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COMPARISON OF PERCOLATION PROBABILITIES

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Abstract

Some recent results of Oxley and Welsh (1979) and McDiarmid (1981) concerning Bernoulli percolation on clutters are generalized. Our results allow to consider quantitative aspects of percolation on graphs and clutters.

PERCOLATION PROBABILITY; CLUTTER; RANDOM GRAPH; ASSOCIATION

1. Introduction

Let I be an ordered finite set and let $\mathscr{C}$ be a clutter of I, i.e. $\mathscr{C}$ is a set of pairwise incomparable subsets of I. Let $X = (X_i)_{i \in I}$ be an $|I|$-dimensional random vector and define for $E \in \mathscr{C}$, $X_E = (X_i)_{i \in E}$, the random vector with indices in E ordered according to the order of I. Let $f = (f_E)_{E \in \mathscr{C}}$, where $f_E : R^{|E|} \to R^1$ is measurable and let $h : R^{|\mathscr{C}|} \to R^1$ also be measurable. Then define the quantitative percolation probability

$$(1) \qquad P_{h,f}(\mathscr{C}, X) = Eh(f_{\mathscr{C}}(X_{\mathscr{C}})),$$

where $f_{\mathscr{C}}(X_{\mathscr{C}}) = (f_E(X_E))_{E \in \mathscr{C}}$ and the expectation is assumed to exist. Definition (1) is extended to the case where f_E are defined on subsets of $R^{|E|}$ in an obvious way.

We give some examples showing the connection to the usual percolation models. For a discussion of these models cf. Hammersley and Welsh (1965), Smythe and Wierman (1978), Oxley and Welsh (1979) and McDiarmid (1981).

(a) If X_i, $i \in I$, are binomial distributed,

$$f_E(X_E) = \prod_{i \in E} X_i \text{ for each } E \in \mathscr{C} \text{ and } h(X_{\mathscr{C}}) = \max_{E \in \mathscr{C}} X_E \text{ for } X_{\mathscr{C}} = (X_E)_{E \in \mathscr{C}} \in R_+^{|\mathscr{C}|}$$

then

$$(2) \qquad P_{h,f}(\mathscr{C}, X) = P\Big(\bigcup_{E \in \mathscr{C}} \bigcap_{i \in E} \{X_i = 1\}\Big) = P(\mathscr{C}, X),$$

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where $P(\mathscr{C}, X)$ is the probability that there exists an open element in the clutter $\mathscr{C}$. ($P(\mathscr{C}, X)$ is called percolation probability by Oxley and Welsh (1979) and McDiarmid (1981).) Comparison results for this case were studied by Oxley and Welsh (1979) and McDiarmid (1981).

(b) Let I be the set of edges of a graph G and let the elements of $\mathscr{C}$ be the edge-sets of minimal paths between two vertices i_0, i_1; let X_i denote the time a particle remains on edge i,

$$f_E(x_E) = \sum_{i \in E} x_i, \quad x_E \in R^{|E|}, \quad \text{and} \quad h(x_{\mathscr{C}}) = \min_{E \in \mathscr{C}} x_E,$$

$x_{\mathscr{C}} = (x_E)_{E \in \mathscr{C}} \in R^{|\mathscr{C}|}$, then $P_{h,f}(\mathscr{C}, X)$ is the expected first-passage time between i_0, i_1 (cf. Hammersley and Welsh (1965)). Assume that a fluid passes from vertex i_0 to i_1 and assume that edge i is only partially open, so that only X_i percent of a fluid arriving at this edge can pass it, then $\Pi_{i \in E} X_i = f_E(X_E)$ is the proportion passing from i_0 to i_1 on the path E. If we choose $h(x_{\mathscr{C}}) = \Sigma_{E \in \mathscr{C}} x_E$, $x_{\mathscr{C}} \in R^{|\mathscr{C}|}$, $P_{h,f}(\mathscr{C}, X)$ is the total quantity of fluid passing from i_0 to i_1.

(c) Let N be a network with random capacities X_i on node i and let $\mathscr{C}$ be the set of minimal cuts separating source i_0 and sink i_1. Then by the maxflow–mincut theorem $P_{h,f}(\mathscr{C}, S)$, with $h(x_{\mathscr{C}}) = \min_{E \in \mathscr{C}} x_E$, $f_E(x_E) = \Sigma_{i \in E} x_i$ is the expected maximal flow between i_0, i_1.

