Homework accompanying the lecture „Basics in Applied Mathematics“

Homework sheet for students who missed the stochastic part

Hand in: Tuesday, 07.01.2025, after the lecture in the mailbox at the Math Institut
(Don't forget to put your name on your homework.
Please hand in your solutions in groups of two.)

If you could not participate in the stochastic-part of the lecture, you will need to get at least **16 of the 32 points** from this exercise sheet. You should also be able to explain every solution you hand in, even if you worked in a group of two people.

Exercise 1 (4 points)

We define the discrete probability space $(\Omega, \mathcal{P}(\Omega), \mathbb{P})$ by

$$\Omega = \{0,1\}^5 = \{(x_1, ...x_5) \mid x_1, ..., x_5 \in \{0,1\}\}$$

and

$$\forall A \in \mathcal{P}(\Omega): \ \mathbb{P}(A) = \frac{|A|}{|\Omega|} \ .$$

a) Define the random variables $X_1, ..., X_5$ by

$$X_i : \Omega \to \mathbb{R} \ \ ; \ \ (x_1, ...x_5) \mapsto x_i \ .$$

What is the distribution of X_1? More precisely, was possible values can X_1 take and what are the probabilities of each value? (1P)

b) Prove that $X_1, ..., X_5$ are independent. (2P)

c) What is the distribution of the random variable $S = X_1 + ... + X_5$? (1P)

Exercise 2 (4 points)

When playing the lottery, you have to select 6 numbers from $\{1, ..., 49\}$ and hope that your selection is identical with the randomly selected 6 winning numbers. As all numbers have the same chance of being selected for winning numbers, the chance of winning is

$$\frac{1}{\binom{49}{6}} = \frac{6!\,43!}{49!} = \frac{1 \cdot 2 \cdot ... \cdot 6}{44 \cdot 45 \cdot ... \cdot 49} = \frac{1}{13983816} \ .$$

a) What are the chances of getting exactly 5 of the winning numbers correct. (2P)

b) If you could select 7 numbers instead of 6, with what probability would these 7 numbers contain the 6 winning numbers? (2P)

Exercise 3 (4 points)

We say a random variable X has Poisson distribution to parameter $\lambda > 0$ (write $X \sim \text{Poi}(\lambda)$), if it takes values in $\{0, 1, 2, ...\}$ and each $k \in \{0, 1, 2, ...\}$ has the probability

$$\mathbb{P}(X = k) = \frac{\lambda^k}{k!} e^{-\lambda} \ .$$

Let X_1, X_2 be two independent random variables with $X_1 \sim \text{Poi}(\lambda_1)$ and $X_2 \sim \text{Poi}(\lambda_2)$.

a) Prove that $X_1 + X_2$ has Poisson distribution to parameter $\lambda_1 + \lambda_2$. (2P)

b) Calculate the conditional probability

$$\mathbb{P}(X_1 + X_2 = m \mid X_2 \leq 1) \ .$$

(You will need to handle $m = 0$ differently than $m \geq 1$.) (2P)

Exercise 4 (4 points)

Say you own a rigged coin that shows head 80% of the time. You have accidentally stored it with 3 other coins, which are fair and can't tell which one was rigged. You take one of the coins and flip it 20 times. Let X denote the number of times the outcome is head. Let R denote the event where you have the rigged coin and $F = R^c$ denote the event that you have a fair coin.

What is the probability of R, depending on X, i.e. give the formula for

$$\mathbb{P}(R \mid X = k) \ .$$

Hint: You know $\mathbb{P}(X = k \mid R) = \binom{20}{k}(0.8)^k(0.2)^{20-k}$ and $\mathbb{P}(X = k \mid F) = \binom{20}{k}(0.5)^k(0.5)^{20-k}$.

Bonus-Exercise 5 (4 bonus points)

In the setting of exercise 4 you have observed heads fifteen times, so $X = 15$. Your friend grabs a different coin from the remaining three and throws it five times, getting only heads. He confidently tells you that he has the rigged coin. What are the chances that he is correct?

Attention: You must also account for the information of your own throws. The possible events are now R_1 (your coin is rigged), R_2 (your friends coin is rigged) and F (both of your coins are fair).

Exercise 6 (4 points)

We say a random variable X has geometric distribution with parameter $p \in [0,1]$, if it takes values in $\{1,2,...\}$ and

$$\mathbb{P}(X=k) = (1-p)^{k-1}p \quad \text{for all } k \in \{1,2,...\} \ .$$

Calculate the mean $\mathrm{E}[X]$ and variance $\mathrm{Var}[X]$.
HINT: For the variance it may help to first calculate $\mathrm{E}[X(X+1)]$.

