Mass transportation and risk bounds under dependence uncertainty

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Risk bounds under dependence uncertainty
Worst case portfolio vectors, ...
Additional structural and ...
Ordering results for risk models
Conclusion
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1. **Risk bounds under dependence uncertainty**
2. **Worst case portfolio vectors, comonotonicity, and mass transportation**
   - A – Law invariant convex risk measures for portfolio vectors
   - B – Risk measures and optimal mass transportation
   - C – Optimal couplings and examples
3. **Additional structural and dependence information**
   - A – Higher dimensional marginals
   - B – Risk bounds under moment constraints
   - C – Positive and negative dependence information
   - D – Partially specified risk factor models
4. **Ordering results for risk models**
5. **Conclusion**

1. Risk bounds under dependence uncertainty

Stochastic Dependence

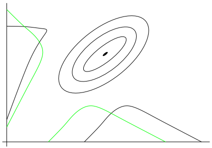
a) dependence modelling

$X = (X_1, \dots, X_n), \quad X_i \in \mathbb{R}^d$
$X_i \sim P_i$ marginal structure

dependence structure: **Copula**

$\rightarrow$ copula models
Sklar's theorem

$f_\vartheta \sim P_\vartheta$



—: f_ϑ marginals;
—: given marginals

b) Hoeffding–Fréchet bounds

stochastic ordering, extremal dependence bounds for risk functionals

Conferences: *Probability with given marginals*
Rome 1990, Seattle 1993, Prague 1996, Barcelona 1998, Montreal 2004, Tartu 2007, Sao Paulo 2010

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VaR-bounds with marginal information

$X = (X_1, \dots, X_n)$ risk vector

marginal information: $X_i \sim F_i$

$\longrightarrow$ high model risk for VaR, TVaR, ...

maximal tail risk

$$M(s) = \sup_{X_i \sim F_i} \left\{ P\left(\sum_{i=1}^n X_i \geq s \right) \right\}$$

$\underline{\mathrm{VaR}}_\alpha(S_n)$	$\mathrm{VaR}_\alpha(S_n^\perp)$	$\mathrm{VaR}_\alpha(S_n^c)$	$\overline{\mathrm{VaR}}_\alpha(S_n)$
lower bound	independent	comonotonic	upper bound

$\mathrm{VaR}_\alpha(Y) = F_Y^{-1}(\alpha)$ upper α-quantile of F_Y

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generalized Hoeffding-Fréchet functional

$\varphi = \varphi(x_1, \dots, x_n)$, $X_i \sim P_i$, $1 \leq i \leq n$

$$M(\varphi) = \sup \left\{ \int \varphi dP; \;\; P \in M(P_1, \dots, P_n) \right\}$$

worst case risk $\sim$ maximal influence of dependence

generalized Hoeffding-Fréchet bounds, Rü (1979); Kellerer (1984), Rachev, Rü (1998); Fréchet (1935/1951); Hoeffding (1940)

Duality theorem for generalized Hoeffding-Fréchet functionals

$$M(\varphi) = \inf\Big\{\sum_{i=1}^n \int f_i dP_i;\ \sum_{i=1}^n f_i(x_i) \ge \varphi(x)\Big\}$$

general n, cost function φ:
Rü (1979, 1981); Gaffke, Rü (1981); Kellerer (1984); Rachev (1984, 1991); Rachev, Rü (1998); ...

Kantorovich (1942, 1948); Kantorovich, Rubinstein (1957):
$\varphi = \varphi(x_1, x_2)$ is a metric (on compact space)

$\rightarrow$ **mass transport problem** Kantorovich–Rubinstein theorem,
$n = 2$ multi-marginal transport problem

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VaR-bounds with marginal information

$\mathrm{VaR}_\alpha \leq \mathrm{TVaR}_\alpha$, convex ordering result: $S_n \leq_{\mathrm{cx}} S_n^c$ comonotonic sum

Theorem (unconstrained bounds)

$$A := \sum_{i=1}^{n} \mathrm{LTVaR}_\alpha(X_i) = \mathrm{LTVaR}_\alpha(S_n^c) \leq \mathrm{VaR}_\alpha(S_n)$$

$$\leq \mathrm{TVaR}_\alpha(S_n) \leq \mathrm{TVaR}_\alpha(S_n^c) = \sum_{i=1}^{n} \mathrm{TVaR}_\alpha(X_i) =: B$$

$$\mathrm{LTVaR}_\alpha(X_i) := \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_u(X_i)du, \quad S_n^c = \text{ comonotonic sum}$$

Bernard, Rü, Vanduffel (2013); Puccetti, Rü (2012); Wang, Wang (2011); Embrechts, Puccetti (2006); Embrechts, Puccetti, Rü (2013); Puccetti, Rü (2013), dual bounds

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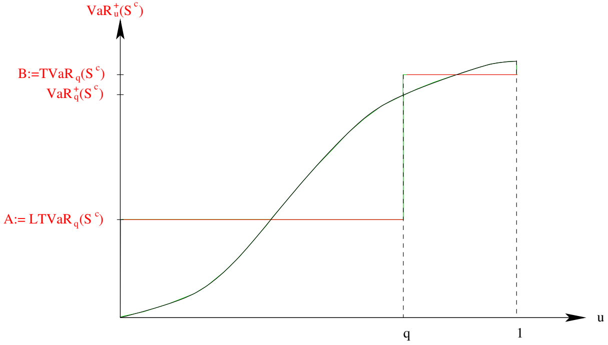
$$\overline{\text{VaR}}_\alpha(S_n) \sim \text{TVaR}_\alpha(S_n^c), \quad n \to \infty$$

and $$\underline{\text{VaR}}_\alpha(S_n) \sim \text{LTVaR}_\alpha(S_n^c), \quad n \to \infty$$

Puccetti, Rü (2012); Puccetti, Wang (2013); Wang, Wang (2014); Embrechts, Wang, Wang (2015)

note: mixing (= negative dependence) in upper domain allows to increase VaR upper bound



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Rearrangement = Dependence

Theorem (Rü (1983))

Let $\mathfrak{F}(F_1, \ldots, F_d)$ *be the set of all joint dfs on* $\mathbb{R}^d$ *with marginals* $F_1, \ldots, F_d$.
Let U *be a random variable with* $F_U = U(0,1)$. *Then:*

$$\mathfrak{F}(F_1, \ldots, F_d) = \left\{F_{(f_1(U),\ldots,f_d(U))};\ f_i \sim_r F_i^{-1},\ 1 \le i \le d\right\}.$$

$$M(s) = \sup\left\{P\left(\sum_{i=1}^n L_i \ge s\right);\ L_i \sim F_i\right\}$$
$$= 1 - \inf\left\{\alpha\,;\ \exists\ f_j^\alpha \sim_r F_j^{-1}\big|_{[\alpha,1]},\ \sum_{j=1}^n f_j^\alpha \ge s\right\}$$

$\rightarrow$ RA-algorithm, precise determination of VaR bounds
Puccetti, Rü (2012)

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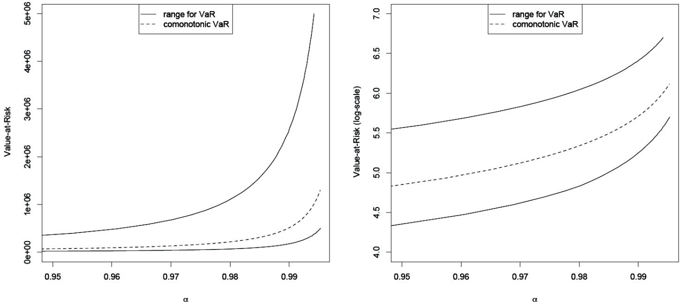
Dependence Uncertainty

$d = 8$	$N = 1.0e05$	*avg time*: 30 secs		
α	$\underline{\mathrm{VaR}}_\alpha(L)$ (RA range)	$\mathrm{VaR}^+_\alpha(L)$ (exact)	$\overline{\mathrm{VaR}}_\alpha(L)$ (exact)	$\overline{\mathrm{VaR}}_\alpha(L)$ (RA range)
0.99	9.00 – 9.00	72.00	141.67	141.66–141.67
0.995	13.13 – 13.14	105.14	203.66	203.65–203.66
0.999	30.47 – 30.62	244.98	465.29	465.28–465.30
$d = 56$	$N = 1.0e05$	*avg time*: 9 mins		
α	$\underline{\mathrm{VaR}}_\alpha(L)$ (RA range)	$\mathrm{VaR}^+_\alpha(L)$ (exact)	$\overline{\mathrm{VaR}}_\alpha(L)$ (exact)	$\overline{\mathrm{VaR}}_\alpha(L)$ (RA range)
0.99	45.82 – 45.82	504	1053.96	1053.80–1054.11
0.995	48.60 – 48.61	735.96	1513.71	1513.49–1513.93
0.999	52.56 – 52.58	1714.88	3453.99	3453.49–3454.48
$d = 648$	$N = 5.0e04$	*avg time*: 8 hrs		
α	$\underline{\mathrm{VaR}}_\alpha(L)$ (RA range)	$\mathrm{VaR}^+_\alpha(L)$ (exact)	$\overline{\mathrm{VaR}}_\alpha(L)$ (exact)	$\overline{\mathrm{VaR}}_\alpha(L)$ (RA range)
0.99	530.12 – 530.24	5832.00	12302.00	12269.74-12354.00
0.995	562.33 – 562.50	8516.10	17666.06	17620.45-17739.60
0.999	608.08 – 608.47	19843.56	40303.48	40201.48-40467.92

Estimates for $\overline{\mathrm{VaR}}_\alpha(L)$ and $\underline{\mathrm{VaR}}_\alpha(L)$ for random vectors of Pareto(2)-distributed risks.



VaR range (5), and comonotonic VaR(8) (in log-scale on the right) for the sum of $d = 8$ GPD risks with parameters following Moscadelli (2004), based on RA for $N = 1 : 0e05$.

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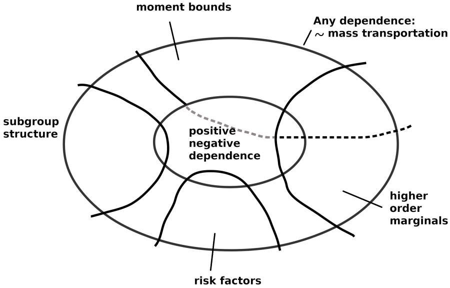

⟶ mass transportation with additional restrictions (generalized moments, multivariate marginals, positive negative dependence, additional structural restrictions)

⟶ additional martingale constraints leads to improved price bounds

⟶ ordering within subclasses

⟶ worst case risks w.r.t. risk measures ∼ non-linear mass transportation, higher dimensional risks

2. Worst case portfolio vectors, comonotonicity, and mass transportation

portfolio vector: $X = (X_1, \dots, X_n)$, $X_i \in \mathbb{R}^d$, $X_i \sim P_i$

$\varrho = \varrho(X)$ risk measure

worst case portfolio = worst case dependence structure

$$\varrho(X) = \sup_{Y_i \sim P_i} \varrho(Y)$$

joint portfolio: $\varrho = \varrho\left(\sum_{i=1}^n X_i\right)$

$d = 1$ Comonotonicity

$X^c = \left(F_1^{-1}(U), \dots, F_n^{-1}(U)\right)$, $F_i \sim P_i$ comonotone vector

$$\sum_{i=1}^n X_i \leq_{cx} \sum_{i=1}^n F_i^{-1}(U), \qquad X_i \in L^1$$

Meilijson, Nadas (1979)

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$$\varrho\left(\sum_{i=1}^n X_i\right) \le \varrho\left(\sum_{i=1}^n F_i^{-1}(U)\right)$$

for all law invariant, convex risk measures ϱ

$$\sup_{\widetilde{X}_i \sim P_i} \varrho\left(\sum_{i=1}^n \widetilde{X}_i\right) = \varrho\left(\sum_{i=1}^n F_i^{-1}(U)\right)$$

X^c ist worst case portfolio vector for any convex, law invariant risk measure ϱ

- $\varrho(\max F_i^{-1}(U)) = \inf_{\widetilde{X}_i \sim P_i} \varrho(\max \widetilde{X}_i)$
- $\sup_{\widetilde{X}_i \sim P_i} \mathrm{VaR}_\alpha\left(\sum_{i=1}^n \widetilde{X}_i\right) = ?$

Comonotonicity notion in $d \ge 2$?

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Comonotonicity and worst case joint portfolios

Unlike $d = 1$ there is no general notion of **comonotonicity** in $d \geq 2$ (Rü 2004)

Theorem (Comonotone improvement theorem of risk sharing, $d = 1$)

$X \in L^1$, $Y = (Y_1, \dots, Y_n) \in \mathcal{A}(X)$ an allocation of X, i.e.

$$Y_i \in L^1,\ \sum_{i=1}^n Y_i = X.$$

Then there exists a ***comonotone allocation*** *$\overline{Y} \in \mathcal{A}(X)$, such that $\overline{Y}_i \leq_{cx} Y_i$, $1 \leq i \leq n$.*

In particular: $\varrho_i(\overline{Y}_i) \leq \varrho_i(Y_i)$ for all convex law invariant risk measures ϱ_i on L^1.

Landsberger, Meilijson (1994); Dana, Meilijson (2003);

Ludkovski, Rü (2008); Filipovic, Svindland (2008);

Kiesel, Rü (2009)

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Comonotonicity $d \geq 1$?

$\boldsymbol{n = 2}$ $\quad \Psi(X) = \left(E\|X\|_2^2\right)^{1/2} \quad L^2$-risk

$\Psi(X_1 + X_2) = \sup \Leftrightarrow X_1, X_2$ worst case portfolio

$\Leftrightarrow E\|X_1 - X_2\|^2 = \inf \quad$ i.e. $X_1 \underset{\text{oc}}{\sim} X_2$

$\Leftrightarrow$: X_1, X_2 comonotone (w.r.t. Ψ)

but no uniformity over risk measures

nonexistence of comonotone vectors:

$d \geq 1$, $P_1, P_2 \dots, P_n \in M^1(\mathbb{R}^d, \mathcal{B}^d)$, $n \geq 3$, then (typically) there do **not** exist $X_i \sim P_i$ such that the pairs

(∗) $\qquad (X_i, X_j)$ are optimal couplings for all i, j.

e.g. $P_i \sim N(\mu_i, \Sigma_i)$ then

$$(*) \;\Leftrightarrow\; \Sigma_i \Sigma_j = \Sigma_j \Sigma_i \quad \forall\ i,\ j$$

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Optimal couplings depend on convex risk measure ϱ

$d \geq 2$. There does not exist dependence structure
i.e. $X \sim P$, $Y \sim Q$, such that

$$\varrho(X+Y) = \sup_{V \sim X, W \sim Q} \varrho(V+W) \text{ worst case}$$

$$\varrho(X+Y) = \inf_{V \sim P, W \sim Q} \varrho(V+W) \text{ best case}$$

for **all** convex risk measures ϱ.

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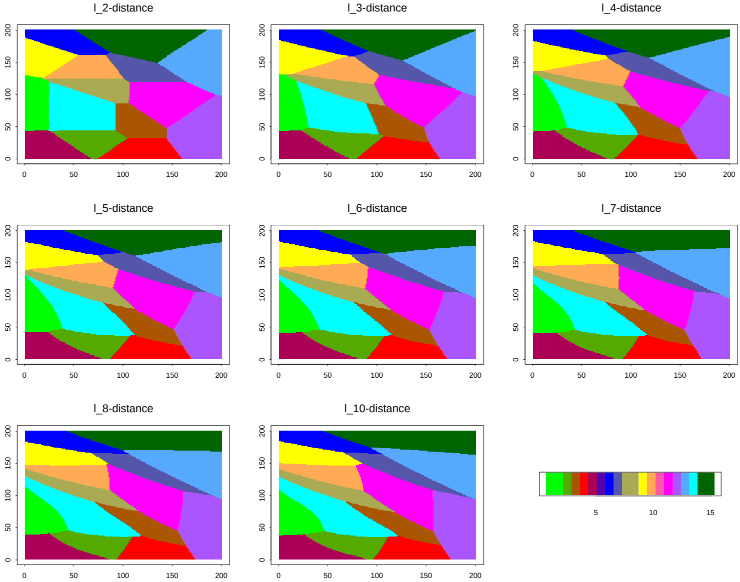

l_2-distance
l_3-distance
l_4-distance
l_5-distance
l_6-distance
l_7-distance
l_8-distance
l_10-distance
0
50
100
150
200
5
10
15

Worst case joint portfolio

$\boldsymbol{d \geq 2}$ ϱ convex risk measure

$X = (X_1, \dots, X_n)$ **ϱ-comonotone**

$\Leftrightarrow$ X **worst case joint portfolio** w.r.t. ϱ i.e.

$$\varrho\left(\sum_{i=1}^n X_i\right) = \sup_{\widetilde{X}_i \sim X_i} \varrho\left(\sum_{i=1}^n \widetilde{X}_i\right)$$

Aim: Characterization.

