Construction and hedging of optimal payoffs in Lévy Models

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Abstract

The construction of lowest cost strategies for a given payoff has found considerable interest in recent literature and it has been shown in applications to real market data, that the cost savings associated with these cost-efficient strategies can be quite substantial. In this paper we provide for a variety of options in the frame of Lévy models cost-efficient counterparts and determine the efficiency loss (resp. gain) in applications to several sets of market data. We discuss specific effects of the cost-efficient payoffs for a series of standard options like puts, calls, self-quanto puts and straddles and butterfly spread options, and develop their pricing. We obtain several new results on dependence of the magnitude of the efficiency loss on various model and option parameters. We show that the cost-efficient payoffs behave slightly improved compared to the standard payoffs concerning hedging properties. We provide concrete hedging simulation schemes for various cost-efficient options. The results of the paper show that cost-efficient payoffs may lead to considerable reduction of cost in markets with pronounced trend.

1 Introduction to cost-efficient payoffs

The concept of distributional analysis of portfolio choice has been introduced by Dybvig (1988a). In a market model $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ with finite time horizon $[0,T]$ let $S = (S_t)_{0\leq t\leq T} \in \mathbb{R}^d$ be a price model for d stocks and $(Z_t)_{0\leq t\leq T}$ a pricing density rendering the discounted process $(e^{-rt}S_tZ_t)_{0\leq t\leq T}$ a P-martingale. The cost of a strategy with terminal payoff X_T then is given by the discounted expected payoff

$$c(X_T) = E[e^{-rT}Z_TX_T]. \tag{1.1}$$

For a given payoff distribution G a strategy with terminal payoff $\underline{X}_T$ with distributed payoff G (i.e. $\underline{X}_T \sim G$) is called *cost-efficient* if it minimizes the cost i.e.

$$c(\underline{X}_T) = \min_{X_T\sim G} c(X_T). \tag{1.2}$$

A strategy with payoff $\overline{X}_T \sim G$ is called *most-expensive* if

$$c(\overline{X}_T) = \max_{X_T \sim G} c(X_T). \tag{1.3}$$

The difference of the costs $\ell(X_T) = c(X_T) - c(\underline{X}_T)$ is called the *efficiency loss* of X_T.

The following characterization of cost-efficient payoffs has been stated in various generality in a series of papers including Dybvig (1988a,b), Jouini and Kallal (2001), Dana (2005), Schied (2004), Burgert and Rüschendorf (2006), Bernard and Boyle (2010); Bernard et al. (2014), Vanduffel et al. (2008, 2012) and Rüschendorf (2012).

Theorem 1.1 (cost-efficient payoffs)

a) For a given payoff distribution G holds

$$c(\underline{X}_T) = e^{-rT} \int_0^1 G^{-1}(u) F_{Z_T}^{-1}(1-u) du\,. \tag{1.4}$$

b) A payoff $\underline{X}_T \sim G$ is cost-efficient if and only if $\underline{X}_T$ and Z_T are antimonotonie. $\overline{X}_T \sim G$ is most expensive if and only if $\overline{X}_T$ and Z_T are comonotonic.

c) If F_{Z_T} is continuous then the cost-efficient resp. most expensive payoffs are given by

$$\underline{X}_T = G^{-1}(1 - F_{Z_T}(Z_T)) \qquad \textit{resp.} \quad \overline{X}_T = G^{-1}(F_{Z_T}(Z_T))\,. \tag{1.5}$$

Theorem 1.1 has been applied in several papers to determine cost-efficient payoffs in particular in the context of the Samuelson model as well as in some classes of exponential Lévy processes (see Bernard et al. (2014), Vanduffel et al. (2012) and Hammerstein et al. (2014)) and has been applied to real market data. In the context of Lévy models the results have been mainly based on the Esscher measure defined by the pricing density

$$Z_t^{\overline{\theta}} = \frac{e^{\overline{\theta} L_t}}{M_{L_t(\overline{\theta})}} \tag{1.6}$$

where M_{L_t} denotes the moment generating function of L_t and $\overline{\theta}$, the Esscher parameter, is a solution to the equation

$$e^r = \frac{M_{L_1}(\overline{\theta}+1)}{M_{L_1}(\overline{\theta})}\,. \tag{1.7}$$

Condition 1.7 implies that the Esscher measure $Q^{\overline{\theta}} = Z_T^{\overline{\theta}} P$ is a risk neutral measure for the discounted stock price process $(e^{-rt} S_t)_{0 \le t \le T}$. It has the pleasant property that w.r.t $Q^{\overline{\theta}}$ L remains a Lévy process with modified parameters.

