On the Construction of Optimal Payoffs

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Abstract

In the framework of continuous-time market models with specified pricing density, optimal payoffs under increasing law-invariant preferences are known to be anti-monotonic with the pricing density. Consequently, optimal portfolio selection problems can be reformulated as optimization problems on real functions under monotonicity conditions. We solve two basic types of these optimization problems, which makes it possible to obtain in a fairly unified way the optimal payoff for several portfolio selection problems of interest. In particular, we completely solve the optimal portfolio selection problem for an investor with preferences as in *cumulative prospect theory* or as in *Yaari's dual theory.*

Extending previous work we also characterize optimal payoffs when the payoff is required to have a fixed copula with some benchmark (state-dependent constraint). Specifically, we show that if one can determine the optimal payoff under a concave law-invariant objective, then one can also determine the optimal payoff when adding the state-dependent constraint.

In the final part of the paper, we consider an extension to (incomplete) market models in which the pricing density is not completely specified. When a sufficient number of payoffs have a known market price, we show that optimal payoffs are anti-monotonic to some pricing density that we explicitly derive from these market prices. As examples we deal with some exponential Lévy market models and some market models involving Itô processes.

Key words: State-dependent preferences, Yaari's dual theory of choice, Incomplete market models, Optimization under monotonicity constraints, Hoeffding–Fréchet bounds,

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1 Introduction

Initiated by the seminal mean-variance framework of Markowitz (1952), the study of optimal payoffs is a central theme in mathematical finance. Whilst Markowitz' approach remains influential in practical applications and in academic research, other approaches to portfolio selection problems have been developed as well. Notable approaches include portfolio selection using the expected utility theory of von Neumann and Morgenstern (1947), the target probability maximization approach of Spivak and Cvitanić (1999) and Browne (1999), the dual theory of Yaari (1987), the rank dependent utility approach of Quiggin (1993), the cost minimizing distributional analysis of Dybvig (1988b) and the behavioral/psychological approaches including the SP/A theory of Lopes (1987) and Shefrin and Statman (2000), and the cumulative prospect theory of Tversky and Kahneman (1992).

In the framework of continuous-time market models with *specified pricing density*[1] and using various methods, such as the stochastic control method (Merton (1969, 1971)), the Lagrange technique (Korn (1997), Xu (2016)) and most noticeably the martingale method using convex duality (Pliska (1986), Karatzas et al. (1987), Cox and Huang (1989), He and Pearson (1991), Cvitanić and Karatzas (1992), Cvitanić and Karatzas (1993), Karatzas et al. (1991), Broadie et al. (1998), Kramkov and Schachermayer (1999)), one can find in the literature solutions to several of these portfolio selection problems. However, the optimal payoff under Yaari preferences is only known under a strong condition and also the optimal portfolio under preferences as in cumulative prospect theory is missing in the literature.

The main issue is that the above mentioned techniques are not well suited to deal with non-concave utility functions and/or distorted probabilities (appearing for instance in rank dependent utility theory, cumulative prospect theory and in Yaari's dual theory). We deal with this issue using a two-step approach. First, we reformulate the portfolio selection problem using a "quantile formulation"[2], which implies that the optimal payoff is anti-monotonic with the pricing density and thus that optimal portfolio selection problems can be reduced to optimization problems on real functions

[1]Note that in the mathematical finance literature one often narrows the scope of these market models to so-called complete market models in which the pricing density is unique, and all payoffs are attainable by a self-financing strategy. However, all results on optimal payoffs that are available in the literature for complete market models are also valid in incomplete markets, under the assumption that all participants agree on using a specific pricing density. In this case, however, optimal payoffs are no longer guaranteed to be attainable.

[2]The quantile formulation of the optimal portfolio selection and related optimization problems has a long history. It is intimately connected with the cost minimizing distributional analysis of Dybvig (1988b) and with Hoeffding–Fréchet bounds. For portfolio optimization, it has, for instance, been used in Föllmer and Schied (2004), Carlier and Dana (2006), Burgert and Rüschendorf (2006), Bernard et al. (2014a,b, 2015b), Bernard and Vanduffel (2014), and Kassberger and Liebmann (2012); see also Bernard et al. (2015a) and Bernard et al. (2019) for applications to the ranking of portfolios and explaining their demand. This technique has recently also been put forward and studied in a series of papers, including He and Zhou (2011), Zhang et al. (2011), Jin and Zhou (2008), Xu and Zhou (2013), and Xu (2016); see also the survey paper of Zhou (2011) and the references herein. With respect to risk optimization, it traces its pedigree back to the early eighties (see Rüschendorf (1983)).

under monotonicity restrictions. Next, we determine solutions for two types of such optimization problems.

The first optimization problem and its solution allows dealing with concave utility functions and distorted probabilities. Its proof is based on results on isotonic regression due to Barlow et al. (1972). As a result, we are able to derive in a straightforward way the optimal payoff under rank dependent utility preferences, and, moreover, under cumulative prospect preferences without requiring the strong assumption as in Jin and Zhou (2008).

The second optimization problem makes it possible to deal with linear utility and distorted probabilities. Here, we use a dual approach in that we essentially cast the problem as a testing problem under monotonicity constraints. For its solution, we use arguments like in the proof of the Neyman-Pearson lemma combined with results on isotonic regression (Theorem 2.4). As a consequence, we completely solve the optimal portfolio selection problem under Yaari's dual theory of choice. This problem has also been solved in He and Zhou (2011) using convex analysis, but their solution requires a strong additional condition that lacks an economic foundation. In contrast, our approach is based on the solution of a fairly general optimization problem on real functions and provides a general solution.[3]

Most studies on optimal portfolio choice center on the maximization of a law-invariant objective (i.e., the investor's objective depends solely on the distributional properties of the payoff), but there have also been extensions to consider state-dependent criteria; see for instance Boyle and Tian (2007), Korn and Lindberg (2014), Björk et al. (2014), Dong and Sircar (2014), and Bernard et al. (2015b). The basic reason for doing so is that under law-invariant preferences optimal payoffs will have their lowest outcomes when the economy is in a downturn (Dybvig (1988a)), and arguably this feature does not fit with the aspirations of many investors. We model state-dependence using dependence constraints, i.e., we prescribe the copula between the payoff and some benchmark asset (Bernard et al. (2014b), Bernard et al. (2015b)). We provide a reduction result (a quantile formulation) that allows dealing with optimal portfolio selection problems for investors with concave preferences and who have an additional state-dependent constraint. Specifically, Bernard et al. (2015b) derive the optimal payoff for an expected utility maximizer under dependence constraints and we generalize their result to other classic theories of decision-making including the rank dependent expected utility theory and Yaari's dual theory of choice.