Diversification:

ϱ coherent, $\varrho\left(\sum X_i\right) \leq \sum \varrho(X_i)$

$\sum \varrho(X_i) - \varrho\left(\sum X_i\right)$ **diversification of (X_i)**

$$D = \sum \varrho(X_i) - \sup_{\widetilde{X}_i \sim X_i} \varrho\left(\sum \widetilde{X}_i\right) = D((X_i))$$

worst case diversification of (X_i)

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No worst case diversification, $\forall\,(X_i)$, $D = 0$
$\Leftrightarrow$: ϱ **strongly coherent** Ekeland, Galichon, Henry (2009):

'to prevent giving an unnecessary premium to conglomerates and avoid imposing an overconservative rule to the banks'

$d = 1$ Kusuoka (2001) ϱ coherent risk measure

Theorem (Kusuoka Theorems)

1. *ϱ law invariant, coherent risk measure*
   $\Leftrightarrow \varrho(X) = \sup_{\mu \in A} \int_{[0,1]} \varrho_\lambda(X) d\mu(\lambda)$, $\varrho_\lambda(X) = \mathrm{TVaR}_\lambda(X)$
2. *ϱ strongly coherent*
   $\Leftrightarrow$ *ϱ comonotone additive*
   $\Leftrightarrow$ *ϱ spectral risk measure*

$$\varrho(X) = \int_{[0,1]} \varrho_\lambda(X) d\mu(\lambda), \quad \varrho_\lambda(x) = \mathrm{TVaR}_\lambda(X)$$

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A – Law invariant convex risk measures for portfolio vectors

$(\Omega, \mathfrak{A}, P)$ nonatomic measure space
$\varrho : L_d^p \to (-\infty, \infty]$ convex risk measure
i.e. monotone, convex, cash invariant
$\Psi(X) = \varrho(-X)$ insurance version, $L_d^p = L_d^p(P)$

Theorem 2.1 (Representation)

a) ϱ proper convex, lsc risk measure on L_d^p

$$\Leftrightarrow \varrho(X) = \sup_{Q \in \mathcal{Q}_{d,p}(P)} \{E_Q(-X) - \alpha(Q)\}$$

penalty $\alpha(Q) = \sup_{X \in L_d^p} \{E_Q(-X) - \varrho(X)\}$

$$\mathcal{Q}_{d,p} = \begin{cases} \mathcal{M}_d^p = \left\{ Q \in \mathcal{M}_d;\ \dfrac{dQ_i}{dP} \in L^q \right\} & 1 \le p < \infty \\ ba_d(P) & p = \infty \end{cases}$$

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b) ϱ finite lsc convex risk measure on L^p_d, $1 \le p \le \infty$
$\Leftrightarrow \varrho(X) = \max_{Q \in \mathcal{Q}} \{E_Q(-X) - \varrho^*(Q)\}$
$\exists\, \mathcal{Q} \subset \mathcal{Q}_{d,p}$, $\mathcal{D} = \left\{ \frac{dQ_i}{dP},\ 1 \le i \le d,\ Q \in \mathcal{Q} \right\} \subset L^q$
weakly closed in $L^q(ba_d(P))$.

Cheridito, Delbaen, Kupper (2004); Ruszczyński, Shapiro (2006); Cheridito, Li (2009); Kaina, Rü (2009); Filipovic, Svindland (2009); Rü (2009)

ϱ **strongly continuous** if representation set $\mathcal{Q} \subset \mathcal{Q}_{d,p}$ is weakly compact in L^q

ϱ finite, coherent risk measure on L^p_d

$\Rightarrow \varrho$ strongly continuous.

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Law invariant convex risk measures

$\varrho : L^p_d \to (-\infty, \infty]$ convex, **law invariant**
i.e. $P^X = P^Y \Rightarrow \varrho(X) = \varrho(Y)$
$d = 1$ Kusuoka (2001); Frittelli, Rosazza-Gianin (2005)

$$\varrho(X) = \sup_{\mu \in M_1((0,1])} \left(\int_{(0,1]} \varrho_\lambda(X) d\mu(\lambda) - \beta(\mu) \right)$$

$\varrho_\lambda(X) = \mathrm{TVaR}_\lambda(X)$ average value at risk

Question: What is the analogon for portfolio risk measures?

Proposition ($d \geq 1$)

ϱ convex risk measure on $L^p_d(P)$

$\Rightarrow \widehat{\varrho}(X) := \sup\{\varrho(\widetilde{X}); \widetilde{X} \in A(X)\}$
is convex, law invariant risk measure

ϱ law invariant $\Leftrightarrow \varrho = \widehat{\varrho}$, $\quad A(X) := \{\widetilde{X} \in L^p_d(P) : \widetilde{X} \stackrel{d}{=} X\}$
equivalence class

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Example (Maximal correlation risk measure, Rü (2006))

$Y \in D_q = \{(Y_1, \dots, Y_d);\ Y_i \geq 0[P],\ E_P Y_i = 1,\ Y_i \in L^q,\ 1 \leq i \leq d\}$

scenario densities

$\Psi_Y(X) := EX \cdot Y$ *correlation coefficient*

(up to normalization)

$$\widehat{\Psi}_Y(X) = \sup_{\widetilde{X} \sim X} E\widetilde{X} \cdot Y = \sup_{\widetilde{Y} \sim \mu} EX \cdot \widetilde{Y} = \Psi_\mu(X)$$

maximal correlation risk measure

(in direction Y resp. μ)

$\rightarrow$ *is law invariant convex (coherent) risk measure*

Remarks

$$d = 1 \quad \widehat{\Psi}_Y(X) = \widehat{\Psi}(X, Y) = \int_0^1 F_X^{-1}(u) F_Y^{-1}(u) du$$

$=$ *weighted average value at risk*

$$\widehat{\Psi}_Y(X) = \sup_{\tilde{Y} \sim Y} EX \cdot \tilde{Y} = \Psi_\mu(X)$$

$$= \widehat{\Psi}(X, Y) = \sup\{\textstyle\int x \cdot y \, d\tau(x, y); \tau \in M(P_X, P_Y)\},\ \mu = \mathcal{L}(Y)$$

(optimal) L^2 *transportation problem*

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Theorem (Generalized Kusuoka Theorem, Rü (2006))

Ψ *convex risk measure on* $L_d^p(P)$ *with penalty function* α
Ψ *is law invariant*
$$\Leftrightarrow \Psi(X) = \sup_{Y \in D_0} (\widehat{\Psi}_Y(X) - \alpha(Y)) = \sup_{\mu \in A}(\Psi_\mu(X) - \alpha(\mu))$$
α *law invariant penalty function,*
$D_0 = \{Y \in D_q; \alpha(Y) < \infty\} \sim A$

Ψ **law invariant coherent risk** measure in $L_d^\infty(P)$ ($L_d^p(P)$)
$$\Leftrightarrow \exists A \subset D_q : \Psi(X) = \sup_{Y \in \widetilde{A}} \widehat{\Psi}_Y(X) = \sup_{\mu \in A} \Psi_\mu(X)$$

maximal correlation risk measures are the building blocks of law invariant risk measures

Ψ law invariant $\Rightarrow$ Ψ Fatou continuous (JST (2005))

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Theorem (Optimal L^2-mass transportation)

$P_i \in M^1(\mathbb{R}^d, \mathfrak{B}^d)$, $i = 1, 2$, $\int \|x\|^2 dP_i(x) < \infty$

a) $\exists$ *optimal* L^2*-coupling of* P_1, P_2
i.e. $\exists X_i \sim P_i : EX_1 \cdot X_2 = \sup_{Y_i \sim P_i} EY_1 \cdot Y_2$
(equivalently $E\|X_1 - X_2\|^2 = \inf_{Y_i \sim P_i} E\|Y_1 - Y_2\|^2$*)*

b) $X_i \sim P_i$ *is an* ***optimal*** $\boldsymbol{L^2}$***-coupling***
$\Leftrightarrow \exists$ *convex, lsc* $f \in L^1(P_1) : X_2 \in \partial f(X_1)$ *a.s.*

c) *If* $P_1 \ll \lambda^d$ *then for* f *as in b)*
$\partial f(X) = \{\nabla f(X)\}$ *a.s. and* $(X, \nabla f(X))$ *is a solution of the* ***Monge problem***

d) *If* $P_1 \ll \lambda^d$ *then* $\exists$ *a* P_1 *a.s. unique gradient* ∇f *of a convex function* f*:* $P_1^{\nabla f} = P_2$

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Remarks

1) b) *Rü, Rachev (1990), Brenier (1991),*
sufficiency Knott, Smith (1984)
charact. optimal transport
Lebesgue cont. bd. supp., 'Breniers Theorem'?

c) *from b) + Rademacher theorem*

d) *Brenier (1991) + particular instance of b) in (1987) on polar factorization*
uniqueness and existence

2) *extension to coupling with general cost* $\int c(x, y)d\mu(x, y)$
Rü (1991), ***c*****-convexity,** ***c*****-subgradients**

$$X_2 \in \partial_c f(X_1) \quad a.s.$$

Smith (1994) c-cyclically monotone support
Gangbo, McCann (1995); Schachermayer, Teichmann (2008); Villani (2008)

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Example

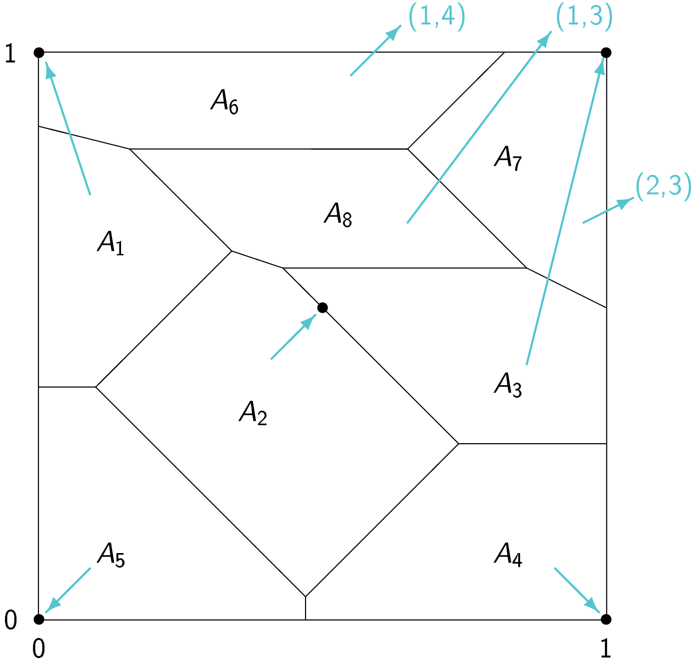
$P = \mathcal{U}_{[0,1]^2}$, $Q = \sum_{j=1}^n \alpha_j \varepsilon_{x_j}$

c-convex functions: $f(x) = \sup_{i \le n}(c(x, x_j) + a_j)$

$$\begin{aligned} A_j &= \{x : f(x) = c(x, x_j) + a_j\} \qquad \text{Voronoi cells} \\ &= \{x : x_j \in \partial_c f(x)\} \end{aligned}$$

Problem: Find shifts a_j such that $P(A_j) = \alpha_j$
particular ex: $c(x, y) = \|x - y\|^2$

$$\begin{aligned} (x_1, \dots, x_8) &= ((0,1), (0.5, 0.5), (1,1), (1,0), \\ &\qquad (0,0), (1,4), (2,3), (1,3)) \\ (\alpha_1, \dots, \alpha_8) &= (0.105, 0.2, 0.125, 0.125, 0.125, 0.12, 0.1, 0.1) \end{aligned}$$



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Worst case joint portfolios and diversification

Ψ finite, convex, law invariant risk measure on L^p_d

$X = (X_1, \ldots, X_n)$, $X_i \in L^p_d$ **worst case portfolio** w.r.t. Ψ if

$$\Psi\left(\frac{1}{n}\sum_{i=1}^n X_i\right) = \sup_{\widetilde{X}_i \sim X_i} \Psi\left(\frac{1}{n}\sum_{i=1}^n \widetilde{X}_i\right)$$

a) $\boldsymbol{\Psi = \Psi_\mu}$ max-correlation risk measure (direction μ)

***X* μ-comonotone**, if for some density vector

$$\exists\, Y \sim \mu,\ Y \in D^q_d : X_i \underset{\text{oc}}{\sim} Y, \quad i \le i \le n$$

$$\Psi_\mu(X_i) = \sup_{\widetilde{X}_i \sim X_i} E\widetilde{X}_i \cdot Y = EX_i \cdot Y$$

$\Rightarrow$

$$\sum_{i=1}^n X_i \underset{\text{oc}}{\sim} Y$$

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Proposition

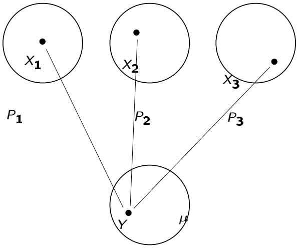
$\Psi = \Psi_\mu$ max-correlation risk measure, $\mu \in \mathcal{M}_d^q$ scenario risk measure, $X_i \sim P_i$

$(X_1, \dots, X_n)$ is worst case dependence structure w.r.t Ψ_μ

$\Leftrightarrow X_1, \dots, X_n$ are μ-comonotone

Ψ_λ is strongly coherent $\sim$ no worst case diversification

EGH (2009), Rü (2009)



. . .