For exponential Lévy models $S_t = S_0 e^{L_t}$ with driving Lévy process $L = (L_t)$ one gets a simpler representation of efficient strategies and for the cost bounds.

Proposition 1.2 (cost-efficient payoffs in Lévy models) *Let $(L_t)_{t \ge 0}$ be a Lévy process with continuous distribution function F_{L_T} at maturity $T > 0$, and assume that a solution $\overline{\theta}$ of* (1.7) *exists.*

If $\bar{\theta} < 0$, the cost-efficient payoff $\underline{X}_T$ and the most-expensive payoff $\overline{X}_T$ with distribution function G are given by

$$\underline{X}_T = G^{-1}(F_{L_T}(L_T)) \text{ and } \overline{X}_T = G^{-1}(1 - F_{L_T}(L_T)) \text{ and}$$

$$E[e^{-rT} Z_T^{\bar{\theta}} X_T] \geq E[e^{-rT} Z_T^{\bar{\theta}} \underline{X}_T] = \frac{1}{M_{L_T}(\bar{\theta})} \int_0^1 e^{\bar{\theta} F_{L_T}^{-1}(1-y) - rT} G^{-1}(1-y) \, \mathrm{d}y \, .$$

If $\bar{\theta} > 0$, the cost-efficient and the most-expensive payoffs are given by

$$\underline{X}_T = G^{-1}(1 - F_{L_T}(L_T)) \text{ and } \overline{X}_T = G^{-1}(F_{L_T}(L_T)) \text{ and}$$

$$E[e^{-rT} Z_T^{\bar{\theta}} X_T] \geq E[e^{-rT} Z_T^{\bar{\theta}} \underline{X}_T] = \frac{1}{M_{L_T}(\bar{\theta})} \int_0^1 e^{\bar{\theta} F_{L_T}^{-1}(y) - rT} G^{-1}(y) \, \mathrm{d}y \, .$$

$\underline{X}_T$ and $\overline{X}_T$ are almost surely unique.

In Lévy models the market is bullish i.e. $E\frac{S_t}{S_0} > e^{rt}$ iff $\overline{\theta} < 0$ and the market is bearish iff $\overline{\theta} > 0$ (see Proposition 2.2 in Hammerstein et al. (2014)). Furthermore, for $\overline{\theta} < 0$ a payoff X_T is cost-efficient iff X_T is an increasing function in L_T and for $\overline{\theta} > 0$, X_T is cost-efficient iff X_T is a decreasing function of L_T. In particular a put is inefficient in increasing markets where $\overline{\theta} < 0$ and a call is inefficient in decreasing markets ($\overline{\theta} > 0$). It is shown (for some examples) that the magnitude of efficiency loss is increasing in the magnitude of the trend in the market described by $|\overline{\theta}|$. As consequence one gets that path dependent options are not cost-efficient and thus can be improved by cost-efficient options.

The main aim in this paper is to determine cost-efficient payoffs for several classes of monotone and nonmonotone options in Lévy models and thus to present a set of examples showing that the method of cost-efficiency can be used in a great variety of applications. We also extend the known results to describe the magnitude of the efficiency loss in dependence on the model parameters and on the hedging costs for the efficient payoff in comparison to the underlying payoffs. We apply and test the results for several real market data modeled by Lévy processes; in particular we consider the normal inverse Gaussian (*NIG*), the variance Gaussian (*VG*) and the normal model and consider two increasing and two decreasing markets. As results we find that cost-efficient payoffs may lead to considerable reduction of costs. The magnitude of the efficiency loss depends essentially on the magnitude of trend in the market described by the absolute value of the Esscher parameter $|\overline{\theta}|$. We show in several examples that cost-efficient payoffs have a similar behavior concerning hedging as the basic payoffs. In particular cost-efficient payoffs do not need extra hedging costs compared to the basic payoffs. The results in this paper are given in the case of real markets and for the case of pricing by the Esscher martingale measure. Some extensions to the multivariate case and to further pricing principles are given in Rüschendorf and Wolf (2014, 2015). For several details in this paper we refer to the dissertation Wolf (2014).