Incomplete market models with non-specified pricing densities lead to more challenging portfolio optimization problems. In addition to the question of attainability of the optimal payoff, it is also no longer clear to which pricing density - if any - the optimal payoff needs to be anti-monotonic. In this regard, the martingale method with

[3] Xu (2016) develops a Lagrangian relaxation technique that allows derivation of an optimal payoff under rank dependent utility preferences. It is claimed that this approach also allows to derive the optimal payoff under cumulative prospect theory preferences or under preferences as in Yaari's dual theory. There is however no proof and moreover the stated approach requires inversion of a concave utility, which is at odds with the use of linear utility in Yaari's dual theory and the use of S-shaped utility functions in cumulative prospect theory.

convex duality has turned out to be successful albeit almost always in the realm of expected utility theory, and it is for instance not clear how to deal with optimal portfolio selection under Yaari preferences. As outlined before, in market models with a specified pricing density, we solve the optimal portfolio selection problem under Yaari preferences by a dual approach, i.e., we reformulate the optimization problem as a testing problem under monotonicity constraints. Therefore, it seems natural to extend such dual approach to incomplete markets with non-specified pricing density. In fact, such dual approach has turned out to be successful for dealing with superhedging and quantile hedging in general incomplete markets, as shown in Föllmer and Leukert (1999), Föllmer and Leukert (2000), Rudloff (2007) and Rudloff and Karatzas (2010). The optimization problem can then be cast as a static problem, which in essence amounts to a testing problem of composite hypotheses, and a representation problem for the reduced claim $\tilde{\varphi}H$ in which $\tilde{\varphi}$ is the optimal test and H is the claim to be hedged. However, for obtaining solutions to the optimal portfolio selection problem under Yaari preferences, it is necessary that the hypothesis of the testing problem is convex and possibly closed, but this condition is not fulfilled due to the inherent monotonicity constraints, implying that the dual approach is not readily extendable to this case.

In order to deal with incomplete markets with non-specified pricing densities, we assume that a sufficient number of payoffs are available in that from their market prices a pricing density can be derived, which can be used for the construction of optimal portfolios. Specifically, we prove that similar to the market setting with fully specified pricing density a reduction principle (quantile formulation) applies for dealing with optimal portfolio choice, i.e., the optimal portfolio is shown to be anti-monotonic to a pricing density that we explicitly derive from the available market prices. As a consequence, we show that solutions to optimal portfolio selection problems derived in the setting with specified pricing measure carry over to the incomplete setting that we consider, and moreover that attainability is ensured.

2 Optimal payoffs under law-invariant preferences in markets with specified pricing density

We study optimal investment strategies for investors with a given finite investment horizon T and no intermediate consumption. To this end, we consider a continuous-time set-up of an arbitrage-free and frictionless financial market given by a market model $S = (S_t)_{0\leqslant t\leqslant T}$ in a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{0\leqslant t\leqslant T}, P)$. We assume that this set-up is rich enough to allow one to construct for every $\mathcal{A}_T$-measurable random variable X_T (i.e., a payoff) a variable V on $(\Omega, \mathcal{A})$ that is independent[4] of X_T and uniformly distributed on $(0, 1)$. We assume that all prices in the market are determined by a a state-price density proces, $\varphi = (\varphi_t)_{0\leqslant t\leqslant T}$, which is adapted to the filtration. This assumption is statisfied in the case of complete market models, but one may also consider incomplete market models in which an empirical pricing measure

[4]This is always possible up to a suitable enlargement of the probability space $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{0\leqslant t\leqslant T}, P)$

arises from a class of pricing measures after suitable calibration to market data. Note that, unlike in the case of a complete market model, it is then no longer guaranteed that optimal payoffs can be attained by a self-financing strategy. In the final part of the paper, we extend our results to incomplete market settings with partially specified pricing densities, without giving up on attainability.

The investor aims at maximizing a law-invariant objective $\Psi(X_T)$ under a budget constraint, $E\varphi_T X_T = x_0$. Formally, we consider the general utility optimization problem:

$$\text{(U)} \qquad \begin{cases} \Psi(X_T) = \max \\ E\varphi_T X_T = x_0 \\ F_{X_T} \in \mathcal{F} \end{cases} \tag{2.1}$$

in which Ψ is a law-invariant, increasing[5] functional on $L^0(\Omega, P)$, F_{X_T} denotes the distribution function of X_T, and $\mathcal{F}$ is a set of admissible distribution functions. If $F_{X_T} = F$, then we define $\Psi(F) := \Psi(X_T)$.

2.1 The quantile formulation

For the construction of optimal payoffs a useful tool is the notion of distributional transform. Given a payoff X_T with distribution F, its distributional transform is defined as

$$U := F(X_T, V), \tag{2.2}$$

where $F(x, v) = P(X_T < x) + vP(X_T = x)$ (see Rüschendorf (1981)). Then

$$U \sim U(0,1) \quad \text{and} \quad X_T = F^{-1}(U) \text{ a.s.}, \tag{2.3}$$

in which the function F^{-1} denotes the quantile function of F (defined as its left inverse). By slight abuse of notation, the distributional transform (2.2) will be further denoted as $U = \tau_F(X_T) = \tau_{X_T}$.

The following well-known result states that a solution to the optimization problem (U) can be found in the set of payoffs X_T that are anti-monotonic[6] to the pricing density φ_T.

Theorem 2.1 (Anti-monotonicity of optimal solutions)**.** *If problem* (U) *has a solution, then there is a solution $\widehat{X}_T$ that is a.s. anti-monotonic with φ_T.*

The anti-monotonicity of a solution $\widehat{X}_T$ with φ_T is a consequence of Hoeffding–Fréchet bounds. For this and related results see Föllmer and Schied (2004), Burgert and Rüschendorf (2006, Proposition 2.4), Carlier and Dana (2006). A recent version

[5] We call a functional Ψ on $L^0(\Omega, P)$ (weakly) increasing if $\Psi(X + a) \geqslant \Psi(X)$ for all $a \geqslant 0$ and $X \in L^0(\Omega, P)$.

[6] Recall that two random variables X and Y are called anti-monotonic if $X(\omega_1) < Y(\omega_2)$ implies $Y(\omega_2) \leqslant X(\omega_1)$. Anti-montonicity between two real functions is defined in a similar way.

of this result is given in Xu (2014, Theorem 7) and Xu (2016). Note that if Ψ is strictly increasing then any solution must be anti-monotonic with φ_T.

The following theorem states that portfolio optimization problems can be reduced to optimization problems on real functions under monotonicity conditions.

Theorem 2.2 (Quantile formulation)**.** *If $\widehat{F}$ is a solution to the restricted utility optimization problem*

$$(U^r) \qquad \begin{cases} \Psi(F) = \max \\ \int_0^1 F_{\varphi_T}^{-1}(1-t)F^{-1}(t)dt = x_0 \\ F \in \mathcal{F} \end{cases} \tag{2.4}$$

and $\widehat{k} = \widehat{F}^{-1}$ denotes its quantile function, then

$$\widehat{X}_T = \widehat{k}(1 - \tau_{\varphi_T}) \tag{2.5}$$

solves problem (U).

Proof. By Theorem 2.1, a solution $\tilde{X}_T$ to problem (U) is anti-monotonic with φ_T. Let $\widehat{F}$ denote its distribution function and $\widehat{k} = \widehat{F}^{-1}$ its quantile function. Defining $\widehat{X}_T = \widehat{k}(1 - \tau_{\varphi_T})$ we obtain that $\widehat{X}_T$ has distribution function $\widehat{F}$ and $\widehat{X}_T$ is also anti-monotonic with φ_T. This implies that $(\tilde{X}_T, \varphi_T)$ and $(\widehat{X}_T, \varphi_T)$ have the same joint distribution. Therefore, $\widehat{X}_T$ also satisfies $E\widehat{X}_T\varphi_T = E\tilde{X}_T\varphi_T = x_0$. Furthermore, by construction $\Psi(\widehat{X}_T) = \Psi(\tilde{X}_T)$ and thus $\widehat{X}_T$ is also a solution to problem (U). We remark that similar arguments can also be found in Proposition 2.4 of Burgert and Rüschendorf (2006). □

In Section 2.2, we solve two particular restricted (utility) optimization problem that are of the form (U^r). In Section 3 and Section 4 we then show that these optimization results can be applied to yield in a unified manner the solution of various very well-known portfolio selections problems that are of the form (U). Specifically, the optimal portfolios for investors with preferences as in expected utility theory, Yaari's dual theory, rank dependent expected utility theory, or as in cumulative prospect theory all follow in straight-forward way as applications.