$X_1, \dots, X_n$ μ-comonotone

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b) General finite l.i.convex risk measures on L_d^p

(**) $\Psi(X) = \max_{\mu \in A} (\Psi_\mu(X) - \alpha(\mu))$

$A \subset \mathcal{M}_d^q$ weakly closed, scenario measures

$$F(\mu) := \frac{1}{n} \sum_{i=1}^n \Psi_\mu(X_i) - \alpha(\mu)$$

average risk functional (w.r.t. μ)

$\mu_0 \in A$ **worst case scenario** if

$$F(\mu_0) = \sup_{\mu \in A} F(\mu)$$

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Theorem (Worst case joint portfolio, Rü (2009, 2012))

$X_i \sim P_i, \quad 1 \leq i \leq n \quad$ *portfolio,*

Ψ *finite, convex, law invariant risk measure as in (**)*

a) *worst case risk* = sup *of average risk functional* $F(\mu)$

$$\sup_{\widetilde{X}_i \sim X_i} \Psi\left(\frac{1}{n}\sum_{i=1}^n \widetilde{X}_i\right) = \sup_{\mu \in A} F(\mu)$$

b) μ_0 *worst case scenario and* (X_i^*) *are* μ_0*-comonotone,*

then $(X_1^*, \dots, X_n^*)$ *is a worst case joint portfolio.*

c) *If* Ψ *strongly continuous then*

$\exists$ *worst case scenario measure* $\mu_0 \in A$

Remark (Worst case total risk)

Ψ ***coherent*** $F_c(\mu) = \sum_{i=1}^{n} \Psi_\mu(X_i)$ *total risk functional.*

$\mu_0 \in A$ ***worst case scenario*** *if* $F_c(\mu_0) = \sup_{\mu \in A} F_c(\mu)$

$$\sup_{\tilde{X}_i \sim X_i} \Psi\left(\sum_{i=1}^{n} \tilde{X}_i\right) = \Psi\left(\sum_{i=1}^{n} X_i^*\right), \; (X_i^*) \; \mu_0\text{-comonotone}$$

Ψ ***convex****:* $\Psi\left(\sum_{i=1}^{n} X_i\right) = \Psi\left(\frac{1}{n}\sum_{i=1}^{n} nX_i\right)$

Corollary (Worst case diversification of total risk)

$$D = \sum_{i=1}^{n} \Psi(X_i) - F_c(\mu_0)$$

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$$\frac{1}{n}\sum_{i=1}^{n}\Psi(X_i) - \Psi\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right) \quad \textbf{diversification effect } (X_i)$$

$$D = \frac{1}{n}\sum_{i=1}^{n}\Psi(X_i) - \sup_{\widetilde{X}_i\sim X_i}\Psi\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right) = D((X_i))$$

worst case diversification

Theorem (Second Kusuoka Theorem)

Ψ *strongly continuous convex risk measure*

Ψ *has no worst case diversification effect (strongly coherent)*
i.e. $\forall\,(X_i)$ *holds* $D((X_i)) = 0$

$\Leftrightarrow$ Ψ *is translated* max *correlation risk measure*

$\Psi = \Psi_\mu - \alpha(\mu)$, $\exists\,\mu \in \mathcal{M}_d^q$, $\alpha(\mu) \in \mathbb{R}^1$

$d = 1$ Kusuoka (2001)

$d \geq 1$ Ekeland, Galichon, Henri (2009); Rü (2009)

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C – Optimal couplings and examples

Worst case dependence structure

$\sim$ 1. worst case scenario measure $\mu_0 \in A$

2. $X_1^*, \ldots, X_n^*$ μ_0-comonotone

i.e. $Y \sim \mu_0$, $X_i^* \underset{\text{oc}}{\sim} Y$

discrete distributions approximation: gradient descent algorithm

$\sim$ combinatorial Voronoi type partitioning
(cf. Aurenhammer, Hoffmann, Aronov (2000))

Rü, Uckelmann(2000); Ekeland, Galichon, Henri (2009)

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1. Location scale families, elliptical distributions

$X \in \mathbb{R}^d, \quad X \sim Q, \quad \Sigma = \text{Cov}X$

$\mathcal{Q} = \{Q_{a,B};\ a \in \mathbb{R}^d,\ B \in \mathcal{A}\}$ **location-scale family**

$Q_{a,B} \sim X_{a,B} := BX + a, \quad \mathcal{A}$ scale family

$\mu = Q = Q_{0,I}, \quad X \sim Q \quad$ and $\quad P_i = Q_{a_i,B_i} \in \mathcal{Q}$

a) $\boldsymbol{\mathcal{A} \subset NN(d)}$

$\Rightarrow$ $X_i := X_{a_i,B_i} \underset{oc}{\sim} X$ and $X_1, \dots, X_n$ are μ-comonotone

worst case risk w.r.t. Ψ_μ max correlation risk

$$\sup_{\widetilde{X}_i \sim X_i} \Psi_\mu \left(\sum_{i=1}^n \widetilde{X}_i \right) = \Psi_\mu \left(\sum_{i=1}^n X_i \right) = tr \left(\Big(\sum_{i=1}^n B_i \Big) \Sigma \right)$$

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b) Q invariant w.r.t. orthogonal transformation

$\mathcal{A} \subset M(d, \mathbb{R}) \rightarrow$ affine transformations

$B \in \mathcal{A}, B = PO$, polar factorization, $P \in NN(d)$, $O \in O(d)$

$BX \sim POX \sim PY$, $Y := OX \sim X$

$\Rightarrow$ optimal coupling as in a) with (P_i).

ex. **elliptical distributions**, $N(\mu, \Sigma)$, unif. distr. on ellipsoids, ...

$P_i \in \mathcal{Q}, \quad \Sigma_i = \mathrm{Cov}(P_i), \quad \Sigma_0 = \mathrm{Cov}(T), \quad T \sim Q$

worst case portfolio: $X_i = S_i T, \quad 1 \le i \le n$

$$S_i = \Sigma_i^{1/2} \left(\Sigma_i^{1/2} \Sigma_0 \Sigma_i^{1/2} \right)^{-1/2} \Sigma_i^{1/2}$$

if $A \subset \mathcal{Q}$, $A \sim$ scenario measures, then worst case scenario

$$tr \left[\left(\sum_{i=1}^{n} S_i^T \right) B\Sigma_0 \right] = \sup_{B \in A}$$

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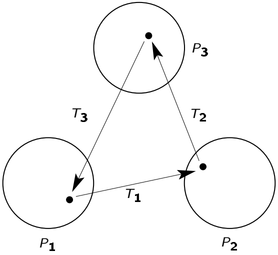
2. Coupling to the sum

Variation risk

$$\Psi(X) = \|X\|_2$$

$$E\left\|\sum_{i=1}^{n} X_i\right\|^2 = \sup \qquad \text{worst case joint portfolio}$$

optimal coupling:



$$T_3 \circ T_2 \circ T_1 = id$$

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$(*) \quad E\|\sum_{i=1}^n X_i\|^2 = \sup!$ optimal coupling

$$\Leftrightarrow \sum_{i=1}^n E\|X_i - S_n\|^2 = \inf!, \quad S_n = \sum_{i=1}^n X_i$$

optimal coupling to the sum principle (Knott and Smith 1994)

equivalently: $law(S_n/n)$ **is a barycenter of** $law(X_i)$

$P_i = N(0, \Sigma_i), \quad \Sigma_i > 0, \quad 1 \leq i \leq n$

assume $S \sim N(0, \Sigma_0)$

$X_i := T_i S, \quad T_i = \Sigma_i^{1/2}(\Sigma_i^{1/2}\Sigma_0\Sigma_i^{1/2})^{1/2}\Sigma_i^{1/2}$

If $\sum_{i=1}^n T_i = id \Leftrightarrow \sum_{i=1}^n (\Sigma_0^{1/2}\Sigma_i\Sigma_0^{1/2})^{1/2} = \Sigma_0$,

then (X_i) is a **worst case portfolio** (optimal n-coupling)

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Theorem

$P_i = N(0, \Sigma_i)$, $\Sigma_i > 0$, $1 \le i \le n$

There exists a solution $\Sigma_0 > 0$ *of* $\sum_{i=1}^{n} (\Sigma_0^{1/2} \Sigma_i \Sigma_0^{1/2})^{1/2} = \Sigma_0$ *and the optimal coupling to the sum is a worst case portfolio*

Theorem

1. $\exists$ *worst case portfolio (i.e. a solution of the matrix equation)*
2. *Optimal coupling to the sum is necessary (in general **not** sufficient)*
3. *If* X_i *are optimally coupled to the sum* S_n, $1 \le i \le n$ *and* $P^{S_n} \ll \lambda^d$ *starlike support, then* (X_i) *is worst case portfolio*

Rü, Uckelmann (2002), worst case scenario measure μ

$=$ distribution of $\sum_{i=1}^{n} X_i$, (X_i) worst case portfolio,

(X_i) comonotone w.r.t. μ.

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3. Additional structural and dependence information

How to reduce risk bounds by using structural and partial dependence information?

- higher order marginals (reduced bounds)
- positive, negative dependence restrictions (improved standard bounds)
- information on variance of S_n, correlations of X_i, X_j
- partial information on risk factors (partially specified risk factor models)
- models with subgroup structure

intuition:

- positive dependence information allows to increase lower risk bounds (but not upper bounds)
- negative dependence information allows to decrease upper risk bounds (but not lower risk bounds)

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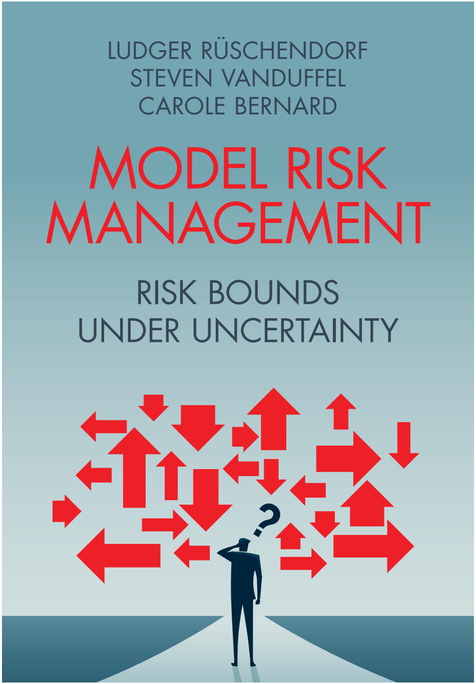

LUDGER RÜSCHENDORF
STEVEN VANDUFFEL
CAROLE BERNARD
MODEL RISK
MANAGEMENT
RISK BOUNDS
UNDER UNCERTAINTY

A – Higher dimensional marginals

$$\mathcal{F}_{\mathcal{E}} = \mathcal{F}(F_J; J \in \mathcal{E}) \subset \mathcal{F}(F_1, \dots, F_n)$$

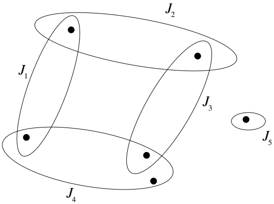
$F_J = F_{X_J}, \quad X_J = (X_j)_{j \in J}$ for $J \in \mathcal{E}$, $\bigcup_{J \in \mathcal{E}} J = \{1, \dots, n\}$

$\mathcal{F}_{\mathcal{E}}$ (resp. $\mathcal{M}_{\mathcal{E}}$) **generalized Fréchet class**

$\mathcal{E} = \{\{1\}, \dots, \{n\}\} \Rightarrow \mathcal{F}_{\mathcal{E}} = \mathcal{F}(F_1, \dots, F_n)$ simple marginal class

$\mathcal{E} = \{\{j, j+1\}, 1 \le j \le n-1\} \to \mathcal{F}_{\mathcal{E}} = \mathcal{F}(F_{1,2}, F_{2,3}, \dots, F_{n-1,n})$ series system

$\mathcal{E} = \{\{1, j\}, 2 \le j \le n\} \to \mathcal{F}(F_{1,2}, F_{1,3}, \dots, F_{1,n})$ starlike system



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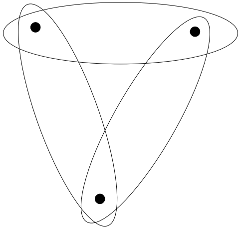
$$\begin{cases} M_{\mathcal{E}}(s) = \sup\{P(X_1 + \cdots + X_n \geq s); F_X \in \mathcal{F}_{\mathcal{E}}\} \\ m_{\mathcal{E}}(s) = \inf\{P(X_1 + \cdots + X_n \geq s); F_X \in \mathcal{F}_{\mathcal{E}}\} \end{cases}$$



marginal problem: $\mathcal{F}_{\mathcal{E}} \neq \emptyset$ (Rü (1991))

decomposable case

$\Leftrightarrow$ (consistency $\Rightarrow$ existence)

duality theorem $\quad \mathcal{M}_{\mathcal{E}} \neq \emptyset$

$$M_{\mathcal{E}}(\varphi) := \sup\left\{\int \varphi dP; P \in \mathcal{M}_{\mathcal{E}}\right\}$$

$$= \inf\left\{\sum_{J\in\mathcal{E}} \int f_J dP_J; \sum_{J\in\mathcal{E}} f_J \circ \pi_J \geq \varphi\right\}, \quad \varphi \text{ usc}$$

Rü (1984); Kellerer (1987)

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Bonferoni type bounds

Proposition

$(E_i, \mathcal{A}_i), (P_J, J \in \mathcal{E})$ *marginal system*

1. $M_{\mathcal{E}}(A_1 \times \cdots \times A_n) \le \min_{J \in \mathcal{E}} P_J(A_J)$
2. $\mathcal{E} = J_2^n = \{(i,j); i,j \le n\}$,

$$q_i = P_i(A_i^c), \quad q_{ij} = P_{ij}(A_i^c \times A_j^c)$$

$$\begin{cases} M_{\mathcal{E}}(A_1 \times \cdots \times A_n) \le 1 - \sum q_i + \sum_{i<j} q_{ij} \\ m_{\mathcal{E}}(A_1 \times \cdots \times A_n) \ge 1 - \sum q_i + \sup_{\tau \in T} \sum_{(i,j) \in \tau} q_{ij} \end{cases}$$

$T =$ *spanning trees of* G_n, *Rü (1991)*

improved upper and lower Fréchet bounds

Conditional bounds

sharp bounds by conditioning in *some decomposable cases*!

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reduced systems

$\mathcal{E} = \{J_1, \ldots, J_m\}$

$\eta_i := \#\{J_r \in \mathcal{E}; i \in J_r\}, \quad 1 \leq i \leq n$

For X risk vector, $F_X \in \mathcal{F}_\mathcal{E}$ define:

$$Y_r := \sum_{i \in J_r} \frac{X_i}{\eta_i}, \quad H_r := F_{Y_r}, \quad r = 1, \ldots, m$$

$\mathcal{H} = \mathcal{F}(H_1, \ldots, H_m)$ Fréchet class

Proposition (reduced bounds)

$\mathcal{F}_\mathcal{E} \neq \emptyset$ consistent marginal system, then for $s \in \mathbb{R}$

$$M_\mathcal{E}(s) \leq M_\mathcal{H}(s) \quad \text{and}$$
$$m_\mathcal{E}(s) \geq m_\mathcal{H}(s)$$

Embrechts, Puccetti (2010); Puccetti, Rü (2012)

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Remark

1. *generalized weighting schemes*

$$Y_r^\alpha = \sum_{i=1}^n \alpha_i^r X_i, \quad \begin{cases} \alpha_i^r > 0 & \textit{iff } i \in J_r \quad \textit{and} \\ \sum_{r=1}^n \alpha_i^r = 1 \end{cases}$$

   $\rightarrow$ *parametrized family of bounds*

2. *Rearrangement algorithm can be used to calculate* $M_{\mathcal{H}}$, $m_{\mathcal{H}}$.

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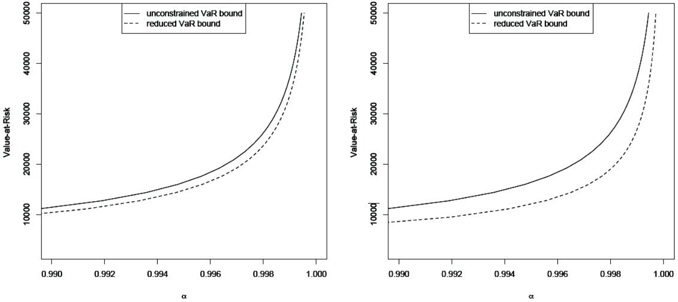
Series case $F_{i,i+1}$ 2-dim Pareto

α	$\mathrm{VaR}^+_\alpha(L)$	$\overline{\mathrm{VaR}}^r_\alpha(L)$, (A)	$\overline{\mathrm{VaR}}^r_\alpha(L)$, (B)	$\overline{\mathrm{VaR}}_\alpha(L)$
0.99	5400.00	8496.13	10309.14	11390.00
0.995	7885.28	12015.04	14788.71	16356.42
0.999	18373.67	26832.2	33710.3	37315.70

Estimates for $\mathrm{VaR}_\alpha(L)$ for a random vector of $d = 600$ Pareto(2)-distributed risks under different dependence scenarios: $\mathrm{VaR}^+_\alpha(L)$ ($(L_1, \dots, L_{600})'$ has copula $C = M$); $\overline{\mathrm{VaR}}^r_\alpha(L)$, (A): the bivariate marginals $F_{2j-1,2j}$ are independent; $\overline{\mathrm{VaR}}^r_\alpha(L)$, (B): the bivariate marginals $F_{2j-1,2j}$ have Pareto copula with $\delta = 1.5$; $\overline{\mathrm{VaR}}_\alpha(L)$: no dependence assumptions are made.



VaR bounds $\overline{\mathrm{VaR}}_\alpha(L)$ (see (5)) and reduced bounds $\overline{\mathrm{VaR}}^r_\alpha(L)$ (see (24a)) for a random vector of $d = 600$ Pareto(2)-distributed risks with fixed bivariate marginals $F_{2j-1,2j}$ generated by a Pareto copula with $\delta = 1.5$, comonotone (left) and by the independence copula (right).