2 Lévy models and some classes of markets data

As in Hammerstein et al. (2014) we focus in this paper on the modeling of market data by three types of Lévy processes, the *NIG*, the *VG* and the normal model. We apply this modeling to market data of 4 stocks (Volkswagen, Allianz, ThyssenKrupp and E.ON), two of them increasing two of them decreasing in the observed period. In spite of the fact that the first two models lead to a better fit of the market data it turns out in the examples that the form of the efficient payoffs is largely independent of the chosen model and the magnitude of the efficiency loss is of similar size in all models and examples of options considered.

2.1 Lévy models

In this subsection we give a short description of the Lévy models used in the applications in this paper. For a detailed introduction to these models and their role in financial modeling we refer to Eberlein (2001) and Schoutens (2003).

NIG-model

The NIG model is a special case of the generalized hyperbolic model $GH(\lambda, \alpha, \beta, \delta, \mu)$ which is obtained by choosing $\lambda = -\frac{1}{2}$ and can be obtained as a normal mean-variance mixture with an inverse Gaussian mixing distribution. More specifically, if $X \sim NIG(\alpha, \beta, \delta, \mu)$, then X can be represented as

$$X \stackrel{d}{=} \mu + \beta Z + \sqrt{Z}\, W, \tag{2.1}$$

where $\mu \in \mathbb{R}$, $W \sim N(0,1)$, and $Z \sim IG(\delta, \sqrt{\alpha^2 - \beta^2})$ is an inverse Gaussian distributed random variable with $\delta > 0$ and $0 \leq |\beta| < \alpha$ that is independent of W. This representation also entails that the infinite divisibility of the mixing inverse Gaussian distribution transfers to the NIG mixture distribution, thus there exists a Lévy process $(L_t)_{t\geq 0}$ with $\mathcal{L}(L_1) = NIG(\alpha, \beta, \delta, \mu)$. The Lebesgue density $d_{NIG(\alpha,\beta,\delta,\mu)}$ is given by

$$\begin{aligned} d_{NIG(\alpha,\beta,\delta,\mu)}(x) &= \int_0^\infty d_{N(\mu+\beta y)}(x) d_{IG(\delta,\sqrt{\alpha^2-\beta^2})}(y)\,\mathrm{d}y \\ &= \mathsf{n}(\alpha,\beta,\delta)\frac{K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})}{\sqrt{\delta^2+(x-\mu)^2}}e^{\beta(x-\mu)}, \end{aligned} \tag{2.2}$$

where K_1 is the modified Bessel function of third kind with index 1, and the norming constant $\mathsf{n}(\alpha, \beta, \delta)$ is given by

$$\mathsf{n}(\alpha,\beta,\delta) = \frac{\alpha\delta}{\pi}e^{\delta\sqrt{\alpha^2-\beta^2}}.$$

The corresponding moment generating function $M_{NIG(\alpha,\beta,\delta,\mu)}$ is of the form

$$M_{NIG(\alpha,\beta,\delta,\mu)}(u) = \int_{-\infty}^\infty e^{ux} d_{NIG(\alpha,\beta,\delta,\mu)}(x)\,\mathrm{d}x \quad = e^{u\mu+\delta\left(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+u)^2}\right)} \tag{2.3}$$

which is defined for all $u \in (-\alpha-\beta, \alpha-\beta)$. The Esscher parameter $\overline{\theta}$ of the risk neutral Esscher measure $Q^{\overline{\theta}}$, i.e. the solution of (1.7) (if it exists) is given by

$$\bar{\theta}_{NIG} = -\frac{1}{2} - \beta + \frac{r-\mu}{\delta}\sqrt{\frac{\alpha^2}{1+(\frac{r-\mu}{\delta})^2} - \frac{1}{4}}.$$

Note that (L_t) remains a NIG Lévy process under $Q^{\overline{\theta}}$ with parameter β replaced by $\beta + \overline{\theta}$, $\overline{\theta} = \overline{\theta}_{NIG}$, i.e. w.r.t. $Q^{\overline{\theta}}$ holds: $L_1 \stackrel{d}{=} NIG(\alpha, \beta + \overline{\theta}, \delta, \mu)$.