2.2 Two optimization problems under a monotonicity condition

Denote by $(\ ,\ ){=}(\ ,\ )_\mu$ the scalar product on $L^2 := L^2([0,1])$ w.r.t. a measure μ on $[0,1]$. For an element $g \in L^2 := L^2([0,1],\mu)$, we denote by g^* the L^2-projection on $M_\uparrow$, the subset of increasing functions of L^2 and by g_* the projection on $M_\downarrow$, the subset of decreasing functions. By results on isotonic regression of Barlow et al. (1972) (Theorems 1.4 and 1.7) the following characterizes g^* and g_*, respectively:

$$\begin{aligned} (g,h) &\leqslant (g^*,h), \qquad \forall h \in M_\uparrow, \\ (g,g^*) &= (g^*,g^*). \end{aligned} \tag{2.6}$$

$$
\begin{aligned}
(g,h) &\geqslant (g_*,h), \qquad \forall h \in M_\uparrow,\\
(g,g_*) &= (g_*,g_*).
\end{aligned} \tag{2.7}
$$

Moreover, for all τ with $\tau \circ g^* \in L^2$, it holds that

$$
(g,\tau(g^*)) = (g^*,\tau(g^*)), \tag{2.8}
$$

and similarly, for all τ with $\tau \circ g_* \in L^2$, we have that

$$
(g,\tau(g_*)) = (g_*,\tau(g_*)). \tag{2.9}
$$

Furthermore, if $a \leqslant g \leqslant b$, then

$$
a \leqslant g^* \leqslant b \quad \text{and} \quad a \leqslant g_* \leqslant b. \tag{2.10}
$$

Hereafter, we solve two optimization problems for real functions under monotonicity restrictions, which will provide a basic tool for the solution of several portfolio optimization problems in the following sections.

First, we consider for a given (utility) function u and given function g the following optimization problem on $L^2([0,1])$:

$$
\text{(M1)} \qquad \begin{cases} (u\circ k,1) = \max \\ (k,g) = x_0 \\ k \uparrow . \end{cases} \tag{2.11}
$$

The next theorem provides a solution.

Theorem 2.3. *Let $g \in L^2$ and let u be a concave differentiable function. Assume that*

$$
\widehat{k} := (u')^{-1}(\lambda g_*), \tag{2.12}
$$

with λ chosen such that $(\widehat{k},g) = x_0$ is well-defined. If $u\circ\widehat{k}$ is integrable, then $\widehat{k}$ solves the optimization problem (M1).

Proof. Since g_* is $\downarrow$ and $(u')^{-1}$ is $\downarrow$ we obtain that $\widehat{k}$ is $\uparrow$. Furthermore, for any $k\uparrow$, by the concavity of u,

$$
\begin{aligned}
u(k(t)) - u(\widehat{k}(t)) &\leqslant u'(\widehat{k}(t))(k(t)-\widehat{k}(t))\\
&= \lambda g_*(t)(k(t)-\widehat{k}(t)).
\end{aligned}
$$

Thus we obtain that

$$
\begin{aligned}
A &:= (u\circ k,1) - (u\circ\widehat{k},1)\\
&\leqslant \lambda(g_*, k-\widehat{k}).
\end{aligned}
$$

Note that $\widehat{k}(t)$ writes as $\widehat{k}(t) = h(g_*(t))$ in which $h(t) = (u')^{-1}(\lambda t)$ is a decreasing function. Hence, $\widehat{k}(t)$ is increasing, as it is the composition of two decreasing functions.

From the projection equation (2.9) we then obtain that $(g_*, h(g_*)) = (g, h(g_*)) = (g, \widehat{k}) = x_0 = (g, k)$ for any k as in (2.11). As a result we find that

$$\begin{aligned}(g_*, k - h(g_*)) &= (g_*, -h(g_*)) + (g_*, k)\\ &= (g, -k) + (g_*, k)\\ &= (g_* - g, k)\\ &\leqslant 0.\end{aligned}$$

The last inequality follows from the inequality in (2.7), as $k \in M_\uparrow$. As a result, we obtain that $A \leqslant 0$, i.e.,

$$(u \circ k, 1) \leqslant (u \circ \widehat{k}, 1). \qquad \square$$

The previous theorem does not allow to deal with u being linear. This case, however, is of great interest when dealing with optimal payoffs under Yaari's theory. Therefore, we formulate the following optimization problem:

$$\text{(M2)} \qquad \begin{cases}(f, 1) = \max\\ (f, g) = x_0\\ f \uparrow, \; a \leqslant f \leqslant b\end{cases} \tag{2.13}$$

We provide a solution in the following theorem.

Theorem 2.4. *Let $g \in L^2$. If the problem (M2) is feasible, then a solution is given as*

$$f_{\gamma,\delta} = \begin{cases} a & \\ \gamma, & \dfrac{1}{g_*} \begin{matrix}<\\=\\>\end{matrix} \delta,\\ b & \end{cases} \tag{2.14}$$

where $\gamma \in [a, b]$ and $\delta \in [\frac{1}{b}, \frac{1}{a}]$ are chosen such that $(f_{\gamma,\delta}, g) = x_0$.

Proof. In a first step, we consider that $g \in M_\downarrow$. Let $h \in M_\uparrow$ satisfy $(h, g) = x_0$, $a \leqslant h \leqslant b$. Then

$$\begin{aligned}(h, 1) - (f_{\gamma,\delta}, 1) &= (h - f_{\gamma,\delta}, 1)\\ &= (h - f_{\gamma,\delta}, 1 - \delta g) + \delta(h - f_{\gamma,\delta}, g)\\ &= (h - f_{\gamma,\delta}, 1 - \delta g) \leqslant 0. \end{aligned} \tag{2.15}$$

For the proof of the inequality, observe that if $h > f_{\gamma,\delta} \geqslant a$, then $1 \leqslant \delta g$ and if $h < f_{\gamma,\delta} \leqslant b$, then $1 \geqslant \delta g$. Furthermore, by assumption, the side condition is nonempty, i.e., there are feasible solutions. This implies the existence of a pair γ, δ with $(f_{\gamma,\delta}, g) = x_0$. This result is proved in a similar way as in the proof of the existence of Neyman–Pearson tests in statistics; see for instance Theorem 1, pp. 73-74 in Lehmann

(2005). Finally, since $g \in M_{\downarrow}$, $f_{\gamma,\delta} \in M_{\uparrow}$. We conclude that $f_{\gamma,\delta}$ as in (2.14) is a solution of Problem (M2).

In a second step we consider the case of a general function g. Let g_* denote its projection on $M_{\downarrow}$. Let $h \in M^{\uparrow}$ satisfy $(h, g) = x_0$, $a \leqslant h \leqslant b$, then

$$(h, 1) - (f_{\gamma,\delta}, 1) = (h - f_{\gamma,\delta}, 1) = (h - f_{\gamma,\delta}, 1 - \delta g) =: d.$$

By (2.9) we have $(f_{\gamma,\delta}, g) = (f_{\gamma,\delta}, g_*)$, since $f_{\gamma,\delta}$ is a decreasing function of g_*. Furthermore, for $h \in M_{\uparrow}$ it holds by (2.7) that

$$(h, g) \geqslant (h, g_*).$$

This implies, that

$$d \leqslant (h - f_{\gamma,\delta}, 1 - \delta g_*).$$

Now, we obtain as in the proof of the first step that $d \leqslant 0$. The existence part is obtained in a similar way as in the first step and the result follows. □

3 Optimal payoff under (rank dependent) expected utility theory and cumulative prospect theory

In this section we show that solutions to several seminal portfolio optimization problems can be obtained in a unified manner from the optimization result presented in Theorem 2.3, i.e., application of this theorem leads in a direct way to the optimal payoff for an investor with preferences as in the expected utility theory (Merton (1971),), the rank dependent expected utility theory (Xia and Zhou (2012), Xu (2016)) and the cumulative prospect theory.