B – Risk bounds under moment constraints

information: $X_i \sim F_i,\ 1 \leq i \leq n$ and $\mathrm{Var}(S_n) \leq s^2$ $(*)$

$\to$ partial information on dependence alternatively information on $\mathrm{Cov}(X_i, X_j)$,
Bernard, Rü, Vanduffel (2016)

$$\begin{cases} M = \sup\{\mathrm{VaR}_\alpha(S_n); \quad S_n \text{ satisfies } (*)\} \\ m = \inf\{\mathrm{VaR}_\alpha(S_n); \quad S_n \text{ satisfies } (*)\} \end{cases}$$

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Theorem

$\alpha \in (0,1), \quad \mathrm{Var}(S_n) \leq s^2$, *then*

$$a := \max\left(\mu - s\sqrt{\frac{\alpha}{1-\alpha}}, A\right) \leq m \leq \mathrm{VaR}_\alpha(S_n) \leq M$$
$$\leq b := \min\left(\mu + s\sqrt{\frac{\alpha}{1-\alpha}}, B\right), \quad \mu = ES_n$$

Remark

VaR *bounds and convex order worst case dependence structure has relation to convex order minima in upper and lower part*

$$\{S_n \geq \mathrm{VaR}_\alpha(S_n)\} \quad \textit{resp.} \quad \{S_n < \mathrm{VaR}_\alpha(S_n)\}$$

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Proposition

$$X_i \sim F_i, \quad F_i^\alpha \sim F_i/[q_i(\alpha), \infty), \quad X_i^\alpha, Y_i^\alpha \sim F_i^\alpha$$

a) $$M = \sup_{X_i \sim F_i} \mathrm{VaR}_\alpha\left(\sum_{i=1}^n X_i\right) = \sup_{Y_i^\alpha \sim F_i^\alpha} \mathrm{VaR}_0\left(\sum_{i=1}^n Y_i^\alpha\right)$$

b) *If* $S^\alpha = \sum_{i=1}^n Y_i^\alpha \leq_{\mathrm{cx}} \sum_{i=1}^n X_i^\alpha$, *then*

$$\mathrm{VaR}_0\left(\sum_{i=1}^n X_i^\alpha\right) \leq \mathrm{VaR}_0(S^\alpha) = \operatorname{ess\,inf}\left(\sum_{i=1}^n Y_i^\alpha\right) \leq B$$

$\longrightarrow$ restriction to convex minima in upper part of distributions

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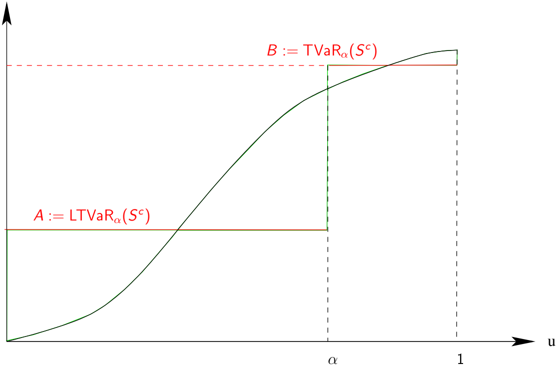
maximizing VaR $\sim$ maximizing minimal support over all $Y_i \sim F_i^\alpha$ is implied by convex order



VaR bounds and convex order

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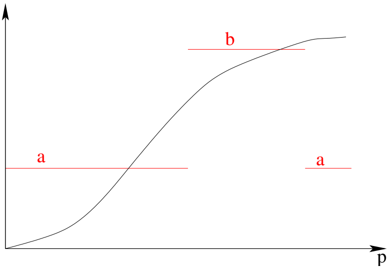
Extended Rearrangement Algorithm (ERA)

two alternating steps

1. choice of domain, starting from largest α-domain
2. Rearrangement in upper α-part and in lower $1-\alpha$-part
3. check variance constraint fulfilled
4. shift of domain and iterate



Variation of ERA: Self determined split of domains.

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Panel A: Approximate sharp bounds obtained by the ERA

(m_d, M_d)		$n = 10$			$n = 100$		
		$\varrho = 0$	$\varrho = 0.15$	$\varrho = 0.3$	$\varrho = 0$	$\varrho = 0.15$	$\varrho = 0.3$
$d = 10{,}000$	$\mathrm{VaR}_{95\%}$	(4.401; 15.72)	(4.091; 21.85)	(3.863; 26.19)	(47.96; 84.72)	(42.48; 188.9)	(39.61; 243.3)
	$\mathrm{VaR}_{99\%}$	(5.486; 28.69)	(4.591; 43.45)	(4.492; 53.22)	(48.99; 129.5)	(46.61; 366.0)	(45.36; 489.5)
	$\mathrm{VaR}_{99.5\%}$	(6.820; 39.48)	(5.471; 59.60)	(4.850; 73.11)	(49.23; 162.8)	(47.54; 499.1)	(46.68; 671.5)

Panel B: Variance-constrained bounds

(a_d, b_d)		$n = 10$			$n = 100$		
		$\varrho = 0$	$\varrho = 0.15$	$\varrho = 0.3$	$\varrho = 0$	$\varrho = 0.15$	$\varrho = 0.3$
$d = 10{,}000$	$\mathrm{VaR}_{95\%}$	(4.398; 16.03)	(4.089; 21.92)	(3.861; 26.23)	(47.96; 84.74)	(42.48; 188.9)	(39.61; 243.4)
	$\mathrm{VaR}_{99\%}$	(4.725; 30.20)	(4.589; 43.64)	(4.490; 53.50)	(48.99; 129.6)	(46.59; 367.3)	(45.33; 491.7)
	$\mathrm{VaR}_{99.5\%}$	(4.800; 40.74)	(4.705; 59.80)	(4.634; 73.77)	(49.23; 162.9)	(47.54; 500.0)	(46.65; 676.3)
$d = +\infty$	$\mathrm{VaR}_{95\%}$	(4.372; 16.94)	(4.037; 23.30)	(3.791; 27.96)	(48.01; 87.75)	(42.09; 200.3)	(38.99; 259.2)
	$\mathrm{VaR}_{99\%}$	(4.725; 32.25)	(4.578; 46.77)	(4.470; 57.41)	(49.13; 136.2)	(46.53; 393.1)	(45.18; 527.4)
	$\mathrm{VaR}_{99.5\%}$	(4.806; 43.63)	(4.702; 64.22)	(4.634; 77.72)	(49.39; 172.2)	(47.56; 536.4)	(46.60; 726.9)

Panel C: Unconstrained bounds independent of ϱ

(A_d, B_d)		$n = 10$	$n = 100$
$d = 10{,}000$	$\mathrm{VaR}_{95\%}$,	(3.646; 30.33)	(36.46; 303.3)
	$\mathrm{VaR}_{99\%}$	(4.447; 57.76)	(44.47; 577.6)
	$\mathrm{VaR}_{99.5\%}$	(4.633; 74.11)	(46.33; 741.1)
$d = +\infty$	$\mathrm{VaR}_{95\%}$	(3.647; 30.72)	(36.47; 307.2)
	$\mathrm{VaR}_{99\%}$	(4.448; 59.62)	(44.48; 596.2)
	$\mathrm{VaR}_{99.5\%}$	(4.635; 77.72)	(46.35; 777.2)

Bounds on Value-at-Risk of sums of Pareto distributed risks ($\theta = 3$)

Application to Credit Risk portfolios

asset correlations ϱ^A – default correlations ϱ^D, loans $X_j \sim \mathcal{B}(p)$

example: $n = 10{,}000$, $p = 0.049$ default probability,
$\varrho^D = 0.0157$ (McNeil et al. (2005)),
$s^2 = np(1-p) + n(n-1)p(1-p)\varrho^D$

	(A_d, B_d)	(a_d, b_d)	(m_d, M_d)	KMV	Beta	CreditMetrics
$VaR_{0.8}$	(0%; 24.50%)	(3.54%; 10.33%)	(3.63%; 10%)	6.84%	6.95%	6.71%
$VaR_{0.9}$	(0%; 49.00%)	(4.00%; 13.04%)	(4.00%; 13%)	8.51%	8.54%	8.41%
$VaR_{0.95}$	(0%; 98.00%)	(4.28%; 16.73%)	(4.32%; 16%)	10.10%	10.01%	10.11%
$VaR_{0.995}$	(4.42%; 100.00%)	(4.71%; 43.18%)	(4.73%; 40%)	15.15%	14.34%	15.87%

The table provides VaR bounds and VaR computed in different models (KMV, Beta, CreditMetrics).

A_d, B_d $\rightarrow$ bounds from marginal information
a_d, b_d $\rightarrow$ bounds with variance constraints

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	$p = 0.25\%$			$p = 1\%$		
	(A, B)	(a, b)	KMV	(A, B)	(a, b)	KMV
$\varrho^A = 0\%$	(0%; 50%)	(0.25%; 0.25%)	0.25%	(0.50%; 100%)	(1.00%; 1.00%)	1.0%
$\varrho^A = 6\%$	(0%; 50%)	(0.23%; 3.27%)	1.2%	(0.50%; 100%)	(0.95%; 10.98%)	4.0%
$\varrho^A = 12\%$	(0%; 50%)	(0.23%; 5.05%)	2.1%	(0.50%; 100%)	(0.92%; 16.27%)	6.3%
$\varrho^A = 18\%$	(0%; 50%)	(0.23%; 6.84%)	2.9%	(0.50%; 100%)	(0.90%; 21.18%)	8.7%
$\varrho^A = 24\%$	(0%; 50%)	(0.21%; 8.76%)	3.8%	(0.50%; 100%)	(0.87%; 26.09%)	11.1%
$\varrho^A = 30\%$	(0%; 50%)	(0.20%; 10.85%)	4.8%	(0.50%; 100%)	(0.85%; 31.13%)	13.7%

Unconstrained and constrained upper and lower 0.995-VaR bounds for several combinations of default probability and correlation and VaR in the (one-factor) KMV model

- significant model error, ex. $\varrho^A = 6\,\%$, $p = 0.25\,\%$, then 99.5 % VaR bounds 0.2 %–3.3 %

Higher order moment constraints

Bernard, Rü, Vanduffel, Yao (2017)

$X_i \sim F_i$, $1 \leq i \leq n$ and $ES_n^k \leq c_k$, $k = 2, \dots, K$

$\to$ strengthened upper bounds for $\mathrm{VaR}_\alpha(S_n)$, modification of RA-algorithm and theoretical bounds

VaR assessment of a corporate portfolio

	$q =$	KMV	Comon.	Unconstrained	$K = 2$	$K = 3$	$K = 4$
	95%	281.3	393.3	(34.0; 2083.3)	(111.8; 483.1)	(111.8; 433.0)	(111.8; 412.8)
$\varrho =$	99%	398.7	2374.1	(56.5; 6973.1)	(115.0; 943.9)	(117.4; 713.3)	(118.2; 610.9)
0.05	99.5%	448.5	5088.5	(89.4; 10119.9)	(116.9; 1285.9)	(118.9; 889.5)	(119.8; 723.2)
	99.9%	573.1	12905.1	(111.8; 14784.9)	(120.2; 2718.1)	(121.2; 1499.6)	(121.8; 1075.9)
	95%	340.6	393.3	(34.0; 2083.3)	(97.3; 614.8)	(100.9; 562.8)	(100.9; 560.6)
$\varrho =$	99%	539.4	2374.1	(56.5; 6973.1)	(111.8; 1245.0)	(115.0; 941.2)	(115.9; 834.7)
0.10	99.5%	631.5	5088.5	(89.4; 10119.9)	(114.9; 1709.4)	(117.6; 1177.8)	(118.5; 989.5)
	99.9%	862.4	12905.1	(111.8; 14784.9)	(119.2; 3692.3)	(120.8; 1995.9)	(121.2; 1472.7)
	95%	388.4	393.3	(34.0; 2083.3)	(91.5; 735.9)	(93.4; 697.0)	(92.0; 727.9)
$\varrho =$	99%	675.8	2374.1	(56.5; 6973.1)	(111.8; 1519.5)	(112.4; 1174.5)	(113.7; 1083.9)
0.15	99.5%	816.1	5088.5	(89.4; 10119.9)	(112.8; 2098.0)	(115.9; 1472.7)	(116.9; 1287.6)
	99.9%	1178.4	12905.1	(111.8; 14784.9)	(118.4; 4531.3)	(120.7; 2501.8)	(120.9; 1916.6)

We report for various asset correlation levels ϱ and confidence levels q the VaRs under the KMV framework (second column), the comonotonic VaRs (third column) and the VaR bounds in the unconstrained and the constrained case (in the last four columns between brackets – K reflects the number of moments of the portfolio sum that are known). The VaR bounds are obtained using Algorithm 1.

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Conclusion:

- impact of variance and higher order moment constraints on VaR bounds
- considerable amount of model risk
- knowledge of marginals + variance (moments) does not always allow to determine VaR's with confidence
- standard risk methods (based on factor models) like KMV, Beta, Credit Metrics report similarly (why? and on what basis?)
- Variance (moment) restriction is a (global) negative dependence assumption; it implies reduction of upper VaR bounds.

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C – Positive and negative dependence information

How does positive/negative dependence information influence risk bounds?

X ***positive upper orthant dependence*** (PUOD)

$$\text{if } \overline{F}_X(x) = P(X > x) \geq \prod_{i=1}^n P(X_i > x_i) = \prod_{i=1}^n \overline{F}_i(x_i)$$

X ***positive lower orthant dependence*** (PLOD)

$$\text{if } F_X(x) \geq \prod_{i=1}^n F_i(x_i), \quad \forall x$$

X POD if X PLOD and PUOD

similary: X NUOD, ...

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One-sided dependence information

$F = F_X$, $\overline{F} = \overline{F}_X$

one-sided dependence information

G increasing function, $F^- \leq G \leq F^+$

$$\begin{cases} G \leq_{\mathrm{PLOD}} F & \rightarrow \textit{positive dependence restriction} \quad \text{(lower tail)} \\ G \leq_{\mathrm{PUOD}} F & \rightarrow \textit{positive dependence restriction} \quad \text{(upper tail)} \end{cases}$$

example: $G(x) = \prod F_i(x_i)$, X is POD

similarly:

$F \leq_{\mathrm{PLOD}} H$, $F \leq_{\mathrm{PUOD}} H$ $\rightarrow$ *negative dependence restriction*

Williamson, Downs (1990); Denuit, Genest, Marceau (1999); Denuit, Dhaene, Ribas (2001); Embrechts, Höing, Juri (2003); Rü (2005); Embrechts, Puccetti (2006); Puccetti, Rü (2012)

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Theorem (improved standard bounds)

X risk vector, marginals $X_i \sim F_i$, $G \uparrow$, $H \downarrow$
$F^- \leq G \leq F^+$, $\overline{F}^- \leq H \leq \overline{F}^+$

a) *Standard bounds:*

$$\left(\bigvee F^-(s)\right)_+ \leq P\Big(\sum_{i=1}^d X_i \leq s\Big) \leq \min\left\{\bigwedge F^+(s), 1\right\}$$

b) *If* $G \leq F_X$, *then*

$$P\Big(\sum_{i=1}^d X_i \geq s\Big) \leq 1 - \bigvee G(s)$$

c) *If* $\overline{F}_X \leq H$, *then*

$$P\Big(\sum_{i=1}^d X_i \geq s\Big) \leq \bigvee H(s)$$

$U(s) := \left\{x \in \mathbb{R}^n; \sum_{i=1}^n x_i = s\right\}$,
$\bigwedge G(s) := \inf_{x \in U(s)} G(x)$ <u>G-infimal convolution</u>,
$\bigvee H(s) := \sup_{x \in U(s)} H(x)$ <u>H-supremal convolution</u>

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Improved Fréchet bounds:

- higher dimensional marginals
  various types of Bonferroni bounds
- parameter uncertainty
- 'known domains'

$$F(x) = \Gamma(x), \quad x \in S$$

(or "$\leq$" or "$\geq$")

$d = 2$ Rachev, Rü (1994); Nelsen, Quesada-Molina, Rodríguez-Lallena, Úbeda-Flores (2001, 2004); Tankov (2011)

$d \geq 2$ Puccetti, Rü, Manko (2016); Lux, Papapantoleon (2016)

digital options on default times for bonds

result: improved VaR-bounds for options

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Model for lower bounds: subgroup structure

Bignozzi, Puccetti, Rü (2014)

$X = (X_1, \dots, X_d)$ risk vector, $F_i = F_{X_i}$

$\{1, \dots, d\} = \bigcup_{j=1}^{k} I_j$ k-subgroups

$Y = (Y_1, \dots, Y_d)$ satisfies:

$$F_Y(x) = \prod_{j=1}^{k} \min_{i \in I_j} G_j(x_i)$$

i.e. – Y has k independent, homogeneous subgroups
– components within subgroups comonotonic