2. Comparison of percolation probabilities

We want to compare percolation probabilities for two different random mechanisms $X = (X_i)_{i \in I}$ and $Y = (Y_i)_{i \in I}$. To do this we need some definitions. Let W_1, W_2 be two k-dimensional random vectors. Define

$$\begin{aligned} &W_1 \leqq_s W_2 \text{ if } P(W_1 \geqq z) \leqq P(W_2 \geqq z) \text{ for all } z \in R^k \\ &(\geqq \text{is the componentwise order on } R^k) \text{ and} \\ &W_1 \leqq^s W_2 \quad \text{if } P(W_1 \leqq z) \geqq P(W_2 \leqq z). \end{aligned} \tag{3}$$

Let $1_{[z,\infty)}$ ($1_{(-\infty,z]}$) denote the indicator function of $[z,\infty)$ ($(-\infty, z]$); let M_1^k be the closure of the set

$$\left\{d_0 + \sum_{j=1}^{n} d_j 1_{[z_j,\infty)};\ d_0 \in R^1,\ d_j \geqq 0,\ z_j \in R^k,\ j \leqq n,\ n \in \mathbb{N}\right\} \tag{4}$$

with respect to pointwise limits of isotone (i.e. monotonically decreasing or increasing) sequences and, similarly,

$$M_2^k \text{ the closure of } \left\{d_0 + \sum_{j=1}^{n} d_j 1_{(-\infty,z_j]};\ d_j \geqq 0,\ d_0 \in R^1,\ z_j \in R^k,\ j \leqq n,\ n \in \mathbb{N}\right\}$$

with respect to pointwise limits of isotone sequences. M_1^k is a subset of the set of Δ-monotone functions and was studied by Rüschendorf (1980). It includes for instance $f_1(x)=\min_{i\le k} x_i$, $x\in R^k$, $f_2(x)=\sum_{i=1}^k x_i$, $x\in R^k$ or $f_3(x)=\prod_{i=1}^k x_i$, $x\in R_+^k$; similarly $h_1(x)=\max_{i\le k} x_i$, $x\in R^k$, $h_2(x)=\sum_{i=1}^k x_i^{d_i}$, $x\in R_+^k$, $d_i\geqq 0$ are elements of M_2^k. Let $I=\sum_{i=1}^n I_i$ be a partition of I and consider the following assumptions.

A$_1$ $\{X_{I_i}\}_{1\leqq i\leqq n}$ are independent random vectors; similarly $\{Y_{I_i}\}_{1\leqq i\leqq n}$ are independent random vectors.

A$_2$ For all $E\in\mathscr{C}$ we have $|E\cap I_i|\leqq 1$, $1\leqq i\leqq n$.

A$_3$ f_E is monotonically non-decreasing for $E\in\mathscr{C}$.

A$_4$ $X_{I_i}\leqq_s Y_{I_i}$, $1\leqq i\leqq n$.

A$_5$ $X_{I_i}\leqq^s Y_{I_i}$, $1\leqq i\leqq n$.

Theorem 1. Under assumptions A$_1$, A$_2$, A$_3$

(5)
(a) $P_{h,f}(\mathscr{C},X)\leqq P_{h,f}(\mathscr{C},Y)$ for $h\in M_1^{|\mathscr{C}|}$ if A$_4$ holds.
(b) $P_{h,f}(\mathscr{C},X)\geqq P_{h,f}(\mathscr{C},Y)$ for $h\in M_2^{|\mathscr{C}|}$ if A$_5$ holds.

Proof. (a) Without loss of generality we may assume that $X_{I_i}=Y_{I_i}$, $2\leqq i\leqq n$. Assume, furthermore, that $X_{I_i}=x_{I_i}$, $2\leqq i\leqq n$, are given. With $\mathscr{C}_i=\{E\in\mathscr{C};\ i\in E\}$ for $i\in I_1$ we have by our independence assumption A$_1$ that $f_E(X_E)=f_E(X_i, x_{E\setminus\{i\}})$, for $E\in\mathscr{C}_i$, is a monotonically non-decreasing function of X_i, $i\in I_1$, conditionally on x_{I_i}, $2\leqq i\leqq n$. Therefore, by A$_4$ conditionally on x_{I_i}, $2\leqq i\leqq n$, we have

$$(f_E(X_E))_{E\in\mathscr{C}}\leqq_s (f_E(Y_E))_{E\in\mathscr{C}}.$$

This implies by definition of $M_1^{|\mathscr{C}|}$ using the theorem on monotone convergence, that $P_{h,f}(\mathscr{C},X)\leqq P_{h,f}(\mathscr{C},Y)$.

(b) follows from (a) by replacing x by $-x$.

Remark 1. (a) If X_i, Y_i are binary our assumptions are identical to condition (C) and the independence assumption of McDiarmid (1981). So Theorem 1 includes part (a) of Theorem 2.1 of McDiarmid (1981) (the GCP theorem) on comparison of $P(\mathscr{C},X)$, $P(\mathscr{C},Y)$ (cf. the introduction). Part (b) of the GCP theorem can be generalized in a similar way.

(b) Our proof of Theorem 1 is on the lines of a proof given by Lehmann (1966) for concordant functions; we do not need results on clutters.

(c) The following sufficient conditions for $\leqq^s$, $\leqq_s$ correspond to those in Lemma 2.2 of McDiarmid (1981): Let F_i be the distribution function of $P^{X_i}=P_i$, $i\in I$, and let U_i be independent and uniformly distributed on $(0,1)$, $1\leqq i\leqq n$.

