Ordering results in classes of elliptical distributions with applications to risk bounds

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Abstract

A classical result of Slepian (1962) for the normal distribution and extended by Das Gupta et al. (1972) for elliptical distributions characterizes one-sided (lower orthant) comparisons of the distributions. Müller and Scarsini (2000) established that these conditions even imply the stronger supermodular comparison in the normal case. In the present paper, we extend this result to elliptical distributions. We also derive a similar comparison result for the directionally convex ordering of elliptical distributions. As application, we obtain extensions and strengthenings of several recent results on risk bounds in partially specified risk factor models with elliptical specifications as well as in elliptical classes under restrictions on the partial correlations.

Keywords supermodular ordering, directionally convex ordering, convex ordering, elliptically contoured distributions, conditionally increasing, multivariate normal distribution, partial correlation, canonical vine

1 Introduction

In the first part of this paper, we extend and strengthen some basic stochastic ordering results for multivariate normal distributions to the frame of elliptical distributions. As consequence, we obtain in the second part of this paper extensions and strengthenings of several recent results on risk bounds in partially specified risk factor models with elliptical specifications as well as in classes of elliptical distributions under restrictions on (partial) correlations.

A classical result of Slepian (1962, Lemma 1.1) characterizes one-sided (lower orthant) comparisons of normal distributions by the increase of the off-diagonal correlations. In the paper of Block and Sampson (1988, Theorem 2.1 and Corollary 2.3) it is stated that an increase of the off-diagonal correlations even implies the supermodular comparison of this distributions. The argument in Block and Sampson (1988) was shown in Müller and Scarsini (2000, Section 4) to be incomplete. In their paper these authors gave a complete proof of the strengthened comparison result for the normal case. In the present paper, we use the ideas in these two papers to establish the supermodular comparison result for the general elliptical case. We also derive a similar characterization for the comparison w.r.t. the directionally convex order.

As application, we study in Section 3 two supermodular and two directionally convex maximization problems. In Section 3.1.1, we obtain a strengthening of the characterization of the lower orthant ordering of the conditionally comonotonic risk vectors, describing risk bounds in partially specified risk factor models, to the stronger supermodular ordering. In Section 3.1.2, we obtain optimal risk bounds for the joint portfolio in elliptical models under the restrictions on the (generalized) partial correlation coefficients that correspond to canonical vine structures. Based on a criterion for the conditional increasingness (CI) of elliptical distributions, we obtain in Section 3.2.1 a solution to the directionally convex maximization problem in partially specified factor models where we also allow the marginal distributions to come from some specifications sets. In elliptical models, we give in Section 3.2.2 a solution to the directionally convex maximization problem with upper bounded (generalized) covariance matrix.

2 Characterization of the supermodular order in classes of elliptical distributions

In this section, we characterize the supermodular ordering and the directionally convex ordering in classes of elliptical distributions.

For a function $f\colon \mathbb{R}^d \to \mathbb{R}$, let $\Delta_i^\varepsilon f(x) := f(x+\varepsilon e_i) - g(x)$ be the difference operator where $\varepsilon > 0$ and e_i denotes the unit vector w.r.t. the canonical base in $\mathbb{R}^d$. Then, f is said to be *supermodular* resp. *directionally convex* if $\Delta_i^{\varepsilon_i}\Delta_i^{\varepsilon_j} f \geq 0$ for all $1 \leq i < j \leq d$ resp. $1 \leq i \leq j \leq d$. For d-dimensional random vectors ξ, ξ', the supermodular ordering $\xi \leq_{sm} \xi'$ resp. the directionally convex ordering $\xi \leq_{dcx} \xi'$ is defined via $\mathbb{E}f(\xi) \leq \mathbb{E}f(\xi')$ for all supermodular resp. directionally convex functions f for which the expectations exist. The *lower orthant ordering* $\xi \leq_{lo} \xi'$ is defined by the pointwise comparison of the corresponding distribution functions, i.e. $F_\xi(x) \leq F_{\xi'}(x)$ for all $x \in \mathbb{R}^d$. Remember that the *convex ordering* $\zeta \leq_{cx} \zeta'$ for real valued random variables ζ, ζ' is defined via $\mathbb{E}\varphi(\zeta) \leq \mathbb{E}\varphi(\zeta')$ for all convex functions φ for which the expectation exists.

For an overview of stochastic orderings, see Müller and Stoyan (2002), Shaked and Shanthikumar (2007) and Rüschendorf (2013).

A d-dimensional random vector X has an *elliptically contoured* (or, shortly, *elliptical*) distribution with parameters μ, Σ and generator ϕ, written

$$X \sim \mathcal{EC}_d(\mu, \Sigma, \phi),$$

if $\mu \in \mathbb{R}^d$, Σ is a $d \times d$ positive semi-definite matrix, and if the characteristic function $\varphi_{X-\mu}$ of $X-\mu$ is a function of the quadratic form $t\Sigma t^T$, i.e. $\varphi_{X-\mu}(t) = \phi(t\Sigma t^T)$.

Elliptical random vectors have a characterization by a stochastic representation of the form

$$X \stackrel{\mathrm{d}}{=} \mu + RU^{(l)}A\,, \tag{1}$$

where $U^{(l)}$, $l \geq 1$, is a random vector of (appropriate) dimension l which is uniformly distributed on the unit sphere in $\mathbb{R}^l$, where R is a non-negative random variable independent of $U^{(l)}$, and where A is a deterministic $l \times d$ matrix such that $\Sigma = A^T A$, see Cambanis et al. (1981).

Necessary and sufficient conditions for such a representation are that $l \geq \mathsf{rank}(\Sigma)$ and $\phi \in \Phi_l$, where Φ_l denotes the class of functions $\psi\colon [0,\infty) \to \mathbb{R}$ with $\psi(u) = \int_{[0,\infty)} \Omega_l(r^2 u)\,\mathsf{d}F(r)$ for Ω_l being the characteristic function of $U^{(l)}$ and F the distribution function of R, see Cambanis et al. (1981, Corollary 2).

Exactly in the case that $\mathbb{E}R^2 < \infty$, or equivalently that $\phi'(0)$ is finite, the covariance matrix exists and is proportional to Σ (see Fang and Zhang (1990, p.67)). It is easy to see that the univariate marginal distribution functions of X are continuous if $\mathsf{rank}(\Sigma) \geq 2$ and R has no point mass in 0 , i.e. $F_R(0) = 0$.

A well-known property of elliptical distributions is that they are closed under marginalization and the margins inherit the elliptical generator (see e.g. Fang and Zhang (1990, Corollary 1 of Theorem 2.6.3)).