3.1 Expected utility theory (EUT)

The most prominent decision theory used in economics is the expected utility theory (EUT) of von Neumann and Morgenstern (1947). Cox and Huang (1989) (see also Merton (1971)) and He and Pearson (1991) consider the following expected utility maximization problem

$$\max_{E\varphi_T X_T = x_0} Eu(X_T), \tag{3.1}$$

in which u is increasing. Problem (3.1) is of the form (U) and by Theorem 2.1, if there exists an optimal solution $\widehat{X}_T$ it can be considered anti-monotonic to the pricing density φ_T. Its quantile function $\widehat{k}$ is thus the solution to the reduced problem

$$\begin{cases} \displaystyle\int_0^1 u(F^{-1}(t))dt = \max \\ \displaystyle\int_0^1 F_{\varphi_T}^{-1}(1-t)F^{-1}(t)dt = x_0 \\ F^{-1} \uparrow . \end{cases} \tag{3.2}$$

Clearly, Problem (3.2) is of the form (M1) with $\mu = \lambda^1$ the Lebesgue measure on $[0,1]$. Under the conditions of Theorem 2.3, we therefore obtain as solution to (3.2)

$$\widehat{k}(t) = (u')^{-1}(\lambda F_{\varphi_T}^{-1}(1-t)), \tag{3.3}$$

where λ is chosen such that $\int_0^1 F_{\varphi_T}^{-1}(1-t)\widehat{k}(t)dt = x_0$. An optimal investment $\widehat{X}_T$ of problem (3.1) is then obtained by

$$\widehat{X}_T = \widehat{k}(1-\tau_{\varphi_T}),$$

where $\tau_{\varphi_T} = \tau_{F_{\varphi_T}}(\varphi_T)$ is the distributional transform of φ_T.

3.2 Rank dependent expected utility theory (RDUT)

The rank dependent expected utility theory (RDUT), introduced in Quiggin (1993), is one of the alternative choice theories that aim at resolving a number of paradoxes associated with expected utility theory. Under the EUT paradigm the payoff X_T is evaluated using the objective function

$$Eu(X_T) = \int u(x)dF_{X_T}(x). \tag{3.4}$$

Under the RDUT paradigm we evaluate payoffs X_T in a similar way, but we distort the distribution F_{X_T} using a distortion function w, which is an increasing, differentiable weight function with $w(0) = 0$, $w(1) = 1$.[7] Hence, we consider the problem

$$\max_{E\varphi_T X_T = x_0} \int u(x)d(1 - w(1 - F_{X_T}(x))), \tag{3.5}$$

in which u is assumed to be increasing. By Theorem 2.1, if there exists an optimal solution $\widehat{X}_T$, it can be taken as anti-monotonic to the pricing density φ_T. Therefore, we find its quantile function as solution to the reduced problem

$$\begin{cases} \displaystyle\int u(x)d(1 - w(1 - F(x))) = \max \\ \displaystyle\int_0^1 F_{\varphi_T}^{-1}(1-t)F^{-1}(t)dt = x_0 \\ F^{-1} \uparrow . \end{cases} \tag{3.6}$$

We aim at writing (3.6) in the form (M1) and solving it using Theorem 2.3. To this end, we first use the transformation $G(x) := 1 - w(1 - F(x)) = h(F(x))$ with

[7] A distortion of the tail probability function of the payment is also used in other theories like Lopes' SP/A model (Lopes (1987), Lopes and Oden (1999)) and Kahnemann and Tversky's influential prospect theory (Kahneman and Tversky (1979) and Tversky and Kahneman (1992)).

$h(z) = 1 - w(1-z)$. Since G^{-1} is $\uparrow$, we rewrite (3.6) as

$$\begin{cases} \int_0^1 u(G^{-1}(t))dt = \max \\ \int_0^1 F_{\varphi_T}^{-1}(1-t)G^{-1}(h(t))dt = x_0 \\ G^{-1} \uparrow . \end{cases} \tag{3.7}$$

Next, using the substitution $z = h(t)$ and thus $\frac{dt}{dz} = \frac{1}{h'(h^{-1}(z))}$, we rewrite the first side condition to obtain the problem

$$\begin{cases} \int_0^1 u(G^{-1}(t))dt = \max \\ \int_0^1 G^{-1}(z)H(z)dz = x_0 \\ G^{-1} \uparrow, \end{cases} \tag{3.8}$$

in which

$$H(z) = F_{\varphi_T}^{-1}(1 - h^{-1}(z))/h'(h^{-1}(z)). \tag{3.9}$$

Problem (3.8) is now of the form (M1), and under the conditions of Theorem 2.3 we obtain its solution as

$$\widehat{k}(t) = (u')^{-1}(\lambda H_*(t)), \tag{3.10}$$

with λ such that $\int_0^1 \widehat{k}(t)H(t)dt = x_0$ and where H_* is the projection of H on $M_\downarrow$. The solution of (3.5) is then given by

$$\widehat{X}_T = \widehat{k}(1 - \tau_{\varphi_T}).$$

This solution conforms with those of Xia and Zhou (2012), who used convex analysis, and of Xu (2016) who used a relaxation method. Our solution merely follows as a straightforward application of the solution to the general optimization result (M1).

3.3 Cumulative prospect theory (CPT)

Kahneman and Tversky (1979) and Tversky and Kahneman (1992) challenge EUT and RDUT in that they provide empirical evidence that investors may consider a reference point when evaluating their terminal wealth, in which case they will perceive a gain only when terminal wealth is higher than the reference level. Moreover, these authors also found that attitude toward gains differs from attitude toward losses in that losses loom larger than gains. To accommodate these observations, they propose an objective function that is no longer concave on the entire domain, but S-shaped, with a convex part below the reference point and a concave part above.

In what follows, we assume without loss of generality that the utility function u satisfies $u(0) = 0$. Furthermore, u is increasing, concave on $(0, \infty)$ and convex on $(-\infty, 0)$. We define a distortion function w as an increasing, differentiable weight function with $w(0) = 0$, $w(1) = 1$.[8] We formulate the following problem:

$$\max_{E\varphi_T X_T = x_0} \int u(x) d(1 - w(1 - F_{X_T}(x))). \tag{3.11}$$

To keep the problem well-posed, we restrict ourselves to payoffs X_T that are bounded from below (see Jin and Zhou (2008)). Denote $X_T^+ = \max(X_T, 0)$ and $X_T^- = -\min(X_T, 0)$.

Theorem 3.1 (Characterization of optimal payoff for a CPT investor)**.** *If problem (3.11) has a solution $\widehat{X_T} = \widehat{X_T}^+ - \widehat{X_T}^-$, then*

a) $\widehat{X_T}^+$ solves problem (3.5) for some budget x_0^+ and

b) $\widehat{X_T}^-$ is of the form $\widehat{X_T}^- = mI_{\{\varphi_T < c\}}$, where $m \geqslant 0, c \geqslant 0$ are such that $E\varphi_T \widehat{X_T}^- = x_0^+ - x_0$.

