Recent developments in dependence modeling 2024

Lecture: Jonathan Ansari
Exercise: Moritz Ritter

Exercise 3

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Submission in either german or english online via moritz.ritter@stochastik.uni-freiburg.de or mailbox 3.15 at the mathematical institute.

Exercise 1 (4 Points)**.** The Clayton Copula with parameter $\theta > 0$ is given by

$$C_\theta(u,v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}.$$

Show that the Clayton copula is an archimedian copula. Let $U_1, U_2 \sim U[0,1]$ independent random variables. Determine the function f_θ such that $(U_1, f_\theta(U_1,U_2)) \sim C_\theta$.

Exercise 2 (4 Points)**.** Show that ζ_2 in Example 2.14 is a dependence measure.

Exercise 3 (4 Points)**.** Prove Proposition 3.4: Let $X = (X_1, \dots, X_d)$ and $Y = (Y_1, \dots, Y_d)$ be d-dimensional random vectors. Then

(i) $X \leq_{st} Y \quad \Longrightarrow \quad X \leq_{icx} Y\,, \quad X \leq_{uo} Y \quad \text{and} \quad X \geq_{lo} Y\,,$

(ii) $X \leq_{ccx} Y \quad \Longrightarrow \quad X \leq_{cx} Y \quad \text{and} \quad X \leq_{dcx} Y$

(iii) $X \leq_{sm} Y \quad \Longrightarrow \quad X \leq_{c} Y \quad \Longrightarrow \quad X \leq_{lo} Y \quad \text{and} \quad X \leq_{uo} Y$

(iv) $X \leq_{sm} Y \quad \Longrightarrow \quad X \leq_{dcx} Y \quad \Longrightarrow \quad \sum_{i=1}^d X_i \leq_{cx} \sum_{i=1}^d Y_i$

Exercise 4 (4 Points)**.** Prove Corolarry 3.19: Let (X,Y) be a bivariate random vector with continuous marginal distribution functions. If (X,Y) is PLOD, then $\mathrm{Cor}(X,Y) \geq 0$, $\tau(X,Y) \geq 0$ and $\rho_S(X,Y) \geq 0$.

Exercise 5 (4 Points; Bonus)**.** Show that the Markov product $A * B$ is a bivariate copula. Moreover, show that for a bivariate copula C it holds that

$$\Pi^2 * C = C * \Pi^2 = \Pi^2\,, \qquad W^2 * C = C^{\sigma_1}\,,$$
$$M^2 * C = C * M^2 = C\,, \qquad C * W^2 = C^{\sigma_2}\,.$$

Exercise 6 (4 Points; Bonus)**.**

(i) Implement the estimator $T_n(Y|X)$ for $T(Y|X)$ following Theorem 2.21.

(ii) Implement the MFOCI Algorithm for $q = 1$, i.e., a single output variable Y.

(iii) Create some examples similar to the slides for several sample sizes.

(iv) Compare your results with the CODEC Package from R.

All your results should be presented in a PDF document.