Further, elliptical distributions are closed under conditioning, see Cambanis et al. (1981, Corollary 5). In contrast to the marginalization property, the generator is not necessarily inherited.

2.1 The supermodular ordering of elliptical distributions

The characterization of the lower orthant ordering of multivariate normal distributions with fixed univariate marginal distributions goes back to Slepian (1962, Lemma 1.1) (see also Tong (1980)). An extension to elliptical distributions is established by Das Gupta et al. (1972, Theorem 5.1). In the bivariate case, this is equivalent to the supermodular ordering.

The following special comparison result is given by Block and Sampson (1988, Theorem 2.1 and Lemma 2.2), see also Müller and Scarsini (2000, Lemma 4.1). The proof is based essentially on a conditioning argument leading to a reduction to a comparison of two-dimensional elliptical distributions. We give the proof since we shall make use of some arguments of it.

Lemma 2.1 *Let $X \sim \mathcal{EC}_d(\mu,\Sigma,\phi)$ and $Y \sim \mathcal{EC}_d(\mu,\Sigma',\phi)$ with $\sigma_{ij} \leq \sigma'_{ij}$ and $\sigma_{kl} = \sigma'_{kl}$ for all $(k,l) \notin \{(i,j),(j,i)\}$ for some $i \neq j$. Then, $X \leq_{sm} Y$.*

Proof: Assume without loss of generality that $(i,j) = (1,2)$. In the first case assume that both Σ and Σ' are positive definite matrices. Write

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

where Σ_{11} is the two-dimensional covariance matrix of (X_1, X_2) and Σ_{22} denotes the $(d-2)$-dimensional covariance matrix of $(X_3, \ldots, X_d)$. Partition Σ', $\mu = (\mu_1, \mu_2)$ and $\mu' = (\mu_1', \mu_2')$ in the same way.

For $z \in \mathbb{R}^{d-3}$, let $\mu_z = \mu_1 + (z - \mu_2)\Sigma_{22}^{-1}\Sigma_{21}$ and

$$\begin{aligned}
\Sigma_{11.2} &= \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\,, \quad \text{resp.} \\
\Sigma_{11.2}' &= \Sigma_{11}' - \Sigma_{12}'\Sigma_{22}'^{-1}\Sigma_{21}'\,.
\end{aligned}$$

Then, for the conditional distributions holds

$$\begin{aligned}
(X_1, X_2)|(X_3, \ldots, X_d) = z \ &\sim \mathcal{EC}_2(\mu_z, \Sigma_{11.2}, \phi_{q(z)})\,, \quad \text{resp.} \\
(Y_1, Y_2)|(Y_3, \ldots, Y_d) = z \ &\sim \mathcal{EC}_2(\mu_z, \Sigma_{11.2}', \phi_{q(z)})\,,
\end{aligned}$$

see Cambanis et al. (1981, Corollary 5), for some generator $\phi_{q(z)}$ depending only on ϕ and $q(z) = (z - \mu_2)\Sigma_{22}^{-1}(z - \mu_2)^T$. Thus, the conditional distributions depend on Σ_{11} resp. Σ_{11}' only through $\Sigma_{11.2}$ resp. $\Sigma_{11.2}'$.

Since by assumption $\Sigma_{11} \leq \Sigma_{11}'$, $\Sigma_{12} = \Sigma_{12}'$, $\Sigma_{21} = \Sigma_{21}'$ and $\Sigma_{22} = \Sigma_{22}'$ it follows that $\Sigma_{11.2} \leq \Sigma_{11.2}'$ componentwise with equality for the diagonal elements. Hence, the characterization of the supermodular ordering in the bivariate case implies

$$(X_1, X_2)|(X_3, \ldots, X_d) = z \ \leq_{sm} \ (Y_1, Y_2)|(Y_3, \ldots, Y_d) = z$$

for almost all z. Then, the concatenation property of the supermodular ordering yields $X|(X_3, \ldots, X_d) = z \ \leq_{sm} \ Y|(Y_3, \ldots, Y_d) = z$ for almost all z. Since $(X_3, \ldots, X_d) \overset{\mathrm{d}}{=} (Y_3, \ldots, Y_d)$, the statement follows from the closure of the supermodular ordering under mixtures, see Shaked and Shanthikumar (1997, Theorem 2.4.).

In the second case assume that at least one of Σ and Σ' is positive semidefinite and not positive definite. Denote by I the identity matrix. Then, the matrices $\Sigma + \frac{1}{n}I$ and $\Sigma' + \frac{1}{n}I$ are positive definite for all $n \in \mathbb{N}$. According to the first case holds for $X_n \sim \mathcal{EC}_d(\mu, \Sigma + \frac{1}{n}I, \phi)$ and $Y_n \sim \mathcal{EC}_d(\mu, \Sigma' + \frac{1}{n}I, \phi)$ that $X_n \leq_{sm} Y_n$ for all $n \in \mathbb{N}$. Then, the statement follows from the closure of the supermodular ordering under weak convergence (see Müller and Scarsini (2000, Theorem 3.5)). ■

In the following theorem we establish that also elliptical distributions are ordered in the off-diagonal elements of the (generalized) covariance matrix w.r.t. the supermodular ordering. This result is the positive answer to the question formulated in Landsman and Tsanakas (2006, Remark 2) whether the supermodular ordering results for multivariate normal distributions (see Müller (2001, Theorem 11) and for Kotz-type distributions (see Ding and Zhang (2004, Theorem 3.11)) can be extended to elliptical distributions of arbitrary dimension. We correct the proof of Block and Sampson (1988, Corollary 2.3) which was shown by Müller and Scarsini (2000, Section 4) to be incomplete.

Theorem 2.2 (**$\leq_{sm}$-ordering elliptical distributions**)
Let $X \sim \mathcal{EC}_d(\mu, \Sigma, \phi)$ and $Y \sim \mathcal{EC}_d(\mu', \Sigma', \phi)$ with $\Sigma = (\sigma_{ij})_{1 \leq i,j \leq d}$, $\Sigma' = (\sigma'_{ij})_{1 \leq i,j \leq d}$. Then, the following statements are equivalent:

(i) $X \leq_{sm} Y$,

(ii) $\mu = \mu'$, $\sigma_{ii} = \sigma'_{ii}$ *f.a.* $1 \leq i \leq d$ *and* $\sigma_{ij} \leq \sigma'_{ij}$ *for all* $i \neq j$.

(iii) X *and* Y *have the same univariate marginals and* $\sigma_{ij} \leq \sigma'_{ij}$ *f.a.* i, j.