Proof. Assume that $\widehat{X_T} = \widehat{X_T}^+ - \widehat{X_T}^-$ is a solution, which by Theorem 2.1 can be taken as anti-monotonic to φ_T, and let $\widehat{X_T}^-$ be bounded by $m > 0$. It is obvious that for $\widehat{X_T}$ to be a solution, it must hold that $\widehat{X_T}^+$ is a solution to problem (3.5) for the budget $x_0^+ := E\varphi_T \widehat{X_T}^+$.

Assume that $\widehat{X_T}^-$ is not concentrated on 0 and m. Construct Y_T^- as a payoff anti-monotonic to φ_T that has the same mean as $\widehat{X_T}^-$ and solely takes the values 0 and m, and define $Y_T^+ = \widehat{X_T}^+$. Then $Y_T = Y_T^+ - Y_T^-$ has a higher objective value than $\widehat{X_T}$ since u is convex on $(-\infty, 0)$. Moreover,

$$E\varphi_T Y_T \leqslant E\varphi_T \widehat{X_T} = x_0$$

since $F_{Y_T}^{-1}$ crosses $F_{\widehat{X_T}}^{-1}$ once from below and $\widehat{X_T}$, Y_T are anti-monotonic to φ_T. Hence, by adding the appropriate positive constant to Y_T we would obtain a payoff that is an admissible solution and improves $\widehat{X_T}$. □

Theorem 3.1 thus provides admissible solutions $\widehat{X_T}$ to the CPT optimization problem (3.11), and these depend on the parameters $x_0^+ \geqslant 0$, $m \geqslant 0$. An optimal solution is then obtained by restricting the optimization in (3.11) to these admissible solutions.

Remark 3.2. *A characterization result of the optimal payoff under CPT preferences was first given in Jin and Zhou (2008). However, a strong assumption was made in that for the positive part X_T^+ of the solution (which solves an RDUT problem) the*

[8] Note that we do not assume the concavity of w, as is predominantly the case in the CPT literature

function H, as stated in (3.9) was assumed to be decreasing and it seems there is no economic reason to do so. By our results in Section 3.2 we obtain a complete characterization of the solution without making this assumption.

In fact, solving the CPT optimal portfolio problem simply amounts to solving, for the positive part of the solution, an RDUT problem (dealt with in Section 3.2), and, for the negative part, an EUT problem with convex utility on a bounded domain (dealt with in the proof of Theorem 3.1). Our solution to the CPT optimization problem thus follows as a straightforward application of the general optimization result (M1) supplemented by elementary arguments. In contrast, Jin and Zhou (2008) used convex analysis to derive their results.

4 Optimal payoff under Yaari's dual theory of choice

Yaari's dual theory of choice can be considered as a limiting case of RDUT in that the utility function $u(x)$ is linear. Formally, we consider the problem of maximizing the Yaari utility

$$\max_{E\varphi_T X_T = x_0} \int x d(1 - w(1 - F_{X_T}(x)). \tag{4.1}$$

As before, the quantile function of the optimal payoff $\widehat{X}_T$ is a solution to the problem

$$\begin{cases} \int_0^1 G^{-1}(t)dt = \max \\ \int_0^1 G^{-1}(z)H(z)dz = x_0 \\ G^{-1} \uparrow, \end{cases} \tag{4.2}$$

where $H(z) = F_{\varphi_T}^{-1}(1 - h^{-1}(z))/h'(h^{-1}(z))$ (see (3.8) with $u(x) = x$ and (3.9)).

Since u is linear, we cannot invoke Theorem 2.3 (note that its proof requires inversion of $u'(x)$). However, our result for problem (M2), i.e., Theorem 2.4 can be invoked to yield a solution. Specifically, the following theorem provides the optimal payoff on a bounded interval under Yaari preferences.

Theorem 4.1 (Optimal payoff for a Yaari investor)**.** *Let $a, b \in \mathbb{R}$ and $H \in L^2$. If the problem*

$$\begin{cases} \int x d(1 - w(1 - F_{X_T}(x)) = \max \\ E\varphi_T X_T = x_0 \\ a \leqslant X_T \leqslant b \end{cases} \tag{4.3}$$

is feasible, then a solution is given by

$$\widehat{X}_T = \widehat{k}(1 - \tau_{\varphi_T}),$$

where

$$\widehat{k}(t) = \begin{cases} a, \\ \gamma, \\ b, \end{cases} \qquad \frac{1}{H_*(t)} \begin{matrix} < \\ = \\ > \end{matrix} \delta. \tag{4.4}$$

Here, H_* is the projection of H on $M_\downarrow$, and $\gamma \in [a, b]$ and $\delta \in [\frac{1}{b}, \frac{1}{a}]$ are chosen such that $\int_0^1 \widehat{k}(t)H(t)dt = x_0$.

Proof. An optimal solution $\widehat{X}_T$ has a quantile function $\widehat{k}$ that is a solution to the reduced problem

$$\begin{cases} \displaystyle\int_0^1 G^{-1}(t)dt = \max \\ \displaystyle\int_0^1 G^{-1}(z)H(z)dz = x_0 \\ G^{-1} \uparrow, \ a \leqslant G^{-1} \leqslant b. \end{cases} \tag{4.5}$$

Problem (4.5) is of the form (M2), which was stated and solved in Theorem 2.4. We thus obtain $\widehat{k}$ as specified in (4.4). The distributional transform then yields the optimal solution $\widehat{X}_T$. □

Remark 4.2. *In Theorem 2.4 (resp. Theorem 4.1) the assumption of square integrability for g (resp. for H) can be relaxed to the assumption of integrability, since by assumption f is bounded and therefore fg remains integrable. The proof is then similar, but requires extensions of results on isotonic regression, as can be found in Chapter 7 of Barlow et al. (1972).*

Remark 4.3. *He and Zhou (2011) also deal with optimal portfolio selection under preferences as in Yaari's dual theory. It is of interest to compare our results with theirs:*

(i) ***Two-point or three-point solution:*** *Under an additional assumption, He and Zhou (2011) solve the Yaari optimization problem (6.5) in the case that $a = 0$ and $b = \infty$. They obtain a solution that has two mass points only, whereas our solution has in general three mass points. This can be explained by the following argument.*

In their analysis, He and Zhou (2011) make the assumption that the function $M(z) := \frac{F_{\varphi_T}^{-1}(1-z)}{w'(1-z)}$ is first strictly increasing and then strictly decreasing. Since $w(z)$ is increasing and thus also $h(z) = 1 - w(1-z)$, this is equivalent to assuming that $\frac{1}{H(z)}$ is first strictly increasing and then strictly decreasing. Hence, their assumption implies that $\frac{1}{H_*(z)}$ will first strictly increase and then become constant.

Let us now fix $a = 0$ and take some $b > 0$. The optimal solution $\widehat{X}_T^b$ has then only two mass points, namely $a = 0$ as well as some $\gamma \leqslant b$. Hence, when $b = \infty$ the optimal payoff $\widehat{X}_T$ has two mass points as well, in which $a = 0$ is the first point and in which the second point, say b^*, follows from optimization of the Yaari utility over the set of two-point payoffs X_T^b (see (3.6)). Denote by $U_{x_0}(b)$ the Yaari utility of the X_T^b, $b > 0$, that is purchased with budget x_0. It follows that

$$U_{x_0}(b) = x_0 \frac{w(p)}{\int_0^p F_{\varphi_T}^{-1}(z)dz}, \text{ with } p > 0 \text{ such that } b = \frac{x_0}{\int_0^p F_{\varphi_T}^{-1}(z)dz}.$$

Direct optimization leads to the solution $b^* := b(p^*)$, where p^* solves

$$F_{\varphi_T}^{-1}(p)w(p) - w'(p)\int_0^p F_{\varphi_T}^{-1}(q)dq = 0. \tag{4.6}$$

This solution conforms with the one presented in Theorem 3.7 of their paper. Thus, if we impose the same condition as in He and Zhou (2011), we recover their solution. In our analysis, however, the is not imposed, and we obtain an optimal payoff that in general has three mass points.