1. If $Y_{I_i}=(F_j^{-1}(U_i))_{j\in I_i}$, $1\leqq i\leqq n$, then $X\leqq_s Y$ and

$$X\leqq^s Y. \tag{6}$$

2. If $|I_i| \leqq 2$, $1 \leqq i \leqq n$, and $Y_{I_i} = (F_j^{-1}(U_i), F_{j'}^{-1}(1-U_i))$ if $I_i = \{j, j'\}$, $Y_{I_i} =$ $_j^{-1}(U_i)$, if $I_i = \{j\}$, then $Y \leqq_s X$ and $Y \leqq^s X$ (cf. Rüschendorf (1980)).
Jsing 1 and 2 above we can generalize some applications concerning compari-ons of different random graphs given by McDiarmid (1981) as, for instance, 'heorem 4.1 and Corollary 2.3.

(d) Under stronger domination assumptions than given in A_4, A_5, for example ochastic order, we do not need A_2 for a similar comparison result. Some eneralizations of Theorem 1 to countable I are obvious.

Now let $\mathscr{C}$ be the disjoint union of $\mathscr{C}_i$, $1 \leqq i \leqq r$. X is called associated, if $f(X)g(X) \geqq Ef(X)Eg(X)$ for all monotonically non-decreasing f, g such that ie expectation exists (cf. Esary, Proschan and Walkup (1967)).

Theorem 2. Let X be associated, $\mathscr{C} = \Sigma_{i=1}^r \mathscr{C}_i$, let $(Y_{\mathscr{C}_i})_{1 \leqq i \leqq r}$ be independent ndom vectors such that $X_{\mathscr{C}_i}$ and $Y_{\mathscr{C}_i}$ have the same distribution and assume A_3. hen,

$$P_{h,f}(\mathscr{C}, X) \geqq P_{h,f}(\mathscr{C}, Y) \quad \text{for } h \in M_1^{|\mathscr{C}|}$$

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$$P_{h,f}(\mathscr{C}, X) \leqq P_{h,f}(\mathscr{C}, Y) \quad \text{for } h \in M_2^{|\mathscr{C}|}.$$

Proof. Since X is associated, also $(X_{\mathscr{C}_i})_{1 \leqq i \leqq r}$ with $X_{\mathscr{C}_i} = (X_E)_{E \in \mathscr{C}_i}$ is associated d, therefore, by A_3 also $(f_{\mathscr{C}_i}(X_{\mathscr{C}_i}))_{1 \leqq i \leqq r}$ is associated (cf. Esary, Proschan and alkup (1967)); this implies that $(f_{\mathscr{C}_i}(Y_{\mathscr{C}_i}))_{1 \leqq i \leqq r} \leqq_s (f_{\mathscr{C}_i}(X_{\mathscr{C}_i}))_{1 \leqq i \leqq r}$ (and also with spect to $\leqq^s$). As in the proof of Theorem 1 this implies Theorem 2.

Remark 2. If X is associated with $P^{X_i} = B(1, p_i)$, $i \in I$, where $B(1, p_i)$ notes binomial distribution with parameter p_i, if $\mathscr{C} = \{E_1, \cdots, E_n\} = \Sigma_{i=1}^r \mathscr{C}_i$, $(x_E) = \Pi_{i \in E} x_i$, $h(x_{\mathscr{C}}) = \max_{E \in \mathscr{C}} x_E$, then $h \in M_2^{|\mathscr{C}|}$ and $P_{hf}(\mathscr{C}, X) = P(\mathscr{C}, X)$. Theorem 2 $P(\mathscr{C}, X) \leqq P(\mathscr{C}, Y) = P(\mathscr{C}_1 \cup \cdots \cup \mathscr{C}_r, Y)$. If $n = r$, $\mathscr{C}_i = \{E_i\}$, $i \leqq n$,

$$P(\mathscr{C}, Y) = 1 - \prod_{i=1}^n P\left\{ \bigcup_{j \in E_i} \{Y_j = 0\} \right\} = 1 - \prod_{i=1}^n \left(1 - \prod_{j \in E_i} p_j\right).$$

is remark generalizes Theorems 3.1 and 3.2 of Oxley and Welsh (1979) who isider the case that $\{X_i\}_{i \in I}$ are independent and $p_i = p_j$, $i, j \in I$. (For a queness part as in Theorem 3.1 of Oxley and Welsh (1979) we had to impose ct monotonicity on $f_{\mathscr{C}_i}$ and on h.)

'or $p_i = p$, $i \in I$, define $P(\mathscr{C}, p) = P(\mathscr{C}, X)$. Oxley and Welsh (1979) derive in eorem 4.1 (sharp) lower bounds for $P(\mathscr{C}, p)$. These bounds can be sharpened ler further restrictions on the clutter $\mathscr{C}$. If for instance $\mathscr{C} = \{E_1, \cdots, E_n\}$, $| = a_i$ and $|E_i \setminus E_j| \geqq 2$ for all $i \neq j$, then

$$P(\mathscr{C}, p) \geqq p^{a_1} + p^{a_2}(1-p^2) + \cdots + p^{a_n}(1-p^2)^{n-1}$$

a similar proof as given by Oxley and Welsh (1979).

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