Assumption: $(*)$ $Y \leq X$, positive dependence restriction

where $\leq$ is $\leq_{uo}$ or $\leq_{lo}$, typically: $F_i = G_j$ for $i \in I_j$

If $k = d$ and $F_j = G_j$ then $(*) \sim$ to PUOD resp. PLOD of X

$k = 1$ and $F_i = G_j \Rightarrow X$ comonotonic

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Example: Pareto portfolio

lower bounds, homogeneous portfolio, d Pareto(2) risks, k subgroups, d/k variables in each subgroup, $Y \leq_{uo} X$

$d = 8$	$k = 1$		$k = 2$		$k = 4$		$k = 8$	
	$\underline{\mathrm{VaR}}_\alpha$	VaR^{lb}_α	$\underline{\mathrm{VaR}}_\alpha$	VaR^{lb}_α	$\underline{\mathrm{VaR}}_\alpha$	VaR^{lb}_α	$\underline{\mathrm{VaR}}_\alpha$	VaR^{lb}_α
$\alpha = 0.990$	9.00	72.00	9.00	36.00	9.00	18.00	9.00	9.00
$\alpha = 0.995$	13.14	105.14	13.14	52.57	13.14	26.28	13.14	13.14
$\alpha = 0.999$	30.62	244.98	30.62	122.49	30.62	61.25	30.62	30.62

lower bounds, inhomogeneous portfolio, $d/2$ Exp(2) risks and $d/2$ Exp(4) risks

$d = 8$	$k = 1$		$k = 2$		$k = 4$		$k = 8$	
	$\underline{\mathrm{VaR}}_\alpha$	VaR^{lb}_α	$\underline{\mathrm{VaR}}_\alpha$	VaR^{lb}_α	$\underline{\mathrm{VaR}}_\alpha$	VaR^{lb}_α	$\underline{\mathrm{VaR}}_\alpha$	VaR^{lb}_α
$\alpha = 0.990$	2.30	13.82	2.30	9.21	2.30	4.61	2.30	2.30
$\alpha = 0.995$	2.65	15.89	2.65	10.60	2.65	5.30	2.65	2.65
$\alpha = 0.999$	3.45	20.72	3.45	13.82	3.45	6.91	3.45	3.45

essential improvement of lower bounds for $k = 1, 2, 4$;
POD alone does not improve lower bounds

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Stronger positive/negative dependence conditions

$X = (X_1, \dots, X_n)$ (sequentially) ***positive cumulative dependent*** (PCD) if

$$P\left(\sum_{i=1}^{k-1} X_i > t_1 \mid X_k > t_2\right) \geq P\left(\sum_{i=1}^{k-1} X_i > t_1\right), \quad 2 \leq k \leq n$$

modification of PCD in Denuit, Dhaene, Ribas (2001)

(sequent.) negative cumulative dependent (NCD) if "$\leq$"

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Proposition

If X is PCD, then

$$S_n^\perp = \sum_{i=1}^n X_i^\perp \leq_{\mathrm{cx}} S_n \leq_{\mathrm{cx}} S_n^c = \sum_{i=1}^n X_i^c$$

Consequence:

Corollary (positive dependence restriction)

If X is PCD, then

a) $\mathrm{TVaR}_\alpha(S_n^\perp) \leq \mathrm{TVaR}_\alpha(S_n) \leq \mathrm{TVaR}_\alpha(S_n^c)$

b) $\underline{\mathrm{LTVaR}_\alpha(S_n^\perp)} \leq \mathrm{LTVaR}_\alpha(S_n) \leq \mathrm{VaR}_\alpha(S_n) \leq \mathrm{TVaR}_\alpha(S_n^c)$

positive dependence information $\rightarrow$ improved lower bounds for VaR and TVaR.

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Proposition (negative dependence restriction)

If X is NCD, then

a) $S_n \leq_{\mathrm{cx}} S_n^{\perp}$ *and*

b) $\mathrm{VaR}_\alpha(S_n) \leq \mathrm{TVaR}_\alpha(S_n) \leq \mathrm{TVaR}_\alpha(S_n^{\perp})$

negative dependence $\rightarrow$ improved upper risk bounds

Remark

a) *Modification with negative depencence of sums of blocks*

b) *PCD is not directly comparable to POD, POD does not imply convex ordering of sum*

c) *A stronger ordering wcs = weak conditionally ordered in sequence; Rü (2004)*

$$X \leq_{\mathrm{wcs}} Y \Rightarrow \sum_{i=1}^{n} X_i \leq_{\mathrm{cx}} \sum_{i=1}^{n} Y_i$$

This allows to extend to more general upper resp. lower restrictions. In particular $\leq_{\mathrm{WAS}} \;\Rightarrow$ PCD.

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Example

Expected shortfall bounds, $Y \leq_{\text{wcs}} X$
($d/2$ Gamma(2,1/2) risks and $d/2$ Gamma(4,1/2))

$d = 8$	unconstrained			$k = 1$		$k = 2$		$k = 4$		$k = 8$	
	$\underline{\text{ES}}_\alpha$	$\overline{\text{ES}}_\alpha$	DU-S	$\text{ES}_\alpha^{\text{lb}}$	ΔDU-S	$\text{ES}_\alpha^{\text{lb}}$	ΔDU-S	$\text{ES}_\alpha^{\text{lb}}$	ΔDU-S	$\text{ES}_\alpha^{\text{lb}}$	ΔDU-S
$\alpha = 0.990$	12.00	38.27	26.27	38.27	−100%	29.15	−65.3%	23.29	−43.0%	19.56	−28.8%
$\alpha = 0.995$	12.00	41.64	29.64	41.64	−100%	31.15	−64.6%	24.52	−42.2%	20.33	−28.1%
$\alpha = 0.999$	12.00	49.27	37.27	49.27	−100%	35.63	−63.4%	27.21	−40.8%	22.02	−26.9%

positive dependence, improvement of lower bounds

$$\text{DU-S} = \overline{\text{VaR}}_\alpha - \underline{\text{VaR}}_\alpha,$$

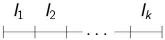
ΔDU-S = reduction of DU-Spread by positive dependence

(Partial) independence structures

Puccetti, Rü, Small, Vanduffel (2014)

Assumption I)

a) independent subgroups $I_1, \ldots, I_k$



b) any dependence within subgroups

$$S = \sum_{i=1}^{k} \sum_{j=1}^{n_i} X_{i,j}\,, \quad Y_i = \sum_{j=1}^{n_i} X_{i,j} \qquad \text{independent}$$

$$S^{c,k} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} F_{i,j}^{-1}(U_i)$$

Theorem

Under independence assumption I)

$$a^I := \mathrm{LTVaR}_\alpha(S^{c,k}) \le \underline{\mathrm{VaR}}_\alpha^I \le \mathrm{VaR}_\alpha \le \overline{\mathrm{VaR}}_\alpha^I$$
$$\le b^I := \mathrm{TVaR}_\alpha(S^{c,k})\,.$$

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Gamma distributed groups:

$d=8$			$k=1$		$k=2$		$k=4$	
	VaR_α^+	$\overline{\mathrm{VaR}}_\alpha$	b^l	e_α	b^l	e_α	b^l	e_α
$\alpha = 0.990$	33.37	38.26	38.27	–	29.15	−23.8%	23.29	−39.1%
$\alpha = 0.995$	36.82	41.63	41.63	–	31.15	−25.2%	24.52	−41.1%
$\alpha = 0.999$	44.59	49.27	49.27	–	35.63	−27.7%	27.21	−44.8%

$d=8$, 4 Gamma(2,1/2), 4 Gamma(4,1/2), $e_\alpha = 1 - \frac{b^l - a^l}{\overline{\mathrm{VaR}}_\alpha - \underline{\mathrm{VaR}}_\alpha}$.

Pareto distributed groups:

$(a^l; b^l)$	$k=1$	$k=2$	$k=5$	$k=10$	$k=25$	$k=50$
$\alpha = 0.95$	(18.23; 153.72)	(20.21; 116.32)	(22.03; 81.54)	(22.95; 63.93)	(23.76; 48.57)	(24.15; 41.09)
$\alpha = 0.99$	(22.24; 297.84)	(23.14; 208.2)	(23.92; 132.28)	(24.28; 95.97)	(24.59; 65.87)	(24.73; 51.98)
$\alpha = 0.995$	(23.17; 388.91)	(23.8; 269.08)	(24.31; 163.37)	(24.55; 115.34)	(24.74; 76.06)	(24.83; 58.25)
$(\underline{\mathrm{VaR}}_\alpha; \overline{\mathrm{VaR}}_\alpha)$						
$\alpha = 0.95$	(18.24; 153.3)					
$\alpha = 0.99$	(22.26; 297.64)					
$\alpha = 0.995$	(23.2; 388)					

Monte Carlo simulation of marginal and independence bounds, Pareto case with $d = 50$, $\theta_i = \theta = 3$ and $c_i = 1$ for $i = 1, \ldots, k$.

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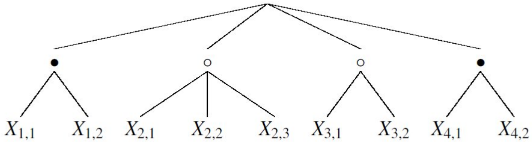
$(\underline{e}^\alpha, \overline{e}^\alpha)$	$k=1$	$k=2$	$k=5$	$k=10$	$k=25$	$k=50$
$\alpha = 0.95$	$(-0.05; -0.27)$	$(10.8; 24.12)$	$(20.78; 46.81)$	$(25.82; 58.3)$	$(30.26; 68.32)$	$(32.4; 73.2)$
$\alpha = 0.99$	$(-0.09; -0.07)$	$(3.95; 30.05)$	$(7.46; 55.56)$	$(9.07; 67.76)$	$(10.47; 77.87)$	$(11.1; 82.54)$
$\alpha = 0.995$	$(-0.13; -0.23)$	$(2.59; 30.65)$	$(4.78; 57.89)$	$(5.82; 70.27)$	$(6.64; 80.4)$	$(7.03; 84.99)$

Monte Carlo simulation of marginal and independence bounds, Pareto case with $d = 50$, $\theta_i = \theta = 3$ and $c_i = 1$ for $i = 1, \dots, k$, $\overline{e}^\alpha = \frac{\overline{\mathrm{VaR}}_\alpha - b^I}{\overline{\mathrm{VaR}}_\alpha}$.

Partial independent substructures:

$$\{1, \dots, n\} = \bigcup_{j=1}^{k} I_j, \ (X_{I_j}) \text{ independent for } j \in H \subset \{1, \dots, k\}$$



Partial independent substructures.

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Theorem (partial independent substructures)

For $\alpha \in (0,1)$ the following VaR *bounds hold:*

$$a^p = a^p(\alpha, H) := \sum_{i \in \{1,\dots,k\} \setminus H} \mathrm{LTVaR}(S_i^c) + \mathrm{LTVaR}\Big(\sum_{i \in H} S_i^c\Big)$$

$$\le \mathrm{VaR}(S_d) \le \sum_{i \in \{1,\dots,k\} \setminus H} \mathrm{TVaR}(S_i^c) + \mathrm{TVaR}\Big(\sum_{i \in H} S_i^c\Big)$$

$$=: b^p(\alpha, H) = b^p.$$

$\sum_{i \in H} S_i^c$ *is an independent sum,*

$$\mathrm{TVaR}(S_i^c) = \sum_{j=1}^{n_i} \mathrm{TVaR}(X_{ij}) \ \ \textit{and} \ \ \mathrm{LTVaR}(S_i^c) = \sum_{j=1}^{n_i} \mathrm{LTVaR}(X_{ij})$$

are simple to calculate.

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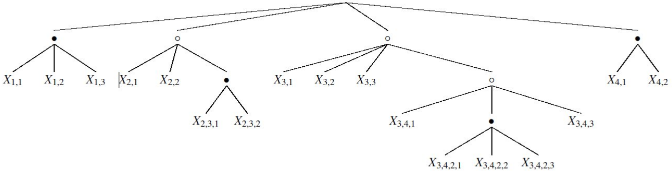
	$\alpha = 0.95$ $F_i \sim \text{Gamma}(\kappa_i^{(1)}, 1)$	$\alpha = 0.995$ $F_i \sim N(\mu_i, 1)$	$\alpha = 0.995$ $F_i \sim N(0, 1)$
$(a^I; b^I)$	$(27.58; 76.02)$	$(149.67; 214.67)$	$(-0.33; 64.66)$
$H = \{2, 3, 4, 5\}$	$(26.83; 90.4)$	$(149.57; 236.76)$	$(-0.44; 86.76)$
$H = \{3, 4, 5\}$	$(25.85; 108.7)$	$(149.47; 257.93)$	$(-0.55; 107.93)$
$H = \{4, 5\}$	$(24.8; 128.81)$	$(149.36; 277.66)$	$(-0.64; 127.66)$
$H = \{5\}$	$(23.75; 148.66)$	$(149.28; 294.6)$	$(-0.73; 144.60)$
$(\underline{\text{VaR}}_\alpha\ ;\ \overline{\text{VaR}}_\alpha\ )$	$(23.76; 148.63)$	$(149.29; 294.59)$	$(-0.71; 144.59)$

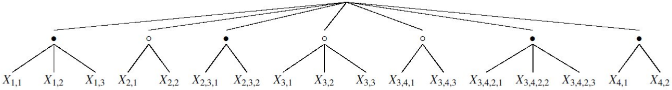
Partial independence bounds with variation of independent substructure, $d = 50$, $k = 5$, $\mu_i = i$.

Remark

a) partial independent graph structures

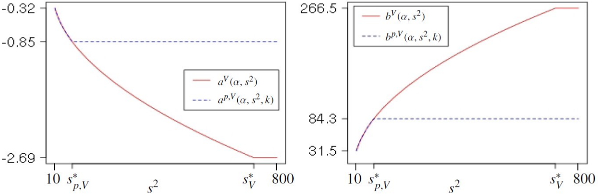


Partial independent graph structures.



Partial independent reduction.

b) combination with variance bounds



(a) $a^{p,V}(\alpha, s^2, k)$ und $a^V(\alpha, s^2)$ as functions of s^2 (b) $b^{p,V}(\alpha, s^2, k)$ and $b^V(\alpha, s^2)$ as functions of s^2

Variance constrained versus independence + variance constrained bounds a^V, $a^{p,V}$ resp. b^V, $b^{p,V}$.

		$d = 10$	$d = 100$
s^*_V	$\alpha = 0.95$	22.39	2239.26
	$\alpha = 0.99$	7.17	717.49
	$\alpha = 0.995$	4.20	420.27

Approximations of critical value s^*_V by Monte Carlo simulation with 10^2 repetitions of 10^5 simulations.

		$d = 100,\ k = 10$				
		$s^2 = 20$	$s^2 = 50$	$s^2 = 100$	$s^2 = 200$	$s^2 = 500$
$(a^{p,V}; b^{p,V})$	$\alpha = 0.95$	$(-1.03; 19.49)$	$(-1.62; 30.82)$	$(-2.29; 43.59)$	$(-3.24; 61.64)$	$(-3.43; 65.23)$
	$\alpha = 0.99$	$(-0.45; 44.5)$	$(-0.71; 70.36)$	$(-0.85; 84.28)$	$(-0.85; 84.28)$	$(-0.86; 84.28)$
	$\alpha = 0.995$	$(-0.32; 63.09)$	$(-0.46; 91.45)$	$(-0.46; 91.45)$	$(-0.45; 91.45)$	$(-0.46; 91.45)$
$(a^V; b^V)$	$\alpha = 0.95$	$(-1.03; 19.49)$	$(-1.62; 30.82)$	$(-2.29; 43.59)$	$(-3.24; 61.64)$	$(-5.13; 97.47)$
	$\alpha = 0.99$	$(-0.45; 44.5)$	$(-0.71; 70.36)$	$(-1.01; 99.5)$	$(-1.42; 140.71)$	$(-2.25; 222.49)$
	$\alpha = 0.995$	$(-0.32; 63.09)$	$(-0.5; 99.75)$	$(-0.71; 141.07)$	$(-1; 199.5)$	$(-1.45; 289.2)$

Approximation of $(a^{p,V}, b^{p,V})$ by Monte Carlo simulation with 10^2 iterations of 10^5 simulations.