(ii) ***Alternative derivation of the solution in He and Zhou (2011)**: The assumption of He and Zhou (2011) that $M(z)$ is strictly unimodal, i.e., first strictly increasing and then strictly decreasing, is quite restrictive and lacks an economic foundation. Moreover, under this unimodality assumption, their result can be obtained in a straightforward way.*

To make this point clear, observe that this unimodality assumption also implies that the quotient of the mixtures $\frac{\int_0^p w'(q)dq}{\int_0^q F_{\varphi_T}^{-1}(q)dq}$ is first strictly increasing and next strictly decreasing (see Theorem 2.3 in Metzger and Rüschendorf (1991)). Hence, we also obtain that $U_{x_0}(b)$ is first strictly increasing and then strictly decreasing in b. Denote its maximum as $U_{x_0}(b^*)$.

Consider next a payoff X_T having n mass points $b_1, b_2, \dots, b_n$. Denote by $X_T^{b_i}$ the two-point payoffs taking the value 0 or b_i, having cost x_i and Yaari utility $U_{x_i}(b_i)$. If X_T is optimum, it satisfies the budget constraint, $x_0 = x_1 + x_2 + \dots + x_n$ and is anti-monotonic with φ_T. It follows that under Yaari preferences the objective value of X_T (its Yaari utility) is linear in the $U_{x_i}(b_i)$.

It is straightforward that any such X_T can be improved by the two point payoff X_T^{b*} (note that in (4.6), p^* does not depend on the initial budget). A standard limit argument shows that this conclusion extends to a generally distributed X_T.

5 Optimal payoffs with state-dependent constraints in markets with specified pricing density

Optimal payoffs under law-invariant preferences were shown in Theorem 2.1 to be decreasing in the state price density φ_T, i.e., income is low when state prices are high. Since high state prices typically correspond to states of economic recession, we conclude that optimal payoffs do not offer protection when investors typically need it most. To account for this (undesirable) feature, we may want to provide the investor with the opportunity to maintain a desired dependence with a benchmark asset (state-dependent constraint). This idea was developed in Bernard et al. (2014a), Bernard et al. (2014b) and Bernard et al. (2015b). Specifically, Bernard et al. (2015b) determine the optimal payoff for an expected utility maximizer under such dependence constraint. In the following, we extend their results by showing that under concave preferences with dependence constraints, optimal portfolio selection problems can still be reduced to optimization function problems under monotonicity conditions (Theorem 5.2). As an application, we obtain the optimal payoff for investors with Yaari or RDUT preferences under a dependence constraint.

Hence, let A_T be a benchmark variable and let C be a copula describing the desired payoff structure of admissible claims X_T with a given benchmark A_T, i.e., the copula $C_{(X_T,A_T)}$ of (X_T, A_T) is prescribed to be identical to C. Consider the following constrained utility optimization problem (UC) with copula constraint:

$$\text{(UC)} \qquad \begin{cases} \Psi(X_T) = \max \\ E\varphi_T X_T = x_0 \\ C_{(X_T,A_T)} = C, F_{X_T} \in \mathcal{F}. \end{cases} \tag{5.1}$$

To deal with problem (UC), we have the following anti-monotonicity result, analogously to Theorem 2.1.

Theorem 5.1. *If problem* (UC) *has a solution, then there exists a solution $\widehat{X}_T$ that is a.s. anti-monotonic with φ_T, conditionally on A_T.*

Proof. If X_T is a solution to problem (UC) then denote by $U = \tau_{\varphi_T|A_T}$ the conditional distributional transform and $\widehat{X}_T = F^{-1}_{X_T|A_T}(1-U))$. Then by arguments as in the proof of Theorem 3.1 in Bernard et al. (2014b), we obtain that $(\widehat{X}_T, A_T) \sim (X_T, A_T))$, i.e., $\widehat{X}_T$ is admissible. Furthermore, since U, X_T are independent we obtain that $\widehat{X}_T$, φ_T are anti-monotonic, conditionally on $A_T = a$. This implies by the Hoeffding–Fréchet bounds that

$$E\widehat{X}_T\varphi_T = EE(\widehat{X}_T\varphi_T \mid A_T) \leqslant EE(X_T\varphi_T \mid A_T) = EX_T\varphi_T$$

. Since ψ is monotone and law-invariant, this implies that also $\widehat{X}_T$ is a solution to (UC)). □

For $C = C_{(X_T,A_T)}$ the copula of (X_T, A_T), let $C_{1|2}$ denote the conditional distribution and let $C_{1|A_T} = C_{1|2=\tau_{A_T}}$ be the factorized copula given that the second component is τ_{A_T}. We define $Z_T := C^{-1}_{1|A_T}(1 - \tau_{\varphi_T|A_T})$, where $\tau_{\varphi_T|A_T}$ is the conditional distributional transform. We obtain that

$$Z_T \sim U(0,1),\ C_{(Z_T,A_T)} = C,\ (Z_T, \varphi_T) \text{ is anti-monotonic, conditionally on } A_T. \tag{5.2}$$

Theorem 5.1 thus implies that for solving problem (UC) we only need to consider payoffs X_T of the form $X_T = g(Z_T)$, $g \uparrow$. In fact, similar to the reduction of (U) to (U^r), the following theorem shows that problem (UC) can also be reduced to a problem that is of the form (U^r). Denote

$$m(Z_T) = E(\varphi_T \mid Z_T) \text{ and } \kappa_T = m_*(Z_T), \tag{5.3}$$

where m_* is the projection of m on $M_\downarrow$.

Theorem 5.2 (Reduction of constrained utility optimization problem)**.** *Assume that in Problem (UC) the functional Ψ is concave. If $\widehat{F}$ is a solution to the restricted utility optimization problem*

$$(UC^r) \qquad \begin{cases} \Psi(F) = \max \\ \int_0^1 m_*(t))F^{-1}(t)dt = x_0 \\ F \in \mathcal{F} \end{cases} \tag{5.4}$$

and $\widehat{k} = \widehat{F}^{-1}$ denotes its quantile function, then

$$\widehat{X}_T = \widehat{k}(1 - \tau_{\kappa_T}) \tag{5.5}$$

solves problem (UC).

Proof. Theorem 5.1 implies that the constrained utility optimization problem (UC) is equivalent to

$$\begin{cases} \Psi(X_T) = \max \\ X_T = g(Z_T), \ g \uparrow \\ EX_T m(Z_T) = x_0 \\ F_{X_T} \in \mathcal{F}, \end{cases} \tag{5.6}$$

which is equivalent to the problem

$$\begin{cases} \Psi(X_T) = \max \\ X_T = g(Z_T), g \uparrow \\ EX_T m_*(Z_T) = x_0 \\ F_{X_T} \in \mathcal{F}. \end{cases} \tag{5.7}$$

To see the equivalence between (5.6) and (5.7), observe that it follows from the projection equations (2.7) that if $g(Z_T)$ is a solution to (5.6), then $Eg(Z_T)m_*(Z_T) \leqslant Eg(Z_T)m(Z_T) = x_0$. Hence, by adding an appropriate constant $c > 0$, one obtains that $g(Z_T) + c$ is admissible, and the maximum value in (5.7) is thus bigger than the one in (5.6).