Proof: '(i) $\Longrightarrow$ (iii)': Since the supermodular ordering is a pure dependence ordering, the univariate marginal distributions must be equal. This implies $\sigma_{ii} = \sigma'_{ii}$ for all i. If the covariance matrix exists, then for $i \neq j$, it follows that $\sigma_{ij} \leq \sigma'_{ij}$ because $f(x) = x_i x_j$ is supermodular. Otherwise, the statement follows by an approximation (w.r.t. weak convergence) of the radial part R by a square-integrable sequence $(R_n)_n$ using the closure of the supermodular ordering under weak convergence (see Müller and Scarsini (2000, Theorem 3.5)).
The implication '(iii) $\Longrightarrow$ (ii)' is immediate.

Assume (ii). Consider two cases. In the first case let us assume that both matrices Σ and Σ' are positive definite. In the same way as in the proof of Das Gupta et al. (1972, Theorem 5.1) resp. Müller and Scarsini (2000, Theorem 4.2.) there exists a finite sequence $\Sigma = \Sigma_1 \leq \ldots \leq \Sigma_k = \Sigma'$ (componentwise) of positive semi-definite matrices such that Σ_{l+1} is obtained from Σ_l by increasing exactly one off-diagonal entry. Hence, statement (i) follows from Lemma 2.1 and from the transitivity of the supermodular ordering.

In the second case assume that at least one of Σ and Σ' is positive semi-definite and not positive definite. Then, the statement follows from the first part and a similar approximation argument as in the second part of the proof of Lemma 2.1. ∎

Remark 2.3 *The supermodular ordering result in Theorem 2.2 is established independently in a recent paper by Yin (2019, Theorem 3.4) submitted to arXiv on Oct 16, 2019. For the proof, this author extends the integral representation argument in Müller (2001, Theorem 11) in the normal case. We remark that our paper and, in particular, Theorem 2.2 is based on the dissertation of the first author published on Apr 09, 2019, see Ansari (2019, Theorem 5.2).*

2.2 The directionally convex ordering of elliptical distributions

In this section, we show a similar characterization for the directionally convex ordering.

Lemma 2.4 *Let $X \sim \mathcal{EC}_d(\mu, \Sigma, \phi)$ and $Y \sim \mathcal{EC}_d(\mu', \Sigma', \phi)$ be integrable with $\sigma_{ij} = \sigma'_{ij}$ for all $(i,j) \neq (1,1)$ and $\sigma_{11} \leq \sigma'_{11}$. Then, $X \leq_{dcx} Y$.*

Proof: As a consequence of Shaked and Shanthikumar (2007, Theorem 3.A.1) and Cambanis et al. (1981, Corollary 5), it holds for the conditional distributions that

$$X_1 \,|\, (X_j = x_j\,, j = 2, \ldots, d) \;\; \leq_{cx} \;\; Y_1 \,|\, (Y_j = x_j\,, j = 2, \ldots, d)$$

for all x_j , $j = 2, \ldots, d$. Since the convex ordering and the directionally convex ordering coincide in the one-dimensional case, we obtain with the concatenation property of the directionally convex ordering that

$$X \,|\, (X_j = x_j\,, j = 2, \ldots, d) \;\; \leq_{dcx} \;\; Y \,|\, (Y_j = x_j\,, j = 2, \ldots, d)$$

for all x_j , $j = 2, \ldots, d$. Then, the closure of the directionally convex ordering under mixtures implies $X \leq_{dcx} Y$, see Müller and Stoyan (2002, Theorem 3.12.6). ■

As a consequence, Lemma 2.4 implies the following result which is essentially based on a conditioning argument and on the characterization of the supermodular ordering in Theorem 2.2, see also Yin (2019, Theorem 3.6).

Theorem 2.5 ($\leq_{dcx}$-ordering of elliptical distributions)
Let $X \sim \mathcal{EC}_d(\mu, \Sigma, \phi)$ and $Y \sim \mathcal{EC}_d(\mu', \Sigma', \phi)$ with $\Sigma = (\sigma_{ij})_{1\leq i,j\leq d}$ and $\Sigma' = (\sigma'_{ij})_{1\leq i,j\leq d}$ be integrable. Then, the following statements are equivalent:

(i) $X \leq_{dcx} Y$,

(ii) $\mu = \mu'$ *and* $\sigma_{ij} \leq \sigma'_{ij}$ *for all* i, j .

Proof: Assume (i). In the first case, let the radial random variable R corresponding to ϕ be square-integrable. Then, the statement follows from the fact that the functions $f(x) = \pm x_i$ and $g(x) = x_i x_i$ for $1 \leq i \leq j \leq d$ are directionally convex. In the second case, approximate the distribution of R weakly by a sequence of square-integrable radial variables. Then, the statement follows from the first case and the approximation argument in Müller and Stoyan (2002, Theorem 3.12.8).
Assume (ii). Let $\xi \sim \mathcal{EC}_d(\mu, \Sigma'', \phi)$ where $\Sigma'' = (\sigma_{ij})_{1\leq i,j\leq d}$ is given by $\sigma''_{ij} = \sigma'_{ij}$ for all $i = j$ and $\sigma''_{ij} = \sigma_{ij}$ for all $i \neq j$. Since componentwise increasing of the diagonal elements does not affect the positive semi-definiteness, Σ'' is positive semi-definite. Thus, Lemma 2.4 implies $X \leq_{dcx} \xi$. Due to Theorem 2.2, it holds that $\xi \leq_{sm} Y$ and thus $X \leq_{dcx} Y$. ■

3 Applications to risk bounds unter elliptical constraints

In this section, we determine and analyse solutions to two supermodular and two directionally convex maximization problems in the class of elliptical distributions resp. under elliptical constraints. In general, solutions to supermodular resp. directionally convex maximization problems do not exist because these orderings are partial orders on the underlying class of distributions that do not form a lattice, see Müller and Scarsini (2006).

Both the supermodular and directionally convex ordering have the useful property that they imply the convex ordering of the sum of the components, i.e.

$$(X_1,\dots,X_d) \leq_{sm} (Y_1,\dots,Y_d) \quad \Longrightarrow \quad \sum_{i=1}^d X_i \leq_{cx} \sum_{i=1}^d Y_i\,, \quad \text{resp.}$$
$$(X_1,\dots,X_d) \leq_{dcx} (Y_1,\dots,Y_d) \quad \Longrightarrow \quad \sum_{i=1}^d X_i \leq_{cx} \sum_{i=1}^d Y_i\,.$$

Thus, a solution to a supermodular resp. directionally convex maximization problem also yields a solution to the maximization problem for the sum in convex ordering under the corresponding constraints. Note that the supermodular ordering is a pure dependence ordering and has the useful property that it is invariant under increasing transformations. In contrast, the directionally convex ordering allows a comparison for univariate marginal distributions with $X_i \leq_{cx} Y_i$.