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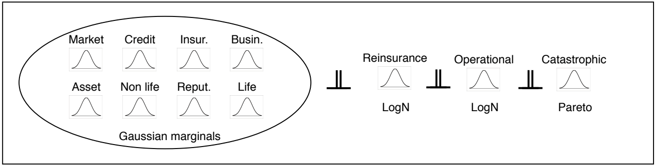
Examples (application to insurance portfolio)

$d = 11,\ k = 4$

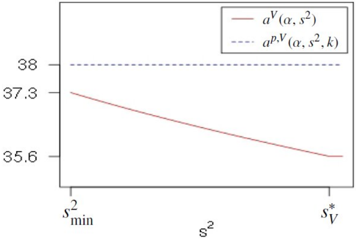


Insurance risk portfolio.

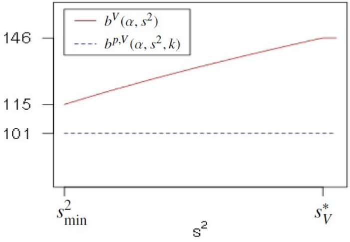
	b^I	VaR_α^+	$\overline{\mathrm{VaR}}_\alpha$	$b^I/\overline{\mathrm{VaR}}_\alpha - 1$
$\alpha = 99\%$	147.34 – 148.46 – 149.66	168.37	209.59	-29.2%
	b^I	VaR_α^+	$\overline{\mathrm{VaR}}_\alpha$	$\Delta\,\mathrm{VaR}_\alpha(L_t^+)$
$\alpha = 99.5\%$	173.37 – 175.18 – 176.96	202.89	249.55	-29.8%
	b^I	VaR_α^+	$\overline{\mathrm{VaR}}_\alpha$	$\Delta\,\mathrm{VaR}_\alpha(L_6^+)$
$\alpha = 99.9\%$	250.41 – 256.04 – 262.47	304.63	367.70	-30.4%

upper bounds b^I, VaR_α^+ = comonotonic VaR and $\overline{\mathrm{VaR}}_\alpha$ for 11-dimensional insurance portfolio

Comparison of independence and variance bounds



(a) $a^{p,V}(\alpha, s^2, k)$ and $a^V(\alpha, s^2)$ as function of s^2



(b) $b^{p,V}(\alpha, s^2, k)$ and $b^V(\alpha, s^2)$ as function of s^2

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Two-sided improved bounds

improved bounds: positive dependence: $G \leq F_X$ or $\overline{F}_X \geq \overline{G}$;
or negative dependence

problem: needs strong positive dependence and d small

two-sided bounds: $\underline{Q} \leq C \leq \overline{Q}$, $\underline{Q}, \overline{Q}$ quasi-copulas

result: **two-sided improved bounds**
based on multiset-inclusion exclusion principle

example:
$$1_{B_1 \cup B_2 \cup B_3} = 1_{B_1} + 1_{B_2} + 1_{B_3} - 1_{B_1 \cap B_2} - 1_{B_2 \cap B_3} - 1_{B_1 \cap B_3} + 1_{B_1 \cap B_2 \cap B_3}$$

needs upper and lower bounds! Bonferoni inequality
parsimonious representation $\rightarrow$ reduction scheme

Lux, Rü (2018) exact duality result, attainment of bounds

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Examples

1. $C^*(u) - \delta \le C \le C^*(u) + \delta$, C^* Gaussian equi-correlated

	$\varrho = -0.1$			$\varrho = 0.4$			$\varrho = 0.8$		
α	i. standard (low : up)	scheme (low : up)	impr. %	i. standard (low : up)	scheme (low : up)	impr. %	i. standard (low : up)	scheme (low : up)	impr. %
0.95	3.4 : 45.0	8.2 : 24.8	60	3.6 : 41.2	7.2 : 28.1	44	7.8 : 31.4	9.2 : 26.2	28
0.99	9.0 : 106.2	15.9 : 56.7	58	9.0 : 105.3	14.9 : 80.8	32	17.4 : 84.9	18.6 : 82.2	6
0.995	13.3 : 153.0	19.0 : 90.0	49	13.3 : 153.0	18.0 : 153.0	3	23.4 : 126.0	22.8 : 125.0	0

Improved standard bounds on VaR of $X_1 + \ldots + X_5$ and VaR estimates via reduction schemes for $\delta = 0.0005$.

2. $C^{\underline{\Sigma}} \le C \le C^{\overline{\Sigma}}$, Gaussian-copula

	$\underline{\varrho} = -0.1$, $\overline{\varrho} = 0.2$			$\underline{\varrho} = 0.3$, $\overline{\varrho} = 0.5$		
α	i. standard (low : up)	scheme (low : up)	impr. %	i. standard (low : up)	scheme (low : up)	impr. %
0.95	3 : 32	8 : 26	38	1 : 30	7 : 29	24
0.99	9 : 74	20 : 52	51	2 : 74	18 : 63	37
0.995	13 : 104	26 : 70	52	3 : 104	25 : 86	40

Improved standard bounds on VaR of $X_1 + \ldots + X_4$ and VaR estimates computed via reduction schemes using $C^{\underline{\Sigma}}$ and $C^{\overline{\Sigma}}$.

3. Subgroup models, $C^{\theta_1} \le C_m \le C^{\theta_2}$ bounds for subgroups copulas by Frank-copulas

	$m = 8$			$m = 4$			$m = 2$		
α	i. standard (low : up)	scheme (low : up)	impr. %	i. standard (low : up)	scheme (low : up)	impr. %	i. standard (low : up)	scheme (low : up)	impr. %
0.95	42 : 113	59 : 86	62	22 : 150	39 : 112	43	12 : 193	28 : 150	33
0.99	82 : 210	108 : 147	70	42 : 264	67 : 175	51	21 : 329	42 : 218	43
0.995	105 : 266	135 : 180	72	53 : 329	83 : 206	55	43 : 403	51 : 252	44

Improved standard bounds and VaR estimates via reduction schemes for $X_1 + \ldots + X_{16}$ given distributions of subgroups.

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D – Partially specified risk factor models

Bernard, Rü, Vanduffel, Wang (2017)

risk vector $X = (X_1, \dots, X_n)$, risk factor Z

$$\textbf{factor model:} \quad X_j = f_j(Z, \varepsilon_j),$$

Z systemic risk factor, ε_j individual risk factors

Assumption: known $H_j \sim (X_j, Z)$, $1 \le j \le n$

but not joint distribution! $\to$ marginals F_j and $Z \sim G$

$H = (H_j), F = (F_j)$, conditional distribution $F_{j|z}$ known

$A(H) = \{(X, Z); (X_j, Z) \sim H_j, \; 1 \le j \le n\}$

partially specified risk factor model

$$\begin{cases} \overline{M}^b(t) = \sup\{P(S \ge t); (X, Z) \in A(H)\} \\ \overline{\mathrm{Var}}_\alpha^b \quad = \sup\{\mathrm{VaR}_\alpha(S); (X, Z) \in A(H)\} \end{cases}$$

similarly $\overline{\mathrm{VaR}}_\alpha^b$, $\overline{\mathrm{TVaR}}_\alpha^b$, $\underline{\mathrm{VaR}}_\alpha^b$, ...

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Proposition (improvement over marginal bounds)

$$\overline{M}^b(t) \le \overline{M}(t) := \sup\{P(S \ge t); X \in A_1(F)\}$$

$$\overline{\mathrm{VaR}}_\alpha^b \le \overline{\mathrm{VaR}}_\alpha, \quad \overline{\mathrm{TVaR}}_\alpha^b \le \overline{\mathrm{TVaR}}_\alpha$$

Let $F_{j|z} = F_{X_j|Z=z}$, $F_z = (F_{j|z})$

$$\overline{M}_z(t) = \sup\left\{P\Big(\textstyle\sum_{j=1}^n X_{j,z} \ge t\Big); (X_{j,z})_j \in A_1(F_z)\right\}$$

similarly $\underline{M}_z(t)$, $\overline{\mathrm{VaR}}_\alpha(S_z), \dots, S_z = \sum X_{j,z}$

Proposition (sharp tail risk bounds)

We have

a) $\overline{M}^b(t) = \int \overline{M}_z(t) dG(z), \quad \underline{M}_b(t) = \int \underline{M}_z(t) dG(z)$

b) $\overline{\mathrm{VaR}}_\alpha^b = (\overline{M}^b)^{-1}(1-\alpha), \quad \underline{\mathrm{VaR}}_\alpha^b = (\underline{M}^b)^{-1}(1-\alpha)$

$$(\overline{M}^b)^{-1}(1-\alpha) = \sup\{t : \overline{M}^b(t) > 1-\alpha\}$$

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Mixture representation:

$X = X_Z$ with $X_z = (X_{j,z}) \in A(F_z)$, $Z \perp (X_{j,z})$.

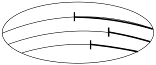
$F_S = \int F_{S_z} dG(z)$

$\alpha \in \Phi$, $b_\alpha := \operatorname{ess\,sup}_{z,G} \mathrm{VaR}_{\alpha(z)}(S_z)$ $\quad \alpha$ defined on range of Z.

Proposition (VaR representation of mixtures)

$$\mathrm{VaR}_\beta(S_Z) = b^* := \inf\left\{ b_\alpha; \alpha \in \Phi, \int \alpha(z) dG(z) \geq \beta \right\}$$

$q_z(\alpha) := \mathrm{VaR}_\alpha(S_z) \uparrow_\alpha$



$Z = z$

$\gamma \in \mathbb{R} : \gamma_z = q_z^{-1}(\gamma) = F_{S_z}(\gamma)$

inverse γ-quantile of $S_z \sim$ probability on $\{Z = z\}$

$$\gamma^*(\beta) := \inf\left\{ \gamma; \int \gamma_z dG(z) \geq \beta \right\},$$

i.e. choose smallest γ such that total probability of test γ_z

$$\int \gamma_z dG(z) \geq \beta.$$

Theorem (worst case VaR in factor model)

a) $\mathrm{VaR}_\beta(S_Z) = \gamma^*(\beta)$

b) $\overline{\mathrm{VaR}}_\beta^b = \overline{\gamma}^*(\beta) = \inf\{\gamma; \int \overline{\gamma}_z dG(z) \geq \beta\}$

$\overline{q}_z(\alpha) = \overline{\mathrm{VaR}}_\alpha(S_z), \overline{\gamma}_z = (\overline{q}_z)^{-1}(\gamma)$

worst case inverse γ-quantile

simplified upper bound:

$t_z(\alpha) = \mathrm{TVaR}_\alpha(S_z^c) = \sum_{j=1}^n \mathrm{TVaR}_\alpha(X_{j,z})$

$\Rightarrow q_z(\beta) \leq t_z(\beta)$

$$\Rightarrow \overline{\gamma}^*(\beta) \leq \gamma_t^*(\beta) = \inf\left\{\gamma; \int t_z^{-1}(\gamma) dG(z) \geq \beta\right\}$$

Worst case for γ_t^* is conditionally comonotonic vector

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Corollary

a) $\overline{\mathrm{VaR}}_\alpha^b = \overline{\gamma}^*(\beta) \le \gamma_t^*(\beta)$

b) $T_z^+ := \mathrm{TVaR}_U(S_z^c)$, $U \sim U(0,1)$, *then*

$$\mathrm{VaR}_\beta(T_Z^+) = \gamma_t^*(\beta)$$

various methods to calculate these bounds

Example (Pareto distributions: p parameter for dependence)

$$\begin{aligned}
X_i^1 &= (1-Z)^{-1/3} - 1 + \varepsilon_i^1 \\
X_i^2 &= I((1-Z)^{-1/3} - 1) + (1-I)(Z^{-1/3} - 1) + \varepsilon_i^2 \\
\varepsilon_i^j &\sim \ \mathrm{Pareto}(\theta_2) \\
\varepsilon_i^1, \varepsilon_i^2 &\sim \ \mathrm{Pareto}(4), \ \ Z \sim U(0,1)
\end{aligned}$$

$$I \sim \mathfrak{B}(1,p), \quad \Delta := 1 - \frac{\mathrm{VaR}_\alpha(T_Z^+) - \mathrm{VaR}_\alpha(T_Z^-)}{\mathrm{TVaR}_\alpha(S^c) - \mathrm{LTVaR}_\alpha(S^c)}$$

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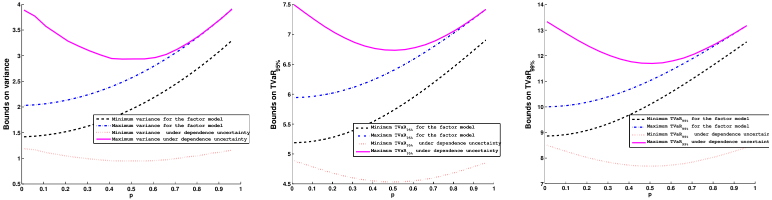


bounds for the variance, TVaR at 95% and TVaR at 99%

p dependence parameter; $p = 0 \sim$ strong negative dependence; $p = 1 \sim$ strong positive dependence

$n = 50$	VaR_α	$\text{TVaR}_\alpha(S^c)$	$\text{VaR}_\alpha\left(T_Z^+\right)$	$\text{LTVaR}_\alpha(S^c)$	$\text{VaR}_\alpha\left(T_Z^-\right)$	Δ
$p = 0.0$	157	378	266	68	149	62%
$p = 0.2$	158	354	267	69	151	59%
$p = 0.4$	164	340	271	70	157	58%
$p = 0.5$	169	338	274	70	161	58%
$p = 0.6$	175	340	278	70	167	59%
$p = 0.8$	189	354	289	69	181	62%
$p = 1.0$	205	378	300	68	198	67%

upper and lower VaR bounds, $\theta_2 = 4$, VaR_α independence

$p \approx 0 \Rightarrow$ strong negative dependence, $p \approx 1 \Rightarrow$ strong positive dependence

Applications and generalizations

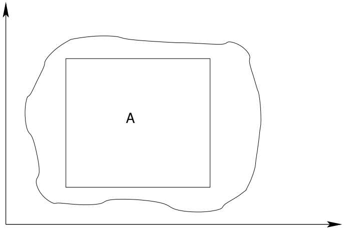
Generalized mixture model: $Z \in D = D_1 + D_2 + D_3$

$$P^X = p_1 P^1 + p_2 P^2 + p_3 P^3, \quad p_i = P(Z \in D_i)$$

$z \in D_1 \Rightarrow P_z^1 = P^1$ fixed distribution

$z \in D_2 \Rightarrow P_z^2 \in \mathcal{F}(F_z)$ risk factor information

$z \in D_3 \Rightarrow P_z^3 \in \mathcal{F}((G_j))$ marginal information



special case:
Bernard, Vanduffel (2014)
central part
$\{Z = 0\} = \{X \in A\} \to P^1$
$\{Z = 1\} = \{X \in A^c\}$
only marginal information

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Consequence:

a) $$\overline{M}(t) = p_1 P^1\Big(\sum_{j=1}^n X_j \geq t\Big) + \int_{D_2} \overline{M}^b_{2,z}(t)dP^Z(z) + p_3\overline{M}_3(t)$$

b) $$S = \sum_{i=1}^n X_i \leq_{\mathrm{cx}} I(Z\in D_1)F_1^{-1}(U) + I(Z\in D_2)S^c_{2,Z} + I(Z\in D_3)S^c_3$$

$S^c_{2,z} = \sum_{j=1}^n F^{-1}_{j|z}(U)$, $S^c_{2,Z} \sim$ conditionally comonotone

Examples (mixture models)

$X_j = f_j(Z,\varepsilon_j)$

Bernoulli mixture model (credit risk)

$$P(X=x \mid Z=z) = \prod_{i=1}^n p_i(z)^{x_i}(1-p_i(z))^{1-x_i}$$

mult. variance mixture model

$X = \mu + \sqrt{W}\varepsilon$, $\varepsilon \sim N(0,\Sigma)$, W stochastic volatility

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B – Risk bounds under moment constraints
C – Positive and negative dependence information
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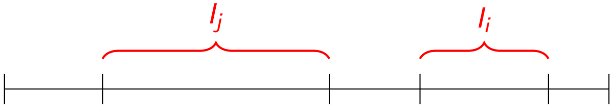
4. Ordering results for risk models