To show the other direction, observe that the solution to (5.7) can be chosen of the form $h(m_*(Z_T))$, with h decreasing. Indeed, as Ψ is concave any solution $g(Z_T)$

to (5.4) can be improved by the conditional expection $E(g(Z_T) \mid m_*(Z_T))$, which is decreasing in $m_*(Z_T)$ because $m_*(Z_T)$ and $g(Z_T)$ are anti-monotonic. Furthermore, with $g = h \circ m_*$, $h \downarrow$ it holds that

$$x_0 = Eg(Z_T)m(Z_T)) = Eg(Z_T)m_*(Z_T). \tag{5.8}$$

Hence, $g(Z_T) = h(m_*(Z_T))$ is admissible for Problem (5.6) and the same objective value is obtained.

As in the law-invariant case, we thus obtain that the constrained utility maximization problem (UC) can be reduced to a quantile formulation:

$$\begin{cases} \Psi(F) = \max \\ \displaystyle\int_0^1 m_*(t)F^{-1}(t)dt = x_0 \\ F \in \mathcal{F}. \end{cases} \tag{5.9}$$

□

The optimal payoff under an additional dependence constraint (problem (UC^r)) is of the same form as in the law-invariant case (problem (U^r)). The difference is merely that in the law-invariant setting the $F_{\varphi_T}^{-1}(1-t)$ in the budget constraint is replaced by $m_*(t)$ (compare (5.4) with (3.2)) – or, equivalently, that in the solution of an optimization problem (UC), the role of φ_T is played by the variable $\kappa_T := m_*(Z_T)$, where m_* is the projection of m on $M_\downarrow$. The results are as follows:

EUT with dependence constraint: Assume that the utility function u is concave and increasing. Under the conditions of Theorem 2.3, a solution to the problem

$$\begin{cases} \displaystyle\int u(x)dF_{X_T}(x) = \max \\ E\varphi_T X_T = x_0 \\ C_{(X_T,A_T)} = C \end{cases} \tag{5.10}$$

is given by

$$\widehat{X}_T = \widehat{k}(1-\tau_{\kappa_T}). \tag{5.11}$$

Here, τ_{κ_T} is the distributional transform of κ_T and $\widehat{k}(t) = (u')^{-1}(\lambda F_{\kappa_T}^{-1}(1-t))$ with $\lambda > 0$ determined by $\int_0^1 F_{\kappa_T}^{-1}(t)\widehat{k}(t)dt = x_0$.

RDUT with dependence constraint: Assume that the utility function u is concave and increasing. Under the conditions of Theorem 2.3, a solution to the problem

$$\begin{cases} \displaystyle\int u(x)d(1-w(1-F_{X_T}(x))) = \max \\ E\varphi_T X_T = x_0 \\ C_{(X_T,A_T)} = C \end{cases} \tag{5.12}$$

is given by

$$\widehat{X}_T = \widehat{k}(1-\tau_{\kappa_T}),$$

where $\widehat{k}(t) = (u')^{-1}(\lambda J_*(t))$ with λ such that $\int_0^1 \widehat{k}(t)J(t)dt = x_0$ and where J_* is the projection of J on $M_{\downarrow}$, with J defined as

$$J(z) = F_{\kappa_T}^{-1}(1 - h^{-1}(z))/h'(h^{-1}(z)). \tag{5.13}$$

Yaari's dual theory with dependence constraint: If the problem

$$\begin{cases} \int xd(1 - w(1 - F_{X_T}(x))) = \max \\ E\varphi_T X_T = x_0 \\ a \leqslant X_T \leqslant b \\ C_{(X_T, A_T)} = C \end{cases} \tag{5.14}$$

is feasible, then a solution is given by

$$\widehat{X}_T = \widehat{k}(1 - \tau_{\kappa_T}) \tag{5.15}$$

with

$$\widehat{k}(t) = \begin{cases} a, \\ \gamma, \\ b, \end{cases} \qquad \frac{1}{J_*(t)} \begin{array}{c} < \\ = \\ > \end{array} \delta. \tag{5.16}$$

Here, J_* is the projection of J (see (5.13)) on $M \downarrow$ and $\gamma \in [a, b]$ and $\delta \in [\frac{1}{b}, \frac{1}{a}]$ are chosen such that $\int_0^1 \widehat{k}(t)H(t)dt = x_0$.

6 Optimal payoffs under law-invariant preferences when pricing densities are partially specified

Let $\mathcal{X}$ be the set of all $\mathcal{A}_T$-measurable payoffs X_T and let $\mathcal{M}$ be the set of all state-price density processes (equivalent martingale measures). So far in this paper, we have assumed that each $X_T \in \mathcal{X}$ is priced by a specified state-price density process $\varphi = (\varphi_t)_{0 \leqslant t \leqslant T} \in \mathcal{M}$, i.e., its market price, $c(X_T)$, at $t = 0$ is given by $c(X_T) = E\varphi_T X_T$. When the market is complete the optimal payoffs we derive can be replicated (hedged) using a self-financing strategy. However, in incomplete markets this is no longer always true. In this section, we do no longer make the assumption that all claims can be priced using a given state-price density process; instead we assume that the state-price density is only partially specified.

Let $\mathcal{X}_a \subset \mathcal{X}$ contain all payoffs X_T with known market price $c(X_T)$. Denote by $\mathcal{M}_a$ the set of associated state-price density processes that are consistent with these market prices (market-consistency), i.e.,

$$\mathcal{M}_a = \{\varphi \in \mathcal{M} \mid c(X_T) = E\varphi_T X_T, X_T \in \mathcal{X}_a\}. \tag{6.1}$$

Any payoff $X_T \in \mathcal{X}_a$ is called *attainable*. In what follows we deal with selection of optimal payoffs in the class $\mathcal{X}_a$. We do so under a structural assumption that we formulate next.

Assumption 1. *There exists $\varphi \in \mathcal{M}_a$ with the property that for all $F \in \mathcal{F}$, $F^{-1}(1 - \tau_{\varphi_T}) \in \mathcal{X}_a$.*

Since this assumption turns out to be crucial in deriving optimal payoffs, we first provide two important cases of financial market settings that comply with the stated structural assumption.

Example 6.1 (Calls are priced with path-independent state price density process)**.** *Denote by $C(K)$ the payoff of a European call with strike K maturing at time T and with underlying the asset (S_t). Define $\mathcal{C} = \{C(K), K \in \mathbb{R}\}$. Assume that*

i.) $\mathcal{C} \subset \mathcal{X}_a$

ii.) $\mathcal{M}_a$ *contains a path-independent state-price density process denoted by (φ_t) i.e., for some function g,*

$$(\varphi_t) = (g(S_t)).$$

The two stated conditions i) and ii) hold for instance in an exponential Lévy market model in which all $C(K)$, $K \in \mathbb{R}$ are priced using the Esscher transform[9].

Condition i) is not unreasonable, as there are often a large (but finite) number of options available in the market, in which case assuming a continuum of strikes could be seen as a reasonable approximation of reality. Carr and Chou (1997) note that this assumption is "analogous to the continuous trading assumption permeating the continuous time literature."

Condition ii) is consistent with a competitive equilibrium in a market in which the participants have law-invariant preferences. Furthermore, Bondarenko (2003) shows that the existence of a path-independent state price density process implies the absence of statistical arbitrage opportunities, where a statistical arbitrage opportunity is defined as a zero-cost trading strategy with a positive expected payoff and nonnegative conditional expected payoffs given the final state S_T of (S_t).