The improvement of bounds for the sum of random variables in convex ordering (where the improvement is w.r.t. the comonotonic sum) has an important practical relevance because it implies improved risk bounds for portfolios in finance or for the aggregate insurance risk whenever the underlying risk measure on $L^1(\Omega,\mathcal{A},P)$ is convex, law-invariant and has the Fatou-property, assuming that $(\Omega,\mathcal{A},P)$ is atomless, see Bäuerle and Müller (2006, Theorem 4.3).

3.1 Supermodular maximization problem

In this section, we study two supermodular maximization problems. For the first one, we assume a partially specified factor model where the bivariate specifications are from classes of elliptical distributions, see Bernard et al. (2017). For the second one, we assume an elliptical model where the (generalized) partial correlations corresponding to a canonical vine are bounded.

3.1.1 $\leq_{sm}$-maximization problem with partial elliptical specifications

For $\mu = 0 \in \mathbb{R}^2$ and $\Sigma = \left(\begin{smallmatrix} 1 & \rho \\ \rho & 1 \end{smallmatrix}\right)$, $\rho \in [-1,1]$, we abbreviate the bivariate distribution $\mathcal{EC}_2(\mu,\Sigma,\phi)$ by $\mathcal{EC}_2(0,\rho,\phi)$. Since the supermodular ordering is invariant under increasing transformations, supermodular ordering results w.r.t. off-diagonal entries of the (generalized) covariance matrix can be formulated w.l.o.g. in the standardized case.

For $\phi \in \Phi_2$ and $\rho_i \in [-1,1]$, $1 \leq i \leq d$, consider the supermodular maximization problem

$$\max\left\{(X_1,\dots,X_d)\,|\,\exists Z\colon (X_i,Z) \sim \mathcal{EC}_2(0,\rho_i,\phi)\ \forall i\right\} \quad \text{w.r.t. } \leq_{sm} \tag{2}$$

with partial elliptical specifications of the components X_i and the common risk factor Z.

Define the function $M\colon [-1,1]^2 \to [-1,1]$ by

$$M(a,b) := ab + \sqrt{1-a^2}\sqrt{1-b^2}\,.$$

Let $X^c_{i,z} = F^{-1}_{i,z}(U)$ with $U \sim U(0,1)$ independent of the common risk factor Z. Then, the conditionally comonotonic random vector $X^c_Z = (X^c_{1,Z}, \dots, X^c_{d,Z})$ solves maximization problem (2) as follows, see Ansari and Rüschendorf (2016, Theorem 2).

Proposition 3.1 *A solution to the supermodular maximization problem* (2) *is given by a vector* $(X^c_Z) \sim \mathcal{EC}_d(0,\Sigma,\phi)$ *where* $\Sigma = (\sigma_{ij})$ *is given by*

$$\sigma_{ij} = \begin{cases} 1 & \text{for } i = j\,, \\ M(\rho_i, \rho_j) & \text{for } i \neq j\,. \end{cases}$$

As a consequence of Theorem 2.2, the following result compares the worst case scenarios in maximization problem (2) w.r.t. the elliptical specifications. It extends the lower orthant ordering result in classes of elliptical distributions in Ansari and Rüschendorf (2016, Proposition 4) to the supermodular ordering.

Theorem 3.2 *Let* $(X_i, Z) \sim \mathcal{EC}_2(0,\rho_i,\phi)$, $(Y_i, Z) \sim \mathcal{EC}_2(0,\rho_i',\phi)$, $1 \leq i \leq d$. *Then, for conditionally comonotonic random vectors* X^c_Z *resp.* Y^c_Z *with these specifications holds*

$$X^c_Z \leq_{sm} Y^c_Z \quad \Longleftrightarrow \quad M(\rho_i,\rho_j) \leq M(\rho_i',\rho_j')\,,\ \forall i \neq j\,. \tag{3}$$

Remark 3.3 *(a) It can easily be verified that* $M(a,b) = 1$ *if and only if* $a = b$. *Thus,* $\rho_i = \rho_j$ *for all* $i \neq j$ *yields* $X^c_Z \stackrel{\mathrm{d}}{=} X^c$, *where* $X^c = (F^{-1}_{X_i}(U))_{1 \leq i \leq d}$ *is comonotonic.*

(b) A sufficient condition for the right hand side in (3) *is*

$$\rho_1 \gtrless \rho_1' \gtrless \rho_2 \gtrless \cdots \gtrless \rho_d \quad \text{and} \quad \rho_i' = \rho_i \text{ for all } 2 \leq i \leq d\,.$$

This is a special case of Ansari and Rüschendorf (2018, Corollary 3.11) in the elliptical setting with $F_R(0) = 0$. *In particular, also*

$$\begin{aligned} &\rho_1 \gtrless \cdots \gtrless \rho_k \gtrless \rho_k' \gtrless \rho_{k+1}' \gtrless \rho_{k+1} \gtrless \cdots \gtrless \rho_d \\ &\text{and} \quad \rho_1' = \cdots = \rho_k' \gtrless \rho_{k+1}' = \cdots = \rho_d \end{aligned} \tag{4}$$

for some $k \in \{1, \dots, d-1\}$ *yields the right hand side in* (3).

3.1.2 $\leq_{sm}$-maximization problem under partial correlation bounds

In this section, we consider a supermodular maximization problem for elliptical distributions under a boundedness assumption on the (generalized) partial correlations corresponding to a canonical vine structure.

For $k \in \mathbb{N}_0$, denote by 1:k the indices $(1,\ldots,k)$. If $k = 0$, then $1{:}\mathrm{k} = \emptyset$. Let $(\sigma_{ij,1:(i-1)})_{1\le i<j\le d} \in [-1,1]^{\frac{d(d-1)}{2}}$. Define the matrix $\Sigma = (\sigma_{ij})_{1\le i,j\le d}$ by $\sigma_{ii} = 1$ for all i and by $\sigma_{ij} = \sigma_{ji} = \sigma_{ij,1:0}$ for $i < j$, where $\sigma_{ij,1:0}$ is iteratively defined through

$$\sigma_{ij,1:(k-1)} = \sigma_{ki,1:(k-1)}\sigma_{kj,1:(k-1)} + \sigma_{ij,1:k}\sqrt{1-\sigma^2_{ki,1:(k-1)}}\sqrt{1-\sigma^2_{kj,1:(k-1)}} \quad (5)$$

for $j = 3,\ldots,d$, $i = 2,\ldots,j-1$ and $k = i-1,\ldots,1$.