A) Subgroup structure models

subgroup models in: Bignozzi, Puccetti, Rü (2015) and Puccetti, Rü, Small, Vanduffel (2015)

subgroups $\{1, \ldots, d\} = \bigcup_{i=1}^{k} I_i$, risk vector X



BPR: $\exists\, Z \leq X$ positive dependence restriction
(or $X \leq Z$ negative dependence restriction)
$\leq$ positive dependence ordering
(e.g. $\leq_{\mathrm{uo}}, \leq_{\mathrm{c}}, \leq_{\mathrm{wcs}}, \leq_{\mathrm{sm}}, \leq_{\mathrm{dcx}}$)

Z independent subgroups, Z_{I_i} comonotonic

PRSV: $\{X_{I_i}\}$ independent, within subgroups any dependence

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stochastic ordering within subgroups and between subgroups

Rü, Witting (2017)

X risk vector, Z comparison vector, split into subgroups

$Y_i = \sum_{j \in I_i} X_j, \quad W_i = \sum_{j \in I_i} Z_j$ subgroup sums

$Y_i \sim G_i, \quad W_i \sim H_i, \quad Y = (Y_1, \dots, Y_k),\ W = (W_1, \dots, W_k)$

$S = \sum_{i=1}^k Y_i, \quad T = \sum_{i=1}^k W_i$

Ordering within subgroups: $G_i \leq H_i$ (resp. $G_i \geq H_i$)

plus **ordering of copulas:** $C_Y \leq C_W$ (resp. $=$ or $\geq$) **between subgroups**

- leads to wide range of ordering results for risks and risk bounds
- combination with partially specified factor assumptions within subgroups

$\rightarrow$ worst | best cases in submodel classes

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Stochastic Ordering

$X = (X_1, \dots, X_m), \quad Y = (Y_1, \dots, Y_m)$
X **conditional increasing** (CI) if

$$X_i \uparrow_{\text{st}} X_J, \quad \forall\, J \subset \{1, \dots, m\} \setminus \{i\}$$

X **conditional increasing in sequence** (CIS) if

$$X_i \uparrow_{\text{st}} (X_1, \dots, X_{i-1}), \quad \forall\, i \le m$$

$X \le_{\text{wcs}} Y$ **weakly conditional in sequence order** if

$$\text{Cov}(1_{(X_i > x_i)}, f(X_{i+1}, \dots, X_m)) \le \text{Cov}(1_{(Y_i > x_i)}, f(Y_{i+1}, \dots, Y_m))$$

for all $f \uparrow$

X **weakly associated in sequence** (WAS) if $X^\perp \le_{\text{wcs}} X$

$$\Leftrightarrow P^{X_{(i+1)}} \le_{\text{st}} P^{X_{(i+1)} | X_i > x_i}, \quad \forall\, i, \forall\, x_i,$$

$$X_{(i+1)} = (X_{i+1}, \dots, X_m)$$

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Theorem (relations between orderings)

a) $CI \Rightarrow CIS \Rightarrow WA \Rightarrow WAS$

b) $\forall\, i : X_i \overset{d}{=} Y_i$ and $X \leq_{\mathrm{wcs}} Y \Rightarrow X \leq_{\mathrm{sm}} Y$

c) $\forall\, i : X_i \leq_{\mathrm{cx}} Y_i$ and $X \leq_{\mathrm{wcs}} Y \Rightarrow X \leq_{\mathrm{dcx}} Y$

d) *If* $C_X = C_Y$ *is CI and* $X_i \leq_{\mathrm{cx}} Y_i, \forall\, i$ *then* $X \leq_{\mathrm{wcs}} Y$

e) $C_X \leq_{\mathrm{sm}} C_Y$ *and* C_Y *is CI,*
then $X_i \leq_{\mathrm{cx}} Y_i \Rightarrow X \leq_{\mathrm{wcs}} Y$

Remark

c), d) implies: $C_X = C_Y$ *is CI,* $X_i \leq_{\mathrm{cx}} Y_i$
$\Rightarrow X \leq_{\mathrm{dcx}} Y$ *(Müller, Scarsini (2001))*

ordering results $\rightarrow$ *cx* ordering of joint portf. sums

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Elliptical Copulas

$S \sim E_d(\mu, \Sigma, \Phi)$ if $\varphi_X(t) = e^{it^\top \mu}\Phi(t^\top \Sigma t)$

$\Rightarrow X \stackrel{d}{=} \mu + RAU, \quad A^\top A = \Sigma,\ U \sim \text{unif}(S_{d-1})$ and $R \perp U$,

$\Sigma \sim$ correlation matrix of X

$A \in \mathbb{R}^{d\times d}$ ***M*-matrix**, if $a_{ij} \le 0, \forall\, i \neq j$ and principal minors positive.

Proposition (CI-property)

a) $X \sim N(0,\Sigma)$, *then: X is CI* $\Leftrightarrow \Sigma^{-1}$ *is an M-matrix*

b) $X \sim E_d(0,\Sigma,\Phi^R)$, $\Phi^R(t) = \int \Phi(\frac{1}{r^2} t^T \Sigma t) dP^R(r)$,
$\Phi \sim$ *radial part of* $N(0,\Sigma)$
Σ^{-1} *M-matrix* $\Rightarrow X$ *is CI*

normal case, Rü (1981)

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Theorem (Dependence ordering in elliptical models)

$X \sim E_d(\mu_1, \Sigma_1, \Phi)$, $Y \sim E_d(\mu_2, \Sigma_2, \Phi)$

a) $\mu_1 \leq \mu_2$, $\Sigma_1 \leq_{\text{psd}} \Sigma_2 \Rightarrow X \leq_{\text{icx}} Y$

b) $\mu_1 = \mu_2$, $\sigma_{ij}^{(1)} \leq \sigma_{ij}^{(2)}, \forall\, i \neq j$, $\sigma_{ii}^{(1)} = \sigma_{ii}^{(2)}, \forall\, i$, *then* $X \leq_{\text{sm}} Y$

c) $\mu_1 = \mu_2$, $\sigma_{ij}^{(1)} \leq \sigma_{ij}^{(2)}$, $\forall\, i, j$, *then* $X \leq_{\text{dcx}} Y$

a) Pan, Qiu, Hu (2016);

b) Block, Sampson (1988); Müller, Scarsini (2000) normal case;

b), c) Ansari, Rüschendorf (2019); Yin (2019) general case

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Dependence structures within subgroups

$C = C_Y$ copula between subgroups fixed

Proposition

$C = C_Y = C_W$ is WAS (or CIS)

a) *If $Y_i \leq_{\text{cx}} W_i$, $1 \leq i \leq k$, then*

$$S = \sum_{i=1}^{k} Y_i \leq_{\text{cx}} T = \sum_{i=1}^{k} W_i$$

in particular: $\text{LTVaR}_\alpha(T) \leq \text{VaR}_\alpha(S) \leq \text{TVaR}_\alpha(T)$

b) *If $W_i \leq_{\text{cx}} Y_i$. $1 \leq i \leq k$, then*

$$T \leq_{\text{cx}} S \text{ and } \text{TVaR}_\alpha(T) \leq \text{TVaR}_\alpha(S)$$

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Remark

In particular

unknown dependence within subgroups*, then*

$X_{I_i} \leq_{\text{sm}} Z_{I_i} = (F_j^{-1}(U_i))_{j \in I_i}$

$$\Rightarrow \quad Y_i \leq_{\text{cx}} W_i = \sum_{j \in I_i} F_j^{-1}(U_i)$$

If $(U_1, \ldots, U_k) \sim C$ *is CIS,*

then: $X \leq_{\text{sm}} Z$ *and* $S \leq_{\text{cx}} T$

partially specified risk factor models within subgroups

Bernard, Rü, Vanduffel, Wang (2016)

$X_j = f_j(Z_i^f, \varepsilon_j)$, $j \in I_i$, *partially specified risk factor models*

$$\Rightarrow Y_i = \sum_{j \in I_i} X_j \leq_{\text{cx}} W_i = \sum_{j \in I_i} X^c_{j|Z_i^f}$$

conditionally comonotone

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Example

d risks, k independent subgroups I_i
partially specified risk factor models within subgroups

half of $X_j : X_j = (1-U_i)^{-1/3} - 1 + \varepsilon_j$

half of $X_j : X_j = p((1-U_i)^{-1/3} - 1) + (1-p)(U_i^{-1/3} - 1) + \varepsilon_j$

$\varepsilon_j \sim Pareto(4), p \in (0,1)$
$C = C^{\perp}$ independent subgroups copula, $C = C_Y = C_W$

	$p = 0.0$	$p = 0.2$	$p = 0.5$	$p = 0.8$	$p = 1.0$
$(\underline{\mathrm{VaR}}_\alpha, \overline{\mathrm{VaR}}_\alpha)$	(68; 392)	(69; 367)	(70; 349)	(69; 368)	(68; 391)

Sharp VaR bounds with marginal information only $d = 100$, $\alpha = 0.95$.

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Example (cont.)

		$p = 0.0$	$p = 0.2$	$p = 0.5$	$p = 0.8$	$p = 1.0$
$k = 1$	$b(\mathrm{TVaR}_\alpha)$	(68; 474)	(69; 376)	(70; 372)	(69; 384)	(68; 402)
	$b(\mathrm{TVaR}^f_\alpha)$	(72; 297)	(72; 301)	(71; 320)	(69; 351)	(68; 376)
	$b(\mathrm{VaR}^f_\alpha)$	(132; 263)	(134; 265)	(145; 273)	(164; 286)	(182; 296)
$k = 2$	$b(\mathrm{TVaR}_\alpha)$	(72; 385)	(74; 295)	(74; 295)	(74; 301)	(73; 313)
	$b(\mathrm{TVaR}^f_\alpha)$	(76; 231)	(75; 234)	(75; 247)	(74; 269)	(73; 287)
	$b(\mathrm{VaR}^f_\alpha)$	(121; 209)	(122; 210)	(130; 216)	(146; 227)	(158; 237)
$k = 5$	$b(\mathrm{TVaR}_\alpha)$	(77; 305)	(77; 222)	(77; 226)	(77; 229)	(77; 234)
	$b(\mathrm{TVaR}^f_\alpha)$	(79; 173)	(79; 174)	(78; 183)	(77; 197)	(77; 208)
	$b(\mathrm{VaR}^f_\alpha)$	(110; 161)	(110; 162)	(116; 167)	(125; 174)	(133; 180)
$k = 10$	$b(\mathrm{TVaR}_\alpha)$	(79; 266)	(79; 186)	(79; 193)	(79; 193)	(79; 195)
	$b(\mathrm{TVaR}^f_\alpha)$	(80; 144)	(80; 145)	(80; 151)	(79; 161)	(79; 169)
	$b(\mathrm{VaR}^f_\alpha)$	(101; 137)	(102; 138)	(107; 141)	(113; 146)	(119; 151)

VaR bounds with and without factor model information for various group sizes, $d = 100$, $\alpha = 0.95$, $k = 1, 2, 5, 10$.

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Dependence structure between subgroups

$C = C_Y, \quad D = C_W$

Proposition

a) $C \leq_{\text{wcs}} D$ *and* $Y_i \leq_{\text{cx}} W_i$, *then*

$$S = \sum_{i=1}^{k} Y_i \leq_{\text{cx}} T = \sum_{i=1}^{k} W_i$$

in particular:

$$\text{LTVaR}_\alpha(T) \leq \text{Var}_\alpha(S) \leq \text{TVaR}_\alpha(S) \leq \text{TVaR}_\alpha(T)$$

b) $W_i \leq_{\text{cx}} Y_i$, $D \leq_{\text{wcs}} C$, *then*

$$T \leq_{\text{cx}} S \text{ and } \text{TVaR}_\alpha(T) \leq \text{TVaR}_\alpha(S).$$

Similar comparison also in terms of $\leq_{\text{sm}}$, $\leq_{\text{dcx}}$

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Example

General dependence within subgroups

a) unconstrained bounds

$d = 50$

	$(\underline{\mathrm{VaR}}_\alpha; \overline{\mathrm{VaR}}_\alpha)$	$(a; b)$
$\alpha = 0.95$	(18; 153)	(18; 154)
$\alpha = 0.99$	(22; 298)	(22; 298)
$\alpha = 0.995$	(23; 388)	(22; 389)

b) $C \leq_{\mathrm{wcs}} D = C^\perp$ independent subgroups

	$k = 2$	$k = 5$	$k = 10$	$k = 25$
$\alpha = 0.95$	(20; 116)	(22; 82)	(23; 64)	(24; 49)
$\alpha = 0.99$	(23; 209)	(24; 132)	(24; 96)	(25; 66)
$\alpha = 0.995$	(24; 266)	(24; 163)	(25; 115)	(25; 76)

negative dependence between groups

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Example (cont.)

c) Upper bound D : Gauss copula resp. t-copula

			$k=2$	$k=5$	$k=10$	$k=25$	$\overline{\Delta}$
Tab. A:	Corr = 0.1	$\alpha = 0.95$	(20; 119)	(22; 88)	(22; 73)	(23; 71)	58
		$\alpha = 0.99$	(23; 214)	(24; 142)	(24; 116)	(24; 110)	130
		$\alpha = 0.995$	(24; 271)	(24; 174)	(24; 135)	(24; 131)	174
Tab. B:	Corr = 0.25	$\alpha = 0.95$	(20; 124)	(21; 98)	(22; 86)	(22; 78)	58
		$\alpha = 0.99$	(23; 222)	(24; 161)	(24; 134)	(24; 115)	107
		$\alpha = 0.995$	(24; 283)	(24; 197)	(24; 160)	(25; 135)	135
Tab. C:	Corr = 0.5	$\alpha = 0.95$	(19; 132)	(20; 116)	(21; 109)	(21; 105)	27
		$\alpha = 0.99$	(23; 242)	(24; 200)	(23; 183)	(24; 172)	70
		$\alpha = 0.995$	(24; 308)	(24; 248)	(24; 225)	(25; 210)	98
Tab. D:	$\nu = 50$,	$\alpha = 0.95$	(20; 119)	(22; 89)	(22; 74)	(23; 63)	56
	Corr = 0.1	$\alpha = 0.99$	(23; 215)	(24; 146)	(24; 114)	(24; 90)	125
		$\alpha = 0.995$	(24; 274)	(24; 179)	(24; 137)	(25; 105)	169
Tab. E:	$\nu = 50$,	$\alpha = 0.95$	(20; 124)	(21; 99)	(22; 88)	(23; 80)	44
	Corr = 0.25	$\alpha = 0.99$	(23; 224)	(24; 164)	(24; 139)	(24; 122)	102
		$\alpha = 0.995$	(24; 285)	(24; 202)	(24; 168)	(24; 144)	143
Tab. F:	$\nu = 10$,	$\alpha = 0.95$	(20; 125)	(21; 102)	(21; 93)	(23; 87)	38
	Corr = 0.25	$\alpha = 0.99$	(23; 230)	(23; 177)	(24; 157)	(24; 144)	86
		$\alpha = 0.995$	(24; 294)	(24; 223)	(24; 196)	(24; 177)	117

VaR bounds in subgroup model with Gauss copula in A, B, and C and with t-copula in D, E, and F. $\overline{\Delta}$ denotes the difference between upper bounds for $k=2$ and $k=25$.

Example (cont.)

d) Upper bound D : Clayton resp. Gumble copula

			$k=2$	$k=5$	$k=10$	$k=25$	$\overline{\Delta}$
Tab. A:	$\vartheta=1$	$\alpha=0.95$	(20; 122)	(22; 94)	(22; 81)	(23; 71)	51
		$\alpha=0.99$	(23; 216)	(24; 147)	(24; 116)	(24; 92)	124
		$\alpha=0.995$	(24; 274)	(24; 179)	(24; 135)	(25; 103)	171
Tab. B:	$\vartheta=3$	$\alpha=0.95$	(20; 130)	(21; 108)	(21; 98)	(22; 90)	40
		$\alpha=0.99$	(23; 227)	(24; 166)	(24; 138)	(24; 119)	108
		$\alpha=0.995$	(24; 285)	(24; 198)	(24; 160)	(25; 132)	153
Tab C:	$\vartheta=10$	$\alpha=0.95$	(19; 140)	(20; 128)	(20; 122)	(20; 118)	22
		$\alpha=0.99$	(23; 244)	(23; 196)	(23; 176)	(24; 162)	82
		$\alpha=0.995$	(24; 304)	(24; 232)	(24; 202)	(24; 180)	124
Tab. D:	$\vartheta=1.5$	$\alpha=0.95$	(19; 140)	(19; 132)	(20; 129)	(20; 127)	13
		$\alpha=0.99$	(23; 272)	(23; 258)	(23; 254)	(23; 250)	22
		$\alpha=0.995$	(23; 353)	(23; 338)	(23; 329)	(23; 327)	26
Tab. E:	$\vartheta=3$	$\alpha=0.95$	(18; 151)	(18; 150)	(18; 149)	(18; 148)	3
		$\alpha=0.99$	(22; 294)	(22; 290)	(22; 290)	(22; 289)	5
		$\alpha=0.995$	(23; 383)	(23; 379)	(23; 379)	(23; 375)	8

VaR bounds in subgroup model with Clayton copula in A, B, and C and Gumbel copula in D and E.

