Homework accompanying the lecture „Basics in Applied Mathematics"

Homework 9

Hand in: Tuesday, 26.11.2024, after the lecture in the mailbox at the Math Institut
(Don't forget to put your name on your homework.
Please hand in your solutions in groups of two.)

Exercise 1 (5 points)

Prove the following properties of the Chebyshev polynomials defined for $t \in [-1,1]$ by $T_n(t) = \cos(n \arccos t)$:

a) $\max_{t\in[-1,1]} |T_n(t)| = 1$

b) With $T_0(t) = 1$ and $T_1(t) = t$,
$$T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t)$$
for all $t \in [-1,1]$. In particular, $T_n \in \mathcal{P}_n|_{[-1,1]}$ applies.

c) For $n \geq 1$ it follows $T_n(t) = 2^{n-1}t^n + q_{n-1}(t)$ with $q_{n-1} \in \mathcal{P}_{n-1}|_{[-1,1]}$.

d) For $n \geq 1$, T_n has the roots $t_j = \cos((j+1/2)\pi/n)$, $j = 0.1, \ldots, n-1$, and the $n+1$ extreme points $s_j = \cos(j\pi/n), j = 0.1, \ldots, n$, with $T_n(s_j) = \pm 1$.

Exercise 2 (5 points)

a) Let $n \in \mathbb{N}$ and $l \in \mathbb{Z}$. Show that
$$\sum_{k=1}^{n-1} \mathrm{e}^{\mathrm{i}lk2\pi/n} = \begin{cases} n & \text{if } n \text{ divides } l, \\ 0 & \text{otherwise.} \end{cases}$$

b) Conclude that the Fourier basis $(\omega^0, \omega^1, \ldots, \omega^{n-1})$ with $\omega^k = (\omega_n^{0k}, \omega_n^{1k}, \ldots, \omega_n^{(n-1)k})^\top$, $k = 0, \ldots, (n-1)$ and the n-th complex root of unity $\omega_n = \mathrm{e}^{\mathrm{i}2\pi/n}$ the property
$$\omega^k \cdot \omega^l = n\delta_{kl}$$
owns.

Exercise 3 (4 points)

Use the representation of the error of the Lagrange interpolation
$$f(x) - p(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{j=0}^{n} (x - x_j),$$
$\xi(x) \in [a,b]$ to prove for the trapezoid or Simpson rule that
$$|I(f) - Q_{\mathrm{Trap}}(f)| \leq \frac{(b-a)^3}{12} \|f''\|_{C^0([a,b])},$$
$$|I(f) - Q_{\mathrm{Sim}}(f)| \leq \frac{(b-a)^5}{192} \|f^{(4)}\|_{C^0([a,b])}.$$

Tip: For Simpson's rule, use the fundamental theorem and a nice property of the polynomial $(x-a)(x-b)(x-\frac{a+b}{2})$.

Programming exercise 4 (4 points)

Use the summed trapezoid and Simpson's rules to determine the integrals in the interval $[0, 1]$ of the functions

$$f(x) = \sin(\pi x)\mathrm{e}^x \quad ; \quad g(x) = x^{1/3}$$

with step sizes $h = 2^{-l}$, $l = 1, 2, \dots, 10$. In each case, calculate the error e_h and determine the experimental convergence rates γ from the approach $e_h = c_1 h^\gamma$ and the formula

$$\gamma \approx \frac{\log(e_h/e_H)}{\log(h/H)}$$

for successive step sizes H and h. Comparatively represent the pairs h, e_h for the different quadrature formulas graphically as polygons in logarithmic axis scaling.