Let $\mathcal{M}^{d\times d}_{\mathrm{cor}}$ be the set of correlation matrices, i.e. the set of positive semi-definite symmetric $d\times d$ matrices with all diagonal elements equal to 1. Then, Σ is a correlation matrix, and, further, every element $\Sigma' \in \mathcal{M}^{d\times d}_{\mathrm{cor}}$ can be decomposed into *(generalized) partial correlations* $(\sigma'_{ij,1:(i-1)})_{1\le i<j\le d} \in [-1,1]^{\frac{d(d-1)}{2}}$ via formula (5), see Proposition 3.4. In the case that Σ is the correlation matrix of a square-integrable random vector $(Y_1,\ldots,Y_d)$, then $\sigma_{ij,1:(i-1)}$ is the partial correlation of Y_i and Y_j given $Y_1,\ldots,Y_{i-1}$.

For $\phi \in \Phi_d$ and $b_i \in [0,1]$, $1 \le i \le d-1$, consider the supermodular maximization problem

$$\max\{X \in \mathcal{EC}_d(0,\Sigma,\phi) \mid \Sigma \in \mathcal{M}^{d\times d}_{\mathrm{cor}} : |\sigma_{ij,1:(i-1)}| \le b_i \text{ f.a. } i<j\} \text{ w.r.t. } \le_{sm} \quad (6)$$

in the class of elliptical distributions with bounded (generalized) partial correlations corresponding to a canonical vine structure. Note that we do not assume square-integrability.

The following result shows that the (generalized) partial correlations $(\sigma_{ij,1:(i-1)})_{1\le i<j\le d}$ are algebraically independent and determine a unique correlation matrix. More precisely, the set of positive definite correlation matrices can be characterized in terms of (generalized) partial correlations that correspond to a *canonical vine* (or *C-vine*) which is a star-shaped *regular vine*. This is a graphical tool to model dependencies, see e.g. Kurowicka and Cooke (2005) and Aas et al. (2009) for definitions.

Proposition 3.4 *(i) There is a one-to-one correspondence between the set of $d\times d$ positive definite correlation matrices and the set of (generalized) partial correlations* $(\sigma_{ij,1:(i-1)})_{1\le i<j\le d} \in (-1,1)^{\frac{d(d-1)}{2}}$.

(ii) The (generalized) partial correlations $(\sigma_{ij,1:(i-1)})_{1\le i<j\le d} \in [-1,1]^{\frac{d(d-1)}{2}}$ *determine a correlation matrix uniquely.*

(iii) If $\Sigma \in \mathcal{M}^{d\times d}_{cor}$ is not of full rank, the corresponding (generalized) partial correlations $(\sigma_{ij,1:(i-1)})_{1\le i<j\le d}$ are not necessarily uniquely determined.

Proof: (i): The (generalized) partial correlations correspond to the structure of a canonical vine. Thus, the statement follows from Bedford and Cooke (2002, Corollary 7.5).

Statement (ii) is a consequence of (i) and (5).
(iii): The determinant of Σ is given by

$$\det(\Sigma) = \prod_{i=1}^{d-1} \prod_{j=i+1}^{d} \left(1 - \sigma_{ij,1:(i-1)}^2\right),$$

see Kurowicka and Cooke (2006, Theorem 4.5). Thus, the determinant vanishes if and only if there exist $1 \leq i < j \leq d$ such that $\sigma_{ij,1:(i-1)} \in \{-1, 1\}$. In this case, (5) implies that the (generalized) partial correlations $\sigma_{lj,1:(l-1)}$, $i < l < j$, are not uniquely determined. ■

The following proposition gives some elementary ordering results for (generalized) partial correlations based on formula (5). To keep the notation simple, we formulate it in the case that $k = 1$, $i = 2$ and $j = 3$.

Proposition 3.5 (Ordering partial correlations)
For the (generalized) partial correlations the following ordering properties hold true:

(i) If $\sigma_{1l}^1 = \sigma_{1l}^2$ for $l = 2, 3$, then

$$\sigma_{23,1}^1 \leq \sigma_{23,1}^2 \quad \Longrightarrow \quad \sigma_{23}^1 \leq \sigma_{23}^2 .$$

(ii) If $\sigma_{23,1}^1 = \sigma_{23,1}^2$, then

$$0 \leq |\sigma_{1l}^1| \leq \sigma_{12}^2 = \sigma_{13}^2 \text{ for } l = 2, 3 \quad \Longrightarrow \quad \sigma_{23}^1 \leq \sigma_{23}^2 . \tag{7}$$

(iii) If $\sigma_{23,1}^1 = \sigma_{23,1}^2 \leq 0$, then

$$0 \leq \sigma_{1l}^1 \leq \sigma_{1l}^2 \text{ for } l = 2, 3 \quad \Longrightarrow \quad \sigma_{23}^1 \leq \sigma_{23}^2 .$$

Proof: Statement (i) follows from the partial correlation formula

$$\sigma_{23} = \sigma_{12}\sigma_{13} + \sigma_{23,1}\sqrt{1 - \sigma_{12}^2}\sqrt{1 - \sigma_{13}^2} =: f(\sigma_{23,1}, \sigma_{12}, \sigma_{13}), \tag{8}$$

see (5). The partial derivative $\partial_2 f = \frac{\partial f}{\partial \sigma_{12}}$ of f w.r.t. the second variable is given by

$$\partial_2 f(\sigma_{23,1}, \sigma_{12}, \sigma_{13}) = \sigma_{13} - \frac{\sigma_{23,1}\sqrt{1 - \sigma_{13}^2}\ \sigma_{12}}{\sqrt{1 - \sigma_{12}^2}} . \tag{9}$$

Then, statement (ii) follows from

$$f(a, c, d) \leq f(a, d, d) \leq f(a, e, e)$$

f.a. $a \in [-1, 1]$, and $0 \leq |c| \leq d \leq e \leq 1$ where the first inequality holds true because

$$|ac| \le d \quad \Longrightarrow \quad ac\sqrt{1-d^2} \le d\sqrt{1-c^2} \quad \Longrightarrow \quad \partial_2 f(a,c,d) \ge 0\,.$$

The second inequality is fulfilled since for $g_a(s) := f(a,s,s)$ holds $g_a'(s) = 2(1-a)s \ge 0$ f.a. $a \le 1$ and $s \ge 0$.
Statement (iii) is a consequence of (9). ■

For the bounds $(b_i)_i$ on the (generalized) partial correlations in maximization problem (6), define the numbers $a_1, \dots, a_{d-1} \in [0,1]$ iteratively by

$$\begin{aligned} a_{i,i-1} &:= b_i && \text{for } i = 1,\dots,d-1\,, \\ a_{i,i-k} &:= a_{i-k+1,i-k}^2 + a_{i,i-k+1}(1-a_{i-k+1,i-k}^2) && \text{for } k = 2,\dots,i\,, \\ a_i &:= a_{i,0} && \text{for } i = 1,\dots,d-1\,. \end{aligned} \tag{10}$$

Denote by δ_{ij} the Kronecker delta and by $i \wedge j$ the minimum of i and j. Then, the supermodular maximization problem (6) has a solution which is given as follows.

