Homework accompanying the lecture „Basics in Applied Mathematics“

Homework 5

Hand in: Tuesday, 19.11.2024, after the lecture in the mailbox at the Math Institut
(Don't forget to put your name on your homework.
Please hand in your solutions in groups of two.)

Exercise 1 (4 points)

Suppose we roll a six sided dice repeatedly. Let N_i be the number of the roll on which we see i for the first time.

(a) Find the joint distribution of N_1 and N_6.

(b) Prove that the marginal distributions of N_1 is the geometric distribution with parameter $p = 1/6$.

Note: The marginal distribution of X given the joint distribution of X, Y is given by

$$p_X(x) \coloneqq \sum_{y \in \Omega^Y} p_{X,Y}(x, y).$$

(c) The conditional distribution of N_6 given $N_1 = i$.

Exercise 2 (4 points)

Prove the linearity of the conditional expectation. More precisely, let X, Y, Z be discrete random variables with existing expectation then for any constants $a, b \in \mathbb{R}$ prove the equality

$$\mathrm{E}[aY + bZ \mid X] = a\,\mathrm{E}[Y \mid X] + b\,\mathrm{E}[Z \mid X]\ .$$

Hint: You may want to first prove that it suffices to show
$\mathrm{E}[aY + bZ \mid X = x] = a\,\mathrm{E}[Y \mid X = x] + b\,\mathrm{E}[Z \mid X = x]$ for every $x \in \Omega^X$ with $\mathbb{P}(X = x) > 0$.

Exercise 3 (4 points)

A continuous random variable on $\mathbb{R}$ is a type of random variable not covered in this lecture. It can take values in an invervall and for each specific $x \in \mathbb{R}$ the probability $\mathbb{P}(X = x)$ is zero. An example would be the uniform distribution on the intervall $[0, 1]$. Suppose $X \sim \text{Uinf}([0, 1])$, then the probability that X lies in the interval $[a, b]$ is precisely $b - a$ for all $0 \leq a \leq b \leq 1$.

The distribution of a continuous random variable X on $\mathbb{R}$ is uniquely defined by its so-called density $f_X : \mathbb{R} \to [0, \infty)$. For every intervall $[a, b]$, with $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$, the probability that X is in $[a, b]$ has the form

$$\mathbb{P}(X \in [a, b]) = \int_a^b f_X(t)\, dt\ .$$

The distribution of a continuous random variable X on $\mathbb{R}$ is also uniquely defined by its (cumulative) distribution function

$$F_X(x) := \mathbb{P}(X \leq x) = \int_{-\infty}^x f_X(t)\, dt\ .$$

Here we also see the reason behind the notation f_x and F_X, since f_X is the derivative of F_X.

a) The density of the uniform distribution is $f_{\text{Unif}([0,1])}(t) = \begin{cases} 1 & \text{for } t \in [0,1] \\ 0 & \text{else} \end{cases}$.
What is the distribution function? Give $F_{\text{Unif}([0,1])}(x)$ for all values $x \in \mathbb{R}$.

b) The exponential distribution with parameter 1 satisfies $F_{\text{Exp}(1)}(x) = \begin{cases} 1 - e^{-x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$.
Find the density $f_{\text{Exp}(1)}$ and prove by calculation that $1 = \int_{\mathbb{R}} f_{\text{Exp}(1)}(t)\, dt$.

(This property is true for all density functions, since $1 = \mathbb{P}(X \in \mathbb{R}) = \int_{\mathbb{R}} f_X(t)\, dt$.)

c) The mean of a continuous random variable is given by $\text{E}[X] = \int_{-\infty}^{\infty} t\, f_X(t)\, dt$. Even better, we have

$$\text{E}[g(X)] = \int_{-\infty}^{\infty} g(t)\, f_X(t)\, dt$$

for any continuous function $g : \mathbb{R} \to \mathbb{R}$.

Calculate $\text{E}[X]$ for $X \sim \text{Exp}(1)$.

d) The (standard) normal distribution $\mathcal{N}(0, 1)$ has the density $f_{\mathcal{N}(0,1)}(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$.
Prove that $\text{E}[X] = 0$ for $X \sim \mathcal{N}(0, 1)$.

Exercise 4 (4 points)

This is a bonus exercise. The 4 achievable points are not counted to the point-total.

Two continuous random variables X, Y on $\mathbb{R}$ can have a joint density $f_{(X,Y)}$, which will satisfy

$$\mathrm{E}[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(s,t) f_{(X,Y)}(s,t)\, ds\, dt$$

for any continuous function g. Let (X,Y) have the jointly normal distribution

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}\Big(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \underbrace{\begin{pmatrix} a & c \\ c & b \end{pmatrix}}_{=:\Sigma}\Big),$$

where $a, b > 0$ and $ab > c^2$. The joint density is

$$f_{(X,Y)}(s,t) = \frac{1}{2\pi\sqrt{\det(\Sigma)}} \exp\Big(-\frac{1}{2}\begin{pmatrix} s & t \end{pmatrix} \cdot \Sigma^{-1} \cdot \begin{pmatrix} s \\ t \end{pmatrix}\Big).$$

Prove that the covariance $\mathrm{Cov}[X,Y]$ is equal to c. You may use:

i) $\mathrm{E}[X] = 0 = \mathrm{E}[Y]$

ii) $\Sigma^{-1} = \frac{1}{ab-c^2}\begin{pmatrix} b & -c \\ -c & a \end{pmatrix}$

iii) $\int_{-\infty}^{\infty} s \exp\Big(-\frac{1}{2}\frac{(s-A)^2}{B}\Big)\, ds = \sqrt{2\pi B}\, A$ for all $A \in \mathbb{R}$ and $B > 0$

iv) $\int_{-\infty}^{\infty} t^2 \exp\Big(-\frac{1}{2}\frac{t^2}{B}\Big)\, dt = \sqrt{2\pi B}\, B$ for all $B > 0$

Programming exercise 5 (2 points)

Let us introduce the uniform distribution $U([0,1])$ on the interval $[0,1]$. Let $Z \sim U([0,1])$ be an continuous random variable, which means that $\mathbb{P}(Z \leq z) = z$ for all $z \in [0,1]$.
We can simulate draws from any one dimensional distribution using draws from Z. This is called the inverse transform sampling.
Guided by a jupyter notebook you will implement such an inverse transform sampling for discrete variables.