Homework accompanying the lecture „Basics in Applied Mathematics“

Homework 6

Hand in: Tuesday, 26.11.2024, after the lecture in the mailbox at the Math Institut
(Don't forget to put your name on your homework.
Please hand in your solutions in groups of two.)

Exercise 1 (Cholesky decomposition ; 2 points)

Let $(v_1, v_2 \dots, v_n)$ be a basis of $\mathbb{R}^n$ and let $V = [v_1|v_2|\dots|v_n] \in \mathbb{R}^{n\times n}$.

a) Show that $G := VV^T$ is symmetric and positive definite.

b) Show that there exists a lower triangular matrix $L \in \mathbb{R}^{n\times n}$ such that for $W := LV$ the identity $W^T W = \mathrm{Id}$ holds.

Exercise 2 (Diagonally dominant matrices ; 2 points)

Let $A \in \mathbb{R}^{n\times n}$ be a strictly diagonally dominant matrix, i.e.

$$\sum_{j=1,\dots,n,\ j\neq i} |a_{ij}| < |a_{ii}|, \quad i = 1, 2, \dots, n.$$

a) Show that the submatrices $A_k := (a_{ij}), 1 \le i, j \le k$, for $k = 1, 2, \dots, n$ are also strictly diagonally dominant.

b) Show that the matrix A is regular (and with (1) of course also all its submatrices are regular).

Hint: To prove (2), show that the kernel of A is the null space.

Exercise 3 (Properties of the operator norm ; 3 points)

Let $\|\cdot\|_n$ and $\|\cdot\|_m$ be norms on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and let $\|\cdot\|_{\mathrm{op}}$ be the induced operator norm on $\mathbb{R}^{m\times n}$. Show:

a) $\|\cdot\|_{\mathrm{op}}$ defines a norm on $\mathbb{R}^{m\times n}$.

b) $\|A\|_{\mathrm{op}} := \sup_{\{x\in\mathbb{R}^n : \|x\|_n = 1\}} \|Ax\|_m = \inf\{c > 0 : \forall x \in \mathbb{R}^n\ \|Ax\|_m \le c\|x\|_n\}$.

c) For $A \neq 0$ and $x \in \mathbb{R}^n$ such that $\|x\|_n \le 1$ and $\|Ax\|_m = \|A\|_{\mathrm{op}}$, it follows that $\|x\|_n = 1$.

d) The infimum and the supremum in (2) are assumed.

Exercise 4 (The induced ℓ^p-matrix norms ; 3 points)

a) Let $A \in \mathbb{R}^{n\times n}$, $\|x\|_p$ be the ℓ^p-norm on $\mathbb{R}^n$ and $\|A\|_p$ be the corresponding induced matrix norm on $A \in \mathbb{R}^{n\times n}$. Show that

$$\|A\|_2^2 \leq \|A\|_1 \|A\|_\infty$$

holds.

b) Show that for every matrix $A \in \mathbb{R}^{n\times n}$ the estimates

$$n^{-1/2}\|A\|_2 \leq \|A\|_1 \leq n^{1/2}\|A\|_2$$
$$n^{-1}\|A\|_\infty \leq \|A\|_1 \leq n\|A\|_\infty$$

and give matrices $A \in \mathbb{R}^{n\times n}$ that show that the estimates cannot be improved (i.e., for each of these four inequalities and any $n \in \mathbb{N}$, find an A such that equality holds).

Programming exercise 5 (Stability ; 2 points)

The function ϕ is defined by

$$\phi(x) = \frac{1}{x} - \frac{1}{x+1} = \frac{1}{x(x+1)}$$

and it is easy to show that ϕ is well conditioned for large values f x. ϕ can be implemented, for example, using the methods

$$\tilde{\phi}_1(x) = \left(\frac{1}{x}\right) - \left(\frac{1}{x+1}\right), \quad \tilde{\phi}_2 = \frac{1}{(x(x+1))},$$

where the brackets determine the order in which the operations are carried out. Calculate $\tilde{\phi}_1$ and $\tilde{\phi}_2$ for different large values of x and compare the results in tabular form/graphically. How would you describe the behavior of $\tilde{\phi}_1$?

Programming exercise 6 (Triangular matrix ; 3 points)

The solutions of linear systems of equations with regular lower and upper triangular matrices are given by the following algorithms for forward and backward substitution:

$$x_j = \frac{1}{u_{jj}}\big(b_j - \sum_{k=1}^{j-1} u_{jk}x_k\big)\,, \qquad x_j = \frac{1}{l_{jj}}\big(b_j - \sum_{k=j+1}^{n} l_{jk}x_k\big)\,.$$

Implement the above algorithms and test your procedures for the systems of equations $A_l x = b_l$, $l = 1, 2, 3$, with

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, b_1 = \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}, b_2 = \begin{pmatrix} 6 \\ 9 \\ 6 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}, b_3 = \begin{pmatrix} 3 \\ 12 \\ 28 \end{pmatrix}$$

Programming exercise 7 (LU-factorization ; 3 points)

One can calculate the LU-factorization of a given LU-decomposable matrix $A = (a_{ij})$ using the following algorithm by Crout:

$$u_{ik} = a_{ik} - \sum_{j=1}^{i-1} l_{ij} u_{jk}, \quad l_{ki} = \left(a_{ki} - \sum_{j=1}^{i-1} l_{kj} u_{ji} \right) / u_{ii}.$$

Implement this algorithm to calculate the LU-decomposition of

$$A_1 = \begin{pmatrix} 5 & 3 & 1 \\ 10 & 8 & 8 \\ 15 & 11 & 10 \end{pmatrix}$$

and, for various very small values of ϵ, for

$$A_2 = \begin{pmatrix} \epsilon & 1 \\ 1 & 0 \end{pmatrix}.$$

Comment on the results.