Theorem 3.6 (Bounded partial correlations)
Let $Y \sim \mathcal{EC}_d(0, \Sigma', \phi)$ where $\Sigma' = (\sigma'_{ij})_{1\le i,j\le d}$ with $\sigma'_{ij} = \delta_{ij} + a_{i\wedge j}(1-\delta_{ij})$. Then, Y is a solution to maximization problem (6).

Proof: Applying the partial correlation formula (5) for $2 \le i < j \le d$ inductively over $k = 2,\dots,i$ yields

$$\begin{aligned} \sigma_{ij,1:(i-k)} &= \sigma_{(i-k+1)i,1:(i-k)}\,\sigma_{(i-k+1)j,1:(i-k)} \\ &\quad + \sigma_{ij,1:(i-k+1)}\sqrt{1-\sigma^2_{(i-k+1)i,1:(i-k)}}\sqrt{1-\sigma^2_{(i-k+11)j,1:(i-k)}} \\ &\le \sigma_{(i-k+1)i,1:(i-k)}\,\sigma_{(i-k+1)j,1:(i-k)} \\ &\quad + a_{i,i-k+1}\sqrt{1-\sigma^2_{(i-k+1)i,1:(i-k)}}\sqrt{1-\sigma^2_{(i-k+1)j,1:(i-k)}} \\ &\le a^2_{i-k+1,i-k} + a_{i,i-k+1}\cdot(1-a^2_{i-k+1,i-k}) \\ &= a_{i,i-k} \end{aligned} \tag{11}$$

using Proposition 3.5 (i) and (ii). This implies with $\sigma_{1,j} \le a_{1,j}$ for $j = 2,\dots,d$ that $\sigma_{ij} \le a_i$ for all $1 \le i < j \le d$. Since $\sigma_{ij} = \sigma_{ji}$ for all $i \ne j$, it follows that $\sigma_{ij} \le a_{i\wedge j}$ for all $i \ne j$. Choosing $(\sigma_{i,j|1:(i-1)})_{1\le i<j\le d} = b_i$ leads to $\sigma_{ij} = a_i$ for all $1 \le i < j \le d$. This defines a correlation matrix (see Proposition 3.4(ii)) which coincides with Σ'. ■

Remark 3.7 *(a) If $b_1 = 1$ in* (6)*, then by construction $a_i = 1$ for all $1 \le i < d$. This yields $\sigma'_{ij} = 1$ f.a. $1 \le i,j \le d$, and, hence, $Y \overset{\mathrm{d}}{=} (X_1^c,\dots,X_d^c)$ is the standard comonotonic upper bound for $X = (X_1,\dots,X_d)$ w.r.t. the supermodular ordering, i.e. there is no improvement of the bounds. This coincides with the fact that $a_1 = 1$ yields $\sigma'_{1i} = 1$ (which means* $\operatorname{Cor}(Y_1,Y_i) = 1$ *in the square-integrable case) f.a. i, and thus $Y = (Y_1,\dots,Y_d)$ is comonotonic. In* *this case, the (generalized) correlations* $(\sigma_{1j})_{1\le j\le d}$ *determine the correlation matrix uniquely and the (generalized) partial correlations* $(\sigma_{ij,1:i})_{2\le i<j\le d}$ *are not uniquely determined, see Proposition 3.4*(iii).

More generally, if $b_i = 1$ *for some* $i \in \{1,\dots,d-1\}$ *in* (6) *then* $a_j = 1$ *for all* $i \le j < d$. *This implies that* $(Y_i,\dots,Y_d)$ *is comonotonic and* $Y|Y_{1:(i-1)} = y$ *is comonotonic conditionally on* $y \in \mathbb{R}^{i-1}$. *The case* $b_2 = 1$ *is a special case of (Ansari and Rüschendorf, Theorem 2.3) in the context of partially specified internal risk factor models.*

(b) If $b_i = 0$ *for all* $1 \le i < d$, *then* $a_i = 0$ *for all* i *and thus* $\Sigma = \Sigma' = (\delta_{ij})_{1\le i,j\le d}$, *i.e.* $X \overset{\mathrm{d}}{=} Y$ *has uncorrelated components. Note that the components are only independent in the case of a multivariate normal distribution, see Cambanis et al. (1981, Section 5(d)).*

(c) The bounds $(b_i)_i$ *in* (6) *on the (generalized) (partial) correlations* $\sigma_{ij|1:(i-1)}$ *yield a positive semi-definite matrix* Σ' *while, in general, upper bounds on the unconditional (generalized) correlations* $(\sigma_{ij})_{ij}$ *do not yield a positive semi-definite matrix.*

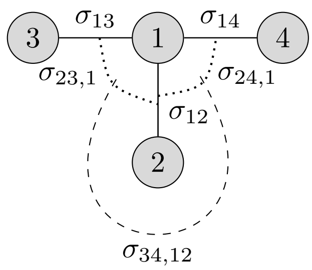

Figure 1: A C-vine on 4 variables that specifies a correlation matrix

The following example illustrates Corollary 3.6.