Variance-Gamma model

A Variance-Gamma distributed random variable $X \sim VG(\lambda, \alpha, \beta, \mu)$ can be represented as a normal mean-variance mixture as in equation (2.1), but in this case the mixing variable $Z \sim \Gamma(\lambda, \frac{\alpha^2-\beta^2}{2})$ is Gamma distributed with shape parameter $\lambda > 0$ and scale parameter $\frac{\alpha^2-\beta^2}{2}$ where $0 \leq |\beta| < \alpha$. Again, the infinite divisibility of $\Gamma(\lambda, \frac{\alpha^2-\beta^2}{2})$ transfers to $VG(\lambda, \alpha, \beta, \mu)$. Analogously as above the corresponding Lebesgue density $d_{VG(\lambda,\alpha,\beta,\mu)}$ is given by

$$d_{VG(\lambda,\alpha,\beta,\mu)}(x) = \mathsf{m}(\lambda, \alpha, \beta)|x-\mu|^{\lambda-\frac{1}{2}} K_\lambda(\alpha|x-\mu|)e^{\beta(x-\mu)} \tag{2.4}$$

with the norming constant

$$\mathsf{m}(\lambda, \alpha, \beta) = \frac{(\alpha^2-\beta^2)^\lambda}{\sqrt{\pi}(2\alpha)^{\lambda-\frac{1}{2}}\Gamma(\lambda)},$$

and the moment generating function is of the form

$$M_{VG(\lambda,\alpha,\beta,\mu)}(u) = e^{u\mu}\frac{\mathsf{m}(\lambda, \alpha, \beta)}{\mathsf{m}(\lambda, \alpha, \beta+u)} = e^{u\mu}\Big(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+u)^2}\Big)^\lambda \tag{2.5}$$

which is defined for all $u \in (-\alpha-\beta, \alpha-\beta)$. Here we have

$$\lim_{u \to \pm\alpha-\beta} M_{VG(\lambda,\alpha,\beta,\mu)}(u) = \infty\,,$$

and as consequence the condition $2\alpha > 1$ is sufficient to guarantee a unique solution $\bar{\theta}$ of equation (1.7) in the VG case. Some lengthy calculations (see Wolf (2014)) show that the Esscher parameter $\bar{\theta}$ i.e. the solution of (1.7) is given by

$$\bar{\theta}_{VG} = \begin{cases} -\frac{1}{1-e^{-\frac{r-\mu}{\lambda}}} - \beta + \operatorname{sign}(r-\mu)\sqrt{\frac{e^{-\frac{r-\mu}{\lambda}}}{\left(1-e^{-\frac{r-\mu}{\lambda}}\right)^2} + \alpha^2}, & r \neq \mu, \\ -\frac{1}{2} - \beta, & r = \mu. \end{cases} \tag{2.6}$$

For $L_t \sim VG(\lambda t, \alpha, \beta, \mu t)$ the law of L_t under the Esscher martingale measure is again Variance-Gamma distributed $L_t \sim VG(\lambda t, \alpha, \beta + \bar{\theta}_{VG}, \mu t)$.

Samuelson model

The classical benchmark model which also is at the basis of the Black–Scholes theory is to assume that the stock price process $(S_0 e^{L_t})_{t\geq 0}$ follows a geometric Brownian motion. In this case, the driving Lévy process is given by

$$L_t = \big(\mu - \frac{\sigma^2}{2}\big)t + \sigma B_t,\; t > 0$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion under the physical measure P, μ is the drift and σ the volatility parameter. Here we have $\mathcal{L}(L_t) = N((\mu - \frac{\sigma^2}{2})t, \sigma^2 t)$, its Lebesgue density is given by

$$d_{L_t}(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{1}{2}\frac{(x-(\mu-\frac{\sigma^2}{2})t)^2}{\sigma^2 t}},$$

and the moment generating function of L_1 equals $M_{N(\mu-\frac{\sigma^2}{2},\sigma^2)}(u) = e^{u(\mu-\frac{\sigma^2}{2})+\frac{\sigma^2 u^2}{2}}$. The Esscher parameter $\bar{\theta}$ is a solution of $e^r = \frac{M_{L_1}(\bar{\theta}_N+1)}{M_{L_1}(\bar{\theta}_N)} = e^{\mu+\bar{\theta}_N\sigma^2}$ and is given by $\bar{\theta}_N = \frac{r-\mu}{\sigma^2}$.

