\frac{1}{K}\Big)\Big) - KP_0(T)Q^T\Big(\ln\Big(\frac{P_T(S)}{P_T(T)}\Big) \ge \ln(K)\Big).$$

4. From now on, we assume that the volatilities $\sigma_s(T) = (\sigma_s^1(T), \dots, \sigma_s^m(T))$ are deterministic. Show that $\ln(\frac{P_T(T)}{P_T(S)})$ is normally distributed under Q^S, and determine its parameters.
5. Show that $\ln(\frac{P_T(S)}{P_T(T)})$ is normally distributed under Q^T, and determine its parameters.
6. Show that the option price is given by

$$\pi = P_0(S)\Phi(d_1) - K\,P_0(T)\Phi(d_2),$$

where Φ is the distribution function of the standard normal distribution, and

$$d_{1,2} = \frac{\ln(\frac{P_0(S)}{KP_0(T)}) \pm \frac{1}{2}\int_0^T \|\Sigma_s(T) - \Sigma_s(S)\|_{\mathbb{R}^m}^2 ds}{\sqrt{\int_0^T \|\Sigma_s(T) - \Sigma_s(S)\|_{\mathbb{R}^m}^2 ds}}.$$