Example 3.8 *Assume that* $X \sim \mathcal{EC}_4(0,\Sigma,\phi)$, $\Sigma = (\sigma_{ij})_{1\le i,j\le 4}$ *with* $\sigma_{ii} = 1$ *for all* i *and with (generalized) partial correlations corresponding to the C-vine in Figure 1. Assume that*

$$
\begin{aligned}
|\sigma_{12}|, |\sigma_{13}|, |\sigma_{14}| &\le 0.5 = b_1 = a_{1,0}\,,\\
|\sigma_{23,1}|, |\sigma_{24,1}| &\le 0.6 = b_2 = a_{2,1}\,,\\
|\sigma_{34,12}| &\le 0.4 = b_3 = a_{3,2}\,.
\end{aligned}
$$

Then, Corollary 3.6 yields

$$\begin{aligned}
a_1 &= a_{1,0} = 0.5\,, \\
a_{2,0} &= a_{1,0}^2 + a_{2,1}(1 - a_{1,0}^2) = 0.7\,, \\
a_2 &= a_{2,0} = 0.7\,, \\
a_{3,1} &= a_{2,1}^2 + a_{3,2}(1 - a_{2,1}^2) = 0.616\,, \\
a_{3,0} &= a_{1,0}^2 + a_{3,1}(1 - a_{1,0}^2) = 0.712\,, \\
a_3 &= a_{3,0} = 0.712\,.
\end{aligned}$$

Hence, for $Y \sim \mathcal{EC}_4(0, \Sigma', \phi)$ *with*

$$\Sigma' = \begin{pmatrix} 1 & 0.5 & 0.5 & 0.5 \\ 0.5 & 1 & 0.7 & 0.7 \\ 0.5 & 0.7 & 1 & 0.712 \\ 0.5 & 0.7 & 0.712 & 1 \end{pmatrix},$$

holds $X \leq_{sm} Y$ *.*

3.2 Directionally convex maximization problem

In this section, we determine solutions to two directionally convex maximization problems. For the first one, we assume a factor model with partial specification sets where the bivariate dependence specification sets of each risk component w.r.t. to a common risk factor are from classes of elliptical distributions and the marginal constraints for each risk factor are from classes of univariate distribution functions with upper bounds in convex order. For the second one, we assume an elliptical model where the (generalized) covariance matrix is componentwise bounded from above by a positive semi-definite symmetric matrix.

3.2.1 $\leq_{dcx}$-maximization problem with marginal constraints in convex order

We consider a partially specified factor model where the dependence specification sets are bivariate elliptical copulas. Further, the univariate margins are not uniquely determined but from some classes of distributions with upper bounds in convex order.

For $F_i \in \mathcal{F}^1$, let $\mathcal{F}_i := \{F \,|\, F \leq_{cx} F_i\}$. Let $\phi \in \Phi_2$ such that $F_R(0) = 0$ and let $-1 \leq \rho_1 < \rho_2 \leq 1$ such that $M(\rho_1, \rho_2) \geq 0$. Denote by $C^{\rho,\phi}$ the (not necessarily uniquely determined) bivariate copula associated with $\mathcal{EC}_2(0, \rho, \phi)$. Then, consider for $k \in \{1, \ldots, d-1\}$ the directionally convex maximization problem

$$\begin{aligned}
\max\{(X_1, \ldots, X_d) \,|\, F_{X_i} \in \mathcal{F}_i\,, \exists Z \colon C_{X_i,Z} = C^{\eta_i,\phi}\,,& \\
\eta_i \leq \rho_1 \text{ for } 1 \leq i \leq k\,,& \\
\eta_i \geq \rho_2 \text{ for } k < i \leq d \;\}& \quad \text{w.r.t. } \leq_{dcx}
\end{aligned} \tag{12}$$

with elliptical dependence constraints and with marginal specifications sets $\mathcal{F}_i$.

To determine a solution, we need the following positive dependence notion. Let $Y = (Y_1, \ldots, Y_d)$ be a d-dimensional random vector. Then Y is said to be *conditionally increasing (CI)* if for all $i \in \{1, \ldots, d\}$, $Y_i \uparrow_{st} Y_J$ for all $J \subset \{1, \ldots, d\} \setminus \{i\}$, i.e. $\mathbb{E}[f(Y_i) \,|\, Y_j = y_j \,, j \in J]$ is increasing in x_j for all $j \in J$, $J \subset \{1, \ldots, d\} \setminus \{i\}$ and for all non-decreasing functions f for which the expectation exists.

Further, a matrix $A \in \mathbb{R}^{d\times d}$ is called an *M-matrix* if all off-diagonal elements are non-positive.

The proof of the CI-criterion for normal distributions in Rüschendorf (1981, Theorem 2) also applies in the case of elliptical distributions which leads to the following result (see Witting (2017, Theorem 1.4.9)).

Proposition 3.9 (CI-criterion for elliptical distributions)
Let $X \sim \mathcal{EC}_d(0, \Sigma, \phi)$ with positive definite matrix Σ . If Σ^{-1} is an M-matrix, then X is conditionally increasing.

Define the matrix $\Sigma = (\sigma_{ij})_{1\leq i,j\leq d}$ by

$$\sigma_{ij} = \begin{cases} 1 & \text{if } 1 \leq i, j \leq k \text{ or } k < i, j \leq d\,, \\ M(\rho_1, \rho_2) & \text{if } 1 \leq i \leq k < j \leq d \text{ or } 1 \leq j \leq k < i \leq d\,. \end{cases}$$

Together with the above CI-criterion, we obtain a solution to maximization problem (12) as follows.

Theorem 3.10 (Directionally convex maximization problem)
Let $Y = (Y_1, \ldots, Y_d) \sim \mathcal{EC}_d(0, \Sigma, \phi)$ an elliptically distributed random vector. Then, the directionally convex maximization problem (12) *has a solution which is given by the vector $Y' = (F_i^{-1}(F_{Y_i}(Y_i)))_{1\leq i\leq d}$.*

Proof: Let $X = (X_1, \ldots, X_d)$ be an admissible vector of the set in (12). Then, Proposition 3.1 implies $X \leq_{sm} X_Z^c$. From (4) we obtain $M(\eta_i, \eta_j) \leq \sigma_{ij}$ for all $i \neq j$. Hence, Theorem 3.2 yields $X_Z^c \leq_{sm} Y$.

Since by assumption $0 \leq M(\rho_1, \rho_2) < 1$, the inverse of $\begin{pmatrix} 1 & M(\rho_i,\rho_j) \\ M(\rho_i,\rho_j) & 1 \end{pmatrix}$ exists and is an M-matrix. Thus, as a consequence of Proposition 3.9 and using that Y is conditionally comonotonic, the copula $C_{Y'} = C_Y$ is conditionally increasing. Thus, it follows from Müller and Scarsini (2001, Theorem 4.5) that $Y \leq_{dcx} Y'$ using $F_{Y_i} \leq_{cx} F_i$. Altogether, this yields $X \leq_{dcx} Y'$. ∎

3.2.2 $\leq_{dcx}$-maximization problem in classes of elliptical distributions with upper bounded covariance matrix

Let $\phi \in \Phi_d$ such that the corresponding radial variable R is integrable. For some positive semi-definite symmetric matrix $\Sigma \in \mathbb{R}^{d\times d}$, consider the directionally convex maximization problem

$$\max\{X \sim \mathcal{EC}_d(\mu, \Sigma', \phi) \,|\, \Sigma' \leq \Sigma \text{ componentwise}\} \quad \text{w.r.t. } \leq_{dcx} \tag{13}$$

in elliptical classes with upper bounded (generalized) covariance matrix.