Under the Esscher martingale measure $Q^{\bar{\theta}}$ holds $\quad L_t \sim N((r-\frac{\sigma^2}{2})t, \sigma^2 t)$.

2.2 Modeling of market data

We apply the Lévy models from (2.1) to model German stock price data for Allianz and Volkswagen and for E.ON and ThyssenKrupp from May 28, 2010, to September 28, 2012, which are shown in Figure 1 and in Figure 2 respectively. The estimated parameters and the corresponding Esscher parameter from the daily log-returns of Allianz and Volkswagen are given in Table 1 and of E.ON and ThyssenKrupp in Table 2. The fitted densities in the three Lévy models are displayed in Figure 3 for Allianz and Volkswagen and in Figure 4 for E.ON and ThyssenKrupp. It stands out that the normal density curve fits worse than the *NIG* and *VG* density. The interest rate used to calculate the Esscher parameter $\bar{\theta}$ is $r = 4.2027 \cdot 10^{-6}$ which corresponds to the continuously compounded daily-Euribor rate of October 1, 2012. From the estimated Esscher parameter we find that the Allianz and Volkswagen data have a positive trend while the ThyssenKrupp and E.On data have a slight negative trend.

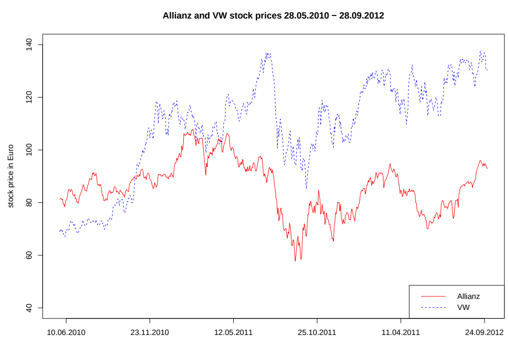

Figure 1: Daily closing prices of Allianz and Volkswagen used for parameter estimation.

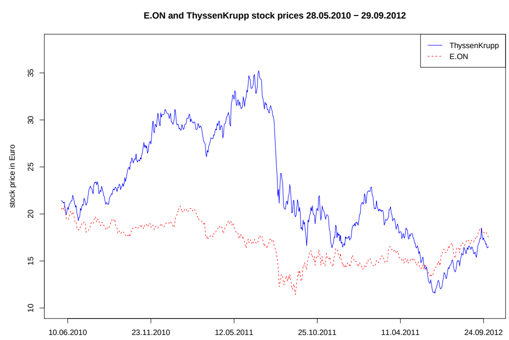

Figure 2: Daily closing prices of E.ON and ThyssenKrupp used for parameter estimation.

3 Cost-efficient payoffs for monotone and nonmonotone options

In this section we apply the results on cost-efficiency to a series of options in the class of Lévy models in Section 2 and apply them to the market data introduced and modeled in Section 2.2. The examples give a good impression on the magnitude of efficiency loss in terms of the parameters and shows that this methodology is also applicable to the improvement of nonmonotone options where calculations typically have to be done numerically.

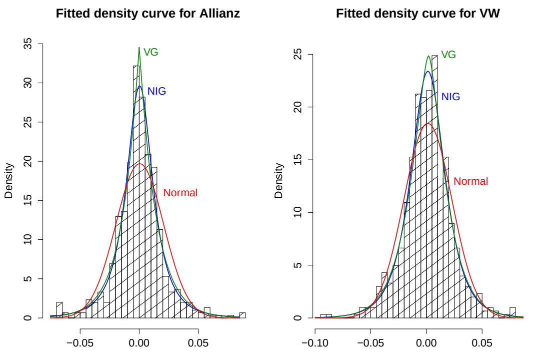

Figure 3: Fitted density curves for Allianz and Volkswagen.

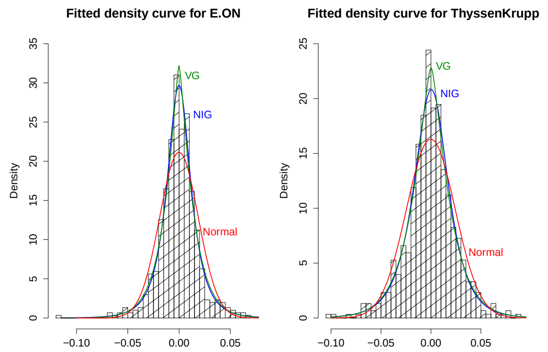

Figure 4: Fitted density curves for E.ON and ThyssenKrupp.