B) General ordering results for risk models

B1) Elliptical models

sm, dcx ordering in elliptical models
$\Rightarrow$ ordering in risk classes

classes of examples: Ansari, Rü (2019,2020, 2023)

1) **Correlation bounds:** $\mathcal{M}_1 = \{X \in E_d(\mu, \Sigma, \Phi); \Sigma \leq \Sigma^u\}$

   Let $Y \sim E_d(\mu, \Sigma^u, \Phi)$, then

Theorem

If $X \in \mathcal{M}_1$ *then* $X \leq_{\mathrm{dcx}} Y$.

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2) Bounded partial correlations

$$\mathcal{M}_2 = \left\{X \in E_d(0, \Sigma, \Phi); \Sigma \in \mathcal{M}^d_{\mathrm{cor}}, |\sigma_{ij,1:(i-1)}| \le b_i, \forall i < j\right\}$$

partial correlations corresponding to C-vine structure

$\forall(\sigma_{ij,1:(i-1)}) \in [-1,1]^{\frac{d(d-1)}{2}}$. Define for $k = i-1, \dots, 1$

$$\sigma_{ij,1:(k-1)} := \sigma_{ki,1:(k-1)}\sigma_{kj,1:(k-1)} + \sigma_{ij,1:k}\sqrt{1-\sigma^2_{ki,1:(k-1)}}\sqrt{1-\sigma^2_{kj,1:(k-1)}} \qquad (*)$$

and **generalized correlations** $\Sigma = (\sigma_{ij})$ by

$$\sigma_{ii} = 1, \quad \sigma_{ij} = \sigma_{ji} = \sigma_{ij,1:0}, \qquad i < j, \text{ then:}$$

$\Sigma \in \mathcal{M}^d_{\mathrm{cor}}$ and $\forall \Sigma' = (\sigma'_{ij}) \in \mathcal{M}^d_{\mathrm{cor}}$ $\exists(\sigma_{ij,1:i-1})$
such that $(*) \longrightarrow \Sigma'$, i.e. $\sigma_{ij} = \sigma'_{ij}$.
If $Y \in E_d(0, \Sigma, \Phi)$, then $\sigma_{ij;1:(i-1)}$ is partial correlation and identical to correlation of Y_i, $Y_j \mid Y_1, \dots, Y_{i-1}$

Proposition

$\exists 1-1$ correspondence between $\mathcal{M}_{\text{cor}}^{d,+}$ (i.e. positive definite correlation matrices) and generalized partial correlations of a C-vine structure

Define recursively

$$a_{i,i-1} = b_i, \quad i \le d-1$$

$$a_{i,k-1} = a^1_{k,k-1} + a_{i,k}(1 - a^1_{k-k-1}), \quad k \le i-1 \qquad (**)$$

$$a_i := a_{i,0}$$

$$\Sigma^u = (\sigma^u_{ij}), \quad \sigma^u_{ii} = 1, \quad \sigma^u_{ij} = a_{i\wedge j}, \quad i \neq j$$

Theorem

Let $Y \sim E_d(0, \Sigma^u, \Phi)$, then: $Y \in \mathcal{M}_2$ and for all $X \in \mathcal{M}_2$:

$$X \le_{\text{sm}} Y$$

Remark: For partial correlations $\sim$ D-vine recursion in $(*)$ does not lead to correlation matrix.

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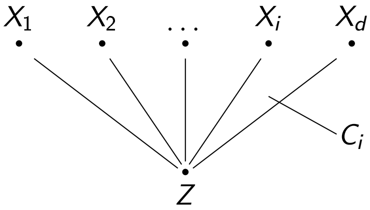
3) **Ordering of worst cases in partially specified risk factor models (PSFM), elliptical specifications**

$\Sigma = \begin{pmatrix} 1 & \varrho \\ \varrho & 1 \end{pmatrix}, \quad E_2(0, \Sigma, \Phi) = E_2(0, \varrho, \Phi)$

$\mathcal{M}_3 = \{X : \exists Z \text{ riskfactor}, (X_i, Z) \sim E_2(0, \varrho_i, \Phi)\}$



$\mathcal{M}_3'$ model with transformed marginals F_i

$M(a, b) = ab + \sqrt{1 - a^2}\sqrt{1 - b^2}, \quad X_{i,z}^c = F^{-1}_{X_i|Z=z}(U)$

X_Z^c cond. comonotone risk vector

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Proposition (Worst case risk model)

1) $X_Z^c \in \mathcal{M}_3$
2) $X \leq_{\text{sm}} X_Z^c$ *for all* $X \in \mathcal{M}_3$
3) $X_Z^c \sim E_d(0, \Sigma, \Phi)$, $\Sigma = (\sigma_{ij})$, $\sigma_{ij} = \begin{cases} 1, & i = j \\ M(\varrho_i, \varrho_j), & i \neq j \end{cases}$

cond. com. is elliptic

Similar result for $\mathcal{M}_3'$

Theorem (Ordering of worst case models)

$(X_i, Z) \sim E_2(0, \varrho_i, \Phi)$, $(Y_i, Z) \sim E_2(0, \varrho_i', \Phi)$

$X_Z^c \leq_{\text{sm}} Y_Z^c \Leftrightarrow M(\varrho_i, \varrho_j) \leq M(\varrho_i', \varrho_j'), \quad \forall i, j$

comparison of cond. com.

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PSFM, partial specifications from elliptical models with different generators and bounds on the correlations, marginals have upper bounds in convex order

4) **Lower and upper bounds on correlations**

Let $M(\varrho_1, \varrho_2) \geq 0$, $b_i > 0$, $Z \sim X_{d+1}$

$$S^{\varrho_1,\varrho_2} = \{\Sigma \in \mathcal{M}_{\text{cor}}^{d+1};\ \sigma_{i,d+1} \leq \varrho_1 < \varrho_2 < \sigma_{j,d+1}, \\ 1 \leq i \leq p < j \leq d\}$$

Φ a given generator

$$\mathcal{M}_4 = \{X : \exists Z, (X, Z) \in E_{d+1}(\mu, \Sigma, \Psi), \Sigma \in S^{\varrho_1,\varrho_2}, \\ \Psi \in \Phi_{\text{rank}(\Sigma)}, R_{2,\Psi} \leq_{\text{st}} R_{2,\Phi}\}$$

elliptical model with bounds on correlations

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PSFM with correlation bounds

Additional flexible marginal classes

For $\eta \in \Phi_2$, satisfying positive dependence condition CI

$\varrho = M(\varrho_1, \varrho_2)$

$$C^{\varrho_1,\eta} = \{C \in \mathcal{C}_2; C \text{ copula of } E_2(0,r,\eta), r \le \varrho_1\}$$
$$D^{\varrho_2,\eta} = \{C \in \mathcal{C}_2; C \text{ copula of } E_2(0,r,\eta), r \ge \varrho_2\}$$

For given $F_i \in \mathcal{F}^1$

$$\mathcal{F}_i = \{F; F \le_{\mathrm{cx}} F_i\}$$

$$\mathcal{M}_5 = \{X : \exists Z, F_{X_i} \in \mathcal{F}_i, C_{X_i,Z} \in C^{\varrho_1,\eta}, C_{X_j,Z} \in D^{\varrho_2,\eta}, \quad 1 \le i \le p < j \le d\}$$

PSF elliptical factor model with bounds on correlations

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Define $\Sigma = (\sigma_{ij})$

$$\sigma_{ij} = \begin{cases} 1 & 1, j \leq p \text{ or } p < i, j \leq d \text{ or } i = j = d+1 \\ M(\varrho_1, \varrho_2) & 1 \leq i \leq p < j \leq d, 1 \leq j \leq p < i \leq d \\ \varrho_1 & 1 \leq i \leq p, j = d+1 \text{ or } 1 \leq j \leq p, i = d+1 \\ \varrho_2 & 1 \leq i \leq d, j = d+1 \text{ or } p < j \leq d, i = d+1 \end{cases}$$

Theorem

1) *For* $(X, Z) \sim E_{d+1}(\mu, \Sigma, \Phi)$ *holds*

$$X \in \mathcal{M}_4 \text{ and } Y \leq_{\mathrm{dcx}} X, \;\; \forall Y \in \mathcal{M}_4$$

2) *For* $(X', Z') \in E_{d+1}(0, \Sigma, \eta)$, η *CI, define* $W = (F_i^{-1}(F_{X_i'}(X_i')))$. *Then it holds:*

$$W \in \mathcal{M}_5 \text{ and } Y \leq_{\mathrm{dcx}} W \text{ for all } Y \in \mathcal{M}_5$$

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B2) General factor models

Ansari, Rü (2020, 2023)

$*$-product of copulas $D^i \in \mathcal{E}_2$, $1 \le i \le d$; $(B_t)_{t\in[0,1]} \subset \mathcal{E}_d$,

$$*_B D^i(u) = \int_0^1 B_t\big(\partial_2 D^1(u_1, t), \ldots, \partial_2 D^d(u_d, t)\big)\, dt$$

continuous case, extension of Durante, Klement (2007) for $d = 2$

- **Sklar Theorem for completely specified factor models**
- **Ordering result w.r.t. conditional copulas**

Proposition

If $(B_t), (C_t)$ *and* $B_t \prec C_t$, $\forall t$

$$\prec \;=\; \le_{\mathrm{lo}}, \le_{\mathrm{uo}}, \le_{\mathrm{sm}}, \le_{\mathrm{dcx}}$$

then $*_B D^i \prec *_C D^i$

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Ordering results w.r.t. specifications

$B = (B_t)$ componentwise convex copula.

Theorem (Ordering for componentwise convex copulas $B = (B_t)$)

If $D^i \leq_{\mathrm{lo}} E^i$, $1 \leq i \leq d$, then

$$\ast_B D^i \leq_{\mathrm{sm}} \ast_B E^i$$

several variants: $\leq_{\mathrm{dcx}}$, $\leq_{\mathrm{lo}}$, ...
particular ordering conditions: Schur ordering, δ ordering,componentwise concave copulas ...

methods: Ky-Fan–Lorentz-Theorem, mass transfer theory, Müller; Meyer and Strulovici

application to: positive, negative dependent copula products; leads to :ordering results in subfamilies of factor models

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5. Conclusion

- Risk bounds with marginal information, portfolio vectors $\sim$ L^2-mass transportation; determine worst case w.r.t. general law invariant convex risk measures
- Risk bounds with marginal information can be calculated, typically (too) wide

  Various reductions by including additional information
- Higher dimensional marginals (reduced bounds)
- Variance constraints, higher order moment constraints good reduction, when constraints are small enough
- partial independence structure (combined with variance information)

– strong reduction of dependence uncertainty ,realistic bounds

- partially specified risk factor models,good reduction
- ordering results $\rightarrow$ worst case models in general classes of factor models

References: risk measures

- L. Rüschendorf, S. Vanduffel, C. Bernard: Model Risk Management: Risk Bounds under Uncertainty. *Cambridge University Press*, 2024. doi:10.1017/9781009367189
- L. Rüschendorf: *Mathematical Risk Analysis: Dependence, Risk Bounds, Optimal Allocations and Portfolios*, 2013
- L. Rüschendorf: Law invariant risk measures for portfolio vectors. *Statistics & Decisions* 24, 97–108, 2006
- L. Rüschendorf: On the distributional transform, Sklar's Theorem, and the empirical copula process. *Journal of Statistical Planning and Inference* 139, 3921–3927, 2009
- L. Rüschendorf: Worst case portfolio vectors and diversification effects. Preprint (2009), *Finance and Stochastics* 16, 155–175, 2012
- G. Puccetti, L. Rüschendorf: Computation of sharp bounds on the distribution of a function of dependent risks. *Journal of Computational and Applied Mathematics* 236, 1833–1840, 2012
- G. Puccetti, L. Rüschendorf: Bounds for joint portfolios of dependent risks. *Statistics & Risk Modeling* 29 (2), 107–132, 2012
- P. Embrechts, G. Puccetti, L. Rüschendorf: Model uncertainty and VaR aggregation. *J. Banking Finance* 37(8), 2750–2764, 2013

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References: risk bounds

- P. Embrechts, G. Puccetti, L. Rüschendorf: Model uncertainty and VaR aggregation. *J. Banking Finance* 37(8), 2750–2765, 2013
- P. Embrechts, B. Wang, R. Wang: Aggregation-robustness and model uncertainty of regulatory risk measures. *Finance Stoch.* 19(4), 763–790, 2015
- G. Puccetti, L. Rüschendorf: Asymptotic equivalence of conservative VaR- and ES-based capital charges. *J. Risk* 16(3), 3–22, 2014
- G. Puccetti, B. Wang, R. Wang: Complete mixability and asymptotic equivalence of worst-possible VaR and ES estimates. *Insurance Math. Econom.* 53(3), 821–828, 2013
- R. B. Nelsen, B. Quesada-Molina, J. A. Rodríguez-Lallena, M. Úbeda-Flores: Bounds on bivariate distribution functions with given margins and measures of association. *Commun. Stat., Theory Methods* 30(6), 1155–1162, 2001
- R. B. Nelsen, B. Quesada-Molina, J. A. Rodríguez-Lallena, M. Úbeda-Flores: Best-possible bounds on sets of bivariate distribution functions. *J. Multivariate Anal.* 90(2), 348–358, 2004

References (cont.)

- C. Bernard, L. Rüschendorf, S. Vanduffel: Value-at-Risk bounds with variance constraints. *Journal of Risk and Insurance* 2015; doi:10.1111/jori.12108
- G. Puccetti, L. Rüschendorf: Bounds for joint portfolios of dependent risks. *Statistics & Risk Modeling* 29(2), 107–132, 2012
- V. Bignozzi, G. Puccetti, L. Rüschendorf: Reducing model risk via positive dependence assumptions. *Insurance Math. Econ.* 61(1), 17–26, 2015
- C. Bernard, L. Rüschendorf, S. Vanduffel, R. Wang: Risk bounds for factor models. *Finance Stoch.* 3, 631–659, 2017
- G. Puccetti, D. Small, L. Rüschendorf, S. Vanduffel: Reduction of Value-at-Risk bounds by independence and variance information. *Scandinavian Actuarial Journal* 2017(3), 245–266, 2017
- T. Lux, A. Papapantoleon: Improved Fréchet–Hoeffding bounds for d-copulas and applications in model-free finance. *Ann. Appl. Prob.* 27 (6), 3633–3671, 2017

References (cont.)

- C. Bernard, L. Rüschendorf, S. Vanduffel, J. Yao: How Robust is the Value-at-Risk of Credit Risk Portfolios? *The European Journal of Finance* 23(6), 507–534, 2017; doi:10.1080/1351847X.2015.1104370
- G. Puccetti, L. Rüschendorf, D. Manko: VaR bounds for joint portfolios with dependence constraints. *Dependence Modeling* 4, 368–381, 2016
- T. Lux, L. Rüschendorf: VaR bounds with two sided dependence information. *Mathematical Finance* 29, 967–1000, 2018
- L. Rüschendorf, J. Witting: VaR bounds in models with partial dependence information on subgroups. *Dependence Modeling* 5, 59–74, 2017
- J. Ansari, L. Rüschendorf: Ordering results in classes of elliptical distributions with applications to risk bounds. *Journal Mult. Analysis* 182, 2021
- J. Ansari, L. Rüschendorf: General ordering results for factor models. *JMVA* 2022

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