As a consequence of Theorem 2.5, we obtain a solution to the above maximization problem as follows.

Corollary 3.11 *Let* $Y \sim \mathcal{EC}_d(\mu, \Sigma, \phi)$ *. Then,* Y *solves the directionally convex maximization problem* (13)*.*

Note that for the comparison of marginal distributions in the elliptical model, we do not need a CI assumption on the solution Y to maximization problem (13). In contrast, in the partially specified factor model in maximization problem (12), the solution needs to be conditionally increasing to allow a more general comparison of the marginal distributions in convex order.

Conclusion

We have extended the characterization of the supermodular resp. directionally convex ordering for multivariate normal distributions to the class of elliptical distributions. The proofs are based essentially on a conditioning argument. As shown in the last section, the results allow applications to the improvement of risk bounds both in elliptical models with knowledge of partial correlation bounds and in partially specified factor models with elliptical dependence constraints.

References

Kjersti Aas, Claudia Czado, Arnoldo Frigessi, and Henrik Bakken. Pair-copula constructions of multiple dependence. *Insur. Math. Econ.*, 44(2):182–198, 2009.

Jonathan Ansari. *Ordering risk bounds in partially specified factor models.* Dissertation, University of Freiburg, 2019.

Jonathan Ansari and Ludger Rüschendorf. Upper risk bounds in internal factor models with constrained specification sets.

Jonathan Ansari and Ludger Rüschendorf. Ordering results for risk bounds and cost-efficient payoffs in partially specified risk factor models. *Methodology and Computing in Applied Probability*, pages 1–22, 2016.

Jonathan Ansari and Ludger Rüschendorf. Ordering risk bounds in factor models. *Dependence Modeling*, 6(1):259–287, 2018.

Richard E. Barlow and Frank Proschan. Statistical theory of reliability and life testing. International Series in Decision Processes. New York etc.: Holt, Rinehart and Winston, Inc. XIII, 290 p. £15.50 (1975)., 1975.

Nicole Bäuerle and Alfred Müller. Stochastic orders and risk measures: consistency and bounds. *Insur. Math. Econ.*, 38(1):132–148, 2006.

Tim Bedford and Roger M. Cooke. Vines – a new graphical model for dependent random variables. *Ann. Stat.*, 30(4):1031–1068, 2002.

Carole Bernard, Ludger Rüschendorf, Steven Vanduffel, and Ruodu Wang. Risk bounds for factor models. *Finance Stoch.*, 21(3):631–659, 2017.

Henry W. Block and Allan R. Sampson. Conditionally ordered distributions. *J. Multivariate Anal.*, 27(1):91–104, 1988.

Stamatis Cambanis, Steel Huang, and Gordon Simons. On the theory of elliptically contoured distributions. *J. Multivariate Anal.*, 11:368–385, 1981.

S. Das Gupta, I. Olkin, L. J. Savage, M. L. Eaton, M. Perlman, and M. Sobel. Inequalities on the probability content of convex regions for elliptically contoured distributions. Proc. 6th Berkeley Sympos. math. Statist. Probab., Univ. Calif. 1970, 2, 241-265 (1972)., 1972.

Ying Ding and Xinsheng Zhang. Some stochastic orders of Kotz-type distributions. *Stat. Probab. Lett.*, 69(4):389–396, 2004.

Kai-Tai Fang and Yao-Ting Zhang. *Generalized multivariate analysis.* Berlin etc.: Springer-Verlag; Beijing: Science Press, 1990.

D. Kurowicka and R. M. Cooke. Distribution-free continuous Bayesian belief nets. In *Modern statistical and mathematical methods in reliability. Selected papers based on the presentation at the 4th international conference on mathematical methods in reliability (MMR 2004), Santa Fe, New Mexico, USA, June 21–25, 2004.*, pages 309–322. Hackensack, NJ: World Scientific, 2005.

Dorota Kurowicka and Roger Cooke. *Uncertainty analysis with high dimensional dependence modelling.* Chichester: John Wiley & Sons, 2006.

Zinoviy Landsman and Andreas Tsanakas. Stochastic ordering of bivariate elliptical distributions. *Statistics & probability letters*, 76(5):488–494, 2006.

Alfred Müller. Stochastic ordering of multivariate normal distributions. *Ann. Inst. Stat. Math.*, 53(3):567–575, 2001.

Alfred Müller and Marco Scarsini. Some remarks on the supermodular order. *Journal of Multivariate Analysis*, 73(1):107–119, 2000.

Alfred Müller and Marco Scarsini. Stochastic comparison of random vectors with a common copula. *Math. Oper. Res.*, 26(4):723–740, 2001.

Alfred Müller and Marco Scarsini. Stochastic order relations and lattices of probability measures. *SIAM J. Optim.*, 16(4):1024–1043, 2006.

Alfred Müller and Dietrich Stoyan. *Comparison methods for stochastic models and risks.* Chichester: Wiley, 2002.

George Poole and Thomas Boullion. A survey on M-matrices. *SIAM Rev.*, 16: 419–427, 1974.

Ludger Rüschendorf. Characterization of dependence concepts in normal distributions. *Ann. Inst. Stat. Math.*, 33:347–359, 1981.

Ludger Rüschendorf. *Mathematical Risk Analysis.* Springer, 2013.

Moshe Shaked and J. George Shanthikumar. Supermodular stochastic orders and positive dependence of random vectors. *J. Multivariate Anal.*, 61(1): 86–101, 1997.

Moshe Shaked and J. George Shanthikumar. *Stochastic Orders.* Springer, 2007.

David Slepian. The one-sided barrier problem for gaussian noise. *Bell System Technical Journal*, 41(2):463–501, 1962.

Yung Liang Tong. Probability inequalities in multivariate distributions. Probability and Mathematical Statistics. New York etc.: Academic Press (A Subsidiary of Harcourt Brace Jovanovich, Publishers)., 1980.

Julian Witting. Value-at-Risk (VaR) Schranken mit zusätzlichen Abhängigkeitsinformationen. Master's thesis, Albert-Ludwigs-Universität Freiburg, 2017.

Chuancun Yin. Stochastic orderings of multivariate elliptical distributions. *arXiv preprint arXiv:1910.07158*, 2019.