Exercise 7 (8 points)

Let $Y_1, Y_2, ...$ and $C_1, C_2, ...$ be independent random variables with the distribution

$$\mathbb{P}(Y_n = 2) = \frac{1}{2} = \mathbb{P}(Y_n = -2)$$
$$\mathbb{P}(C_n = 1) = \frac{1}{3} \ ; \ \frac{2}{3} = \mathbb{P}(C_n = 0) \ .$$

Define the random variables $X_1, X_2, ...$ recursively by

$$X_1 = Y_1 \quad \text{and} \quad X_{n+1} = \mathbb{1}_{C_n=1} Y_{n+1} + \mathbb{1}_{C_n=0} X_n = \begin{cases} Y_{n+1} & \text{, if } C_n = 1 \\ X_n & \text{else} \end{cases} \ .$$

The sequence $X_1, X_2, ...$ could heuristically be generated by the following process: Randomly generate $X_1 = Y_1$ and flip a $(\frac{1}{3}, \frac{2}{3})$-coin C_1. If $C_1 = 0$ (Tails), then X_2 is set to be X_1. Otherwise X_2 is generated independently. Continue like this such that every X_{n+1} is either independently generated or copied from the value of X_n depending on the coin flip C_n.

a) By induction over $i \in \mathbb{N}$ prove that $\mathrm{E}[X_i^2] = 2$ and $\mathrm{E}[X_i] = 0$.

b) By induction over $j \in \mathbb{N}_0$, prove that the mean $\mathrm{E}[X_i X_{i+j}]$ has the form $2\left(\frac{2}{3}\right)^j$.

c) By induction over n prove that the variance $\mathbb{V}\left[\sum_{i=1}^n X_i\right]$ is no grater than $9n$.

d) Prove that $\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^n X_i\right| > \varepsilon\right) \xrightarrow{n\to\infty} 0$ for all $\varepsilon > 0$.

HINT: You may use without proof that X_i, C_n, Y_{n+1} are jointly independent for all $i \leq n \in \mathbb{N}$ and also the fact that $\mathrm{E}[f(X)g(Y)h(Z)] = \mathrm{E}[f(X)]\,\mathrm{E}[g(Y)]\,\mathrm{E}[h(Z)]$ holds for all independent random variables X, Y, Z and functions f, g, h.
Keep in mind that X_i and X_{i+j} are not independent, so $\mathrm{E}[X_i X_{i+j}] \neq \mathrm{E}[X_i]\,\mathrm{E}[X_{i+j}]$, however the product $X_i X_{i+j}$ is independent of $\mathbb{1}_{C_{i+j}=0} = f(C_{i+j})$, so $\mathrm{E}[\mathbb{1}_{C_{i+j}=0} X_i X_{i+j}] = \mathrm{E}[\mathbb{1}_{C_{i+j}=0}]\,\mathrm{E}[X_i X_{i+j}]$.

Exercise 8 (4 points)

For any sequence of random variables

$$X_1, X_2, \dots \ : \ \Omega \to \mathbb{R}$$

show

$$\{\omega \in \Omega \ : \ X_n(\omega) \xrightarrow{n\to\infty} 0\} = \bigcap_{K=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n \geq N} \{\omega \in \Omega \ : \ |X_n(\omega)| < 1/K\}$$

and also

$$\mathbb{P}(X_n \xrightarrow{n\to\infty} 0) = \lim_{K \nearrow \infty} \mathbb{P}\big(|X_n| \geq 1/K \text{ for finitely many } n \in \mathbb{N}\big) \ .$$

Bonus-Exercise 9 (4+3 bonus points)

Two continuous random variables X, Y on $\mathbb{R}$ can have a joint density $f_{(X,Y)}$, which will satisfy

$$\mathrm{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(s,t) f_{(X,Y)}(s,t) \, ds \, dt$$

for any continuous function g. Let (X, Y) have the jointly normal distribution

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}\Big(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \underbrace{\begin{pmatrix} a & c \\ c & b \end{pmatrix}}_{=:\Sigma} \Big) \ ,$$

where $a, b > 0$ and $ab > c^2$. The joint density is

$$f_{(X,Y)}(s,t) = \frac{1}{2\pi\sqrt{\det(\Sigma)}} \exp\Big(-\frac{1}{2} \begin{pmatrix} s & t \end{pmatrix} \cdot \Sigma^{-1} \cdot \begin{pmatrix} s \\ t \end{pmatrix} \Big) \ .$$

Prove that the covariance $\mathrm{Cov}[X, Y]$ is equal to c. You may use:

i) $\mathrm{E}[X] = 0 = \mathrm{E}[Y]$

ii) $\Sigma^{-1} = \frac{1}{ab-c^2} \begin{pmatrix} b & -c \\ -c & a \end{pmatrix}$

iii) $\int_{-\infty}^{\infty} s \exp\Big(-\frac{1}{2} \frac{(s-A)^2}{B} \Big) \, ds = \sqrt{2\pi B} \, A$ for all $A \in \mathbb{R}$ and $B > 0$

iv) $\int_{-\infty}^{\infty} t^2 \exp\Big(-\frac{1}{2} \frac{t^2}{B} \Big) \, dt = \sqrt{2\pi B} \, B$ for all $B > 0$

For extra 3 bouns-points ($\frac{1}{2}$P for (i) and (ii) ; 1P for (iii) and (iv) each) you may prove these properties.