Exercises for the lecture "Probability Theory I"

Sheet 4

Submission deadline: Wednesday, 18.06.2025, until 10:15 o'clock in the mailbox in the math institute (You may deliver the exercise solutions in pairs.)

Exercise 1

(4 points)

- (a) Let $(X_n)_{n \in \mathbb{N}}$ a martingale w.r.t. $(\mathcal{F}_n)_{n \in \mathbb{N}}$ and $(c_n)_{n \in \mathbb{N}}$ previsible with $|c_n| \leq k_n$ for all $n \in \mathbb{N}$. Prove that the martingale transform $(Y_n)_{n \in \mathbb{N}}$ is a martingale w.r.t. $(\mathcal{F}_n)_{n \in \mathbb{N}}$.
- (b) Let $(X_n)_{n \in \mathbb{N}}$ a sub-/supermartingale w.r.t. $(\mathcal{F}_n)_{n \in \mathbb{N}}$ and $(c_n)_{n \in \mathbb{N}}$ previsible with $0 \leq c_n \leq k_n$ for all $n \in \mathbb{N}$. Prove that in this case the martingale transform $(Y_n)_{n \in \mathbb{N}}$ is a sub-/supermartingale w.r.t $(\mathcal{F}_n)_{n \in \mathbb{N}}$.

Exercise 2

(4 points)

Let $(\mathcal{F}_n)_{n \in \mathbb{N}}$ and $(\mathcal{G}_n)_{n \in \mathbb{N}}$ filtrations with $\mathcal{F}_n \subseteq \mathcal{G}_n$, $n \in \mathbb{N}$ and $(X_n)_{n \in \mathbb{N}}$ a stochastic process adapted to both of them.

- (a) Let (X_n)_{n∈ℕ} be a martingale w.r.t. (G_n)_{n∈ℕ}. Show that (X_n)_{n∈ℕ} is also a martingale w.r.t. (F_n)_{n∈ℕ}.
 In particular: If (X_n)_{n∈ℕ} is a martingale w.r.t. (G_n)_{n∈ℕ}, then (X_n)_{n∈ℕ} is also a martingale w.r.t. the filtration induced by itself.
- (b) Find an example of $(\mathcal{F}_n)_{n \in \mathbb{N}}$, $(\mathcal{G}_n)_{n \in \mathbb{N}}$ and $(X_n)_{n \in \mathbb{N}}$, such that $(X_n)_{n \in \mathbb{N}}$ is a martingale w.r.t. $(\mathcal{F}_n)_{n \in \mathbb{N}}$ but not w.r.t. $(\mathcal{G}_n)_{n \in \mathbb{N}}$.
- (c) Let $(\mathcal{H}_n)_{n\in\mathbb{N}}$ be another filtration such that $\mathcal{G}_n = \sigma(\mathcal{F}_n, \mathcal{H}_n)$ and X_n is independent of \mathcal{H}_m given \mathcal{F}_m for all $n \ge m \ge 0$, i.e. $\mathbb{P}(X_n \in A | \sigma(\mathcal{F}_m, \mathcal{H}_m)) = \mathbb{P}(X_n \in A | \mathcal{F}_m)$. Prove the following: If $(X_n)_{n\in\mathbb{N}}$ is a martingale w.r.t. $(\mathcal{F}_n)_{n\in\mathbb{N}}$ then $(X_n)_{n\in\mathbb{N}}$ is a martingale w.r.t. $(\mathcal{G}_n)_{n\in\mathbb{N}}$ as well.
- (d) Construct an example of a martingale $(X_n)_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} X_n = \infty$ almost surely.

Exercise 3

(4 points)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $(\mathcal{F}_n)_{n \in \mathbb{N}}$ a filtration and τ, σ stopping times with respect to this filtration.

- (a) Prove that both $\tau \wedge \sigma$ and $\tau \vee \sigma$ are stopping times.
- (b) Let $\tau, \sigma \geq 0$. Prove that $\tau + \sigma$ is a stopping time.

(please turn over)

Define the σ -algebra der τ -past as

$$\mathcal{F}_{\tau} := \{ A \in \mathcal{A} \mid A \cap \{ \tau \le n \} \in \mathcal{F}_n \text{ for all } n \in \mathbb{N} \}.$$

- (c) Prove that τ is \mathcal{F}_{τ} -measurable.
- (c) Prove that $\mathcal{F}_{\tau} \cap \mathcal{F}_{\sigma} = \mathcal{F}_{\tau \wedge \sigma}$.

Exercise 4

Let $(X_n)_{n \in \mathbb{N}}$ identically distributed and independent random variables with

$$\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}.$$

We define $S_0 = 0$ and $S_n := \sum_{i=1}^n X_i$.

- (a) Prove that \mathbb{P} -almost surely $\limsup_{n\to\infty} S_n = \infty$ and $\liminf_{n\to\infty} S_n = -\infty$. HINT: Consider the sets $\{S_{n_k+k} - S_{n_k} = k\}$ for suitable n_k and $k \in \mathbb{N}$ and use the theorem of Borel-Cantelli.
- (b) Let $a, b \in \mathbb{N}$ and $\tau := \inf\{n \ge 0 \mid S_n \in \{-a, b\}\}$. Determine $\mathbb{P}(S_{\tau} = -a)$. HINT: First, determine $\mathbb{E}[S_{\tau}]$.

Exercises for self-monitoring

- (1) Define filtration and (sub-/super-)martingale.
- (2) Let $(Y_n)_{n \in \mathbb{N}}$ independent random variables satisfying $\mathbb{E}[Y_n] = 1$ for all $n \in \mathbb{N}$ and $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$. Show that $(X_n)_{n \in \mathbb{N}}$ with $X_n = \prod_{k=1}^n Y_k$ is a martingale w.r.t. $(\mathcal{F}_n)_{n \in \mathbb{N}}$.
- (3) Let $(X_n)_{n \in \mathbb{N}}$ be a martingale w.r.t. $(\mathcal{F}_n)_{n \in \mathbb{N}}$. Show that $\mathbb{E}[X_n] = \mathbb{E}[X_1]$ for all $n \in \mathbb{N}$.
- (4) Let $(\mathcal{F}_n)_{n \in \mathbb{N}}$ be a filtration and $m \in \mathbb{N}$. Prove that $\tau = m$ is a stopping time.
- (5) Let $(X_n)_{n \in \mathbb{N}}$ be a martingale and τ a stopping time. Prove that $(X_{\tau \wedge n})_{n \in \mathbb{N}}$ is a martingale as well.
- (6) Think heuristically about why martingales play a role in the description of fair games. What does (5) then tell you about fair games?
- (7) State the optional sampling theorem.
- (8) State Wald's equation.
- (9) Let $(X_n)_{n \in \mathbb{N}}$ and $(Y_n)_{n \in \mathbb{N}}$ be martingales and $a, b \in \mathbb{R}$. Is aX + bY then necessarily a martingale?
- (10) Let $(X_n)_{n \in \mathbb{N}}$ and $(Y_n)_{n \in \mathbb{N}}$ be supermartingales. Is $Z := X \wedge Y = (\min(X_n, Y_n))_{n \in \mathbb{N}}$ then a sub- oder supermartingale?

(4 points)