Let us show that the conditions i) and ii) imply ***Assumption 1****. Hence, consider some payoff $Y_T = F^{-1}(1 - \tau_{\varphi_T})$, $F \in \mathcal{F}$. Then, Y_T is path-independent and thus attainable since any path-independent payoff can be obtained by holding a static portfolio of European calls and puts. This follows from an extended version of the Breeden and Litzenberger (1978) decomposition result; see e.g. Müller and Stoyan (2002)).* □

Other examples of this type can be constructed; we merely need that there is some state price density process φ (possibly path-dependent) that prices all payoffs $X_T \in \mathcal{X}_a$ and that there are enough payoffs X_T of the form $X_T = f(\varphi_T)$ such that a Breeden–Litzenberger type of decomposition result can be invoked to satisfy Assumption 1.

[9]Exponential Lévy market model with Esscher pricing has been introduced in the mathematical finance literature by Gerber et al. (1994) and Madan and Milne (1991) and can be considered as one of the classic models; see Eberlein et al. (1995), Chan (1999), Kallsen and Shiryaev (2002), Esche and Schweizer (2005), Hubalek and Sgarra (2006), Vanduffel et al. (2009), and Benth and Sgarra (2012) for studies of its properties and further motivation.

Example 6.2 (Itô processes with deterministic coefficients)**.** *We assume the market consists of a risk-free bank account and m risky assets with dynamics described by Itô processes. The price process of the bank account satisfies*

$$\frac{dB_t}{B_t} = r_t dt, \quad .$$

in which $r_t \in \mathbb{R}$ is the instantaneous risk-free rate. The m risky assets are driven by a k-dimensional ($k \geqslant m$) Brownian motion $W_t = \left(W_t^1 \dots W_t^k\right)'$,

$$\frac{dS_t^i}{S_t^i} = (r_t + b_t^i)dt + \sum_{j=1}^{k} \sigma_t^{ij} dW_t^j, \quad i = 1, \dots, m,$$

where $b_t := (b_t^1, \dots, b_t^m)' \in \mathbb{R}^m$ and $\sigma_t := (\sigma_t^{ij}) \in \mathbb{R}^{m \times k}$ has full row-rank. When $k = m$ the market is complete and σ_t is invertible. When $k > m$, we are in an incomplete market setting: σ_t is no longer invertible but $\sigma_t {\sigma_t}'$ is.

Denote by $\pi_t = (\pi_t^1, \dots, \pi_t^m)' \in \mathbb{R}^m$ the portfolio vector at time $t > 0$, i.e., π_t^i denotes the fraction of total wealth that at time $t > 0$ is invested in asset S^i . The wealth process is given by the dynamics

$$\frac{dX_t^{\pi_t}}{X_t^{\pi_t}} = (r_t + {b_t}'\pi_t)dt + {\pi_t}' \boldsymbol{\sigma_t} dW_t, \quad X_0^{\pi_0} = x > 0, \tag{6.2}$$

When choosing the weights π_t as

$$\pi_t = (\sigma_t {\sigma_t}')^{-1} b_t := \zeta_t, \tag{6.3}$$

the wealth process corresponds to the so-called Growth Optimal Portfolio (GOP) strategy (since the expected logreturn is maximized) and subject to some technical conditions (Karatzas et al. (1998), Fontana and Runggaldier (2013)), it can be used as a numéraire, i.e., we have that

$$(\varphi_t) = \left(\frac{X_0^{\zeta_0}}{X_t^{\zeta_t}}\right)$$

specifies a state price density process; see also Platen and Heath (2006) for a detailed study on the properties of the GOP and its use in pricing.

In a complete market set-up ($k = m$), (φ_t) is the unique state price density process, but in the incomplete setting ($k > m$), it is merely a particular choice amongst many others. However, in a similar way as in the complete case, the payoff $F^{-1}(1 - \tau_{\varphi_T})$, $F \in \mathcal{F}$ can be replicated by pursuing delta-hedging in the GOP and the bank account. To see this, observe that any payoff of the type $F^{-1}(1 - \tau_{\varphi_T})$ is a deterministic function of $X_T^{\zeta_T}$. Since the wealth process $(X_t^{\zeta_t})$ for the GOP is of exactly the same type as in the case of a complete market model governed by a one-dimensional Itô process with deterministic coefficients, such payoff can be delta-hedged by trading in the GOP and the bank account. Hence, $F^{-1}(1 - \tau_{\varphi_T}) \in \mathcal{X}_a$ and ***Assumption 1*** *is fulfilled.* $\square$

Next we consider the law-invariant optimization problem (U) suitably adapted to the incomplete market setting that we consider, i.e., we consider the optimization problem (UM) defined as

$$\text{(UM)} \qquad \begin{cases} \Psi(X_T) = \max \\ c(X_T) = x_0 \\ X_T \in \mathcal{X}_a. \end{cases} \tag{6.4}$$

Theorem 6.3. *Under Assumption 1 and if the market consistent utility optimization problem (UM) has a solution, there exists a solution $\widehat{X}_T$ that is a.s. anti-monotonic with φ_T.*

Proof. Assume that $\widehat{X_T}$ is optimum and not anti-monotonic with φ_T. Denote its distribution by F and define $Y_T = F^{-1}(1-\tau_{\varphi_T})$. Note that Y_T is attainable and since $\widehat{X_T}$ is also attainable, we obtain that

$$\begin{aligned} c(\widehat{X_T}) &= E[\varphi_T \widehat{X_T}] \\ &\geqslant E[\varphi_T Y_T]. \end{aligned}$$

The inequality follows from the Hoeffding–Fréchet bounds since Y_T is anti-monotonic with φ_T and $F \sim X_T^* \sim Y_T$. Hence, Y_T improves on $\widehat{X_T}$ and the improvement is strict since $\widehat{X_T}$ is not antimonotonic with φ_T. This is a contradiction. □

Theorem 6.3 implies that in order to solve the optimization problem (UM) we only need to solve the utility optimization problem (U) in which φ is a state price density process satisfying the conditions of Theorem 6.3. Hence, all results we obtained in Sections 2 and 3 readily carry over to the incomplete market setting we consider. Specifically, the optimal payoffs that we derived in Sections 2 and 3 for an EUT investor (Section 2.3.1), a RDUT investor (Section 2.3.2), a CPT investor (Section 2.3.3) or a Yaari investor (Section 4) find their counterpart in the incomplete setting with a partially specified pricing measure in a straightforward way. We formulate such result for the Yaari case:

Corollary 6.4 (Optimal payoff for a Yaari investor in an incomplete market with partially specified pricing measure)**.** *Let $a, b \in \mathbb{R}$. Under the assumptions of Theorem 6.3, a solution to*

$$\begin{cases} \displaystyle\int xd(1 - w(1 - F_{X_T}(x)) = \max \\ c(X_T) = x_0 \\ a \leqslant X_T \leqslant b \end{cases} \tag{6.5}$$

is given by

$$\widehat{X}_T = \widehat{k}(1 - \tau_{\varphi_T}),$$

where

$$\widehat{k}(t) = \begin{cases} a, & \\ \gamma, & \frac{1}{H_*(t)} \begin{matrix} < \\ = \\ > \end{matrix} \delta. \\ b, & \end{cases} \tag{6.6}$$

Here, H_* is the projection of H on $M_\downarrow$, and $\gamma \in [a,b]$ and $\delta \in [\frac{1}{b}, \frac{1}{a}]$ are chosen such that $\int_0^1 \widehat{k}(t)H(t)dt = x_0$.

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