1. **Discrete Markets in One Period**
   Let $\Omega = \{\omega_1, \dots, \omega_n\}$, $\mathcal{F} = 2^\Omega$ and $\mathbb{P}$ be a strictly positive probability measure. Consider a market model (S^0, S^1) in one period. We assume $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $n \geq 2$, that the values $S^1_1(\omega_1), \dots, S^1_1(\omega_n)$ are pairwise distinct and $(S^0_0, S^0_1) = (1, 1+r)$ for some $r > -1$. Define $a := \min_i S^1_1(\omega_i)$ and $b := \max_i S^1_1(\omega_i)$.
   Show, without using the fundamental theorem of asset pricing, that this model does not admit arbitrage if and only if $a < (1+r)S^1_0 < b$.

   Points for Question 1: 4

2. **Structure of Trading Strategies**
   Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t=0,\dots,T})$ be a filtered probability space, that carries a $(d+1)$-dimensional price process. Assume $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Denote by $\Theta := \{\bar{\xi} : \bar{\xi} \text{ is a } (d+1)\text{-dimensional predictable process}\}$ the set of trading strategies.
   (a) Show that Θ is a linear space. (1)
   (b) Show that, for every $t \in \{0, \dots, T\}$, the maps (1)
   $$\Theta \ni \bar{\xi} \mapsto \mathrm{G}_t(\bar{\xi}), \quad \Theta \ni \bar{\xi} \mapsto \mathrm{V}_t(\bar{\xi})$$
   are linear.
   (c) Let $\Theta_{sf} \subseteq \Theta$ be the subset of self-financing strategies. Show that Θ_{sf} is a linear space that contains the constant (in time and $\omega \in \Omega$) processes. (1)
   (d) Suppose the market contains an arbitrage. Show that there exists an arbitrage with initial value equal to zero. (1)

   Points for Question 2: 4

3. **Embedding of L^p-Spaces**
   Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $p \in [1, \infty] \cup \{0\}$, $q \in [1, \infty]$ with $q > p$. Here $L^0(\mathbb{P})$ is endowed with the metrizable topology of convergence in probability.
   (a) Show that (3)
   $$L^q(\mathbb{P}) \ni X \mapsto X \in L^p(\mathbb{P})$$
   is well-defined, injective and continuous.
   (b) Conclude that if $A \subseteq L^p(\mathbb{P})$ is closed, then $A \cap L^q(\mathbb{P}) \subseteq L^q(\mathbb{P})$ is closed. (1)

   Points for Question 3: 4

You can achieve a total of **12** points for this sheet.