Allianz	λ	α	β	δ	μ	$\bar{\theta}$
NIG	-0.5	35.01998	-0.36857	0.01478	0.000376	-1.01266
VG	1.03086	72.01061	0.55168	0.0	$1.941 \cdot 10^{-8}$	-1.04116
Normal	$\mu = 4.2757 \cdot 10^{-4}$, $\sigma = 0.02026$					-1.03143
Volkswagen	λ	α	β	δ	μ	$\bar{\theta}$
NIG	-0.5	48.85903	-0.84151	0.02313	0.001451	-2.70867
VG	1.60198	82.94782	-2.16537	0.0	0.00206	-2.73948
Normal	$\mu = 0.00129$, $\sigma = 0.02162$					-2.74475

Table 1: Estimated parameters from daily log-returns of Allianz and Volkswagen for the NIG-, the VG-, and the Samuelson model.

E.ON	λ	α	β	δ	μ	$\bar{\theta}$
NIG	-0.5	44.831	-0.639	0.016	$-5.25 \cdot 10^{-5}$	0.297816
VG	1.276	86.399	-0.63	0.0	$-6.17 \cdot 10^{-5}$	0.322992
Normal	$\mu = -0.0001$, $\sigma = 0.018878$					0.293082
ThyssenKrupp	λ	α	β	δ	μ	$\bar{\theta}$
NIG	-0.5	42.01665	-2.08815	0.02554	0.000846	0.203533
VG	1.43896	69.05434	-0.92983	0.0	0.000137	0.210135
Normal	$\mu = -0.000128$, $\sigma = 0.02447$					0.220797

Table 2: Estimated parameters from daily log-returns of E.ON and ThyssenKrupp for the NIG-, the VG-, and the Samuelson model.

3.1 Put options

(Long) put options are inefficient in increasing markets where $\bar{\theta} < 0$. Thus calculation of cost-efficient options is only of interest in the Volkswagen (VW), Allianz (Al) examples. We start with an example which was already analyzed in Hammerstein et al. (2014). We give a short presentation of these results, extend them in various respect and compare with the following options.

For a put option with strike K and maturity $T > 0$, i.e.

$$X_T^{\mathrm{Put}} = (K - S_T)_+ = (K - S_0 e^{L_T})_+ \tag{3.1}$$

the payoff distribution is given by

$$G_{\text{Put}}(x) = P\big(X_T^{\text{Put}} \le x\big) = \begin{cases} 1, & \text{if } x \ge K, \\ 1 - F_{L_T}\big(\ln\big(\frac{K-x}{S_0}\big)\big), & \text{if } 0 \le x < K, \\ 0, & \text{if } x < 0. \end{cases} \tag{3.2}$$

Applying Proposition 1.2 for $\bar{\theta} < 0$ the cost-efficient payoff that generates the same distribution G_{Put} as the long put is given by

$$\underline{X}_T^{\text{Put}} = G_{\text{Put}}^{-1}(F_{L_T}(L_T)) = \big(K - S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))}\big)_+ \tag{3.3}$$

with payoff function $\underline{\omega}^{\text{Put}}(y) := \big(K - S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(\ln(\frac{y}{S_0})))}\big)_+$.

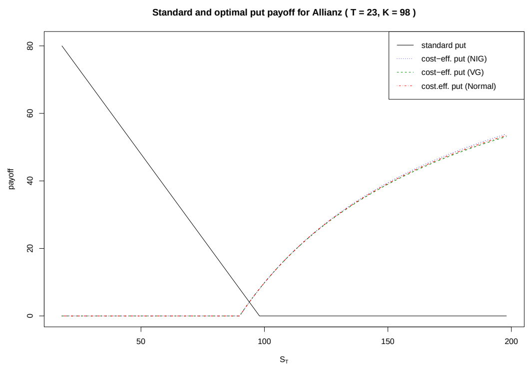

Figure 5: Classical put and its cost-efficient counterparts for Allianz. $S_0 = 93.42$, closing price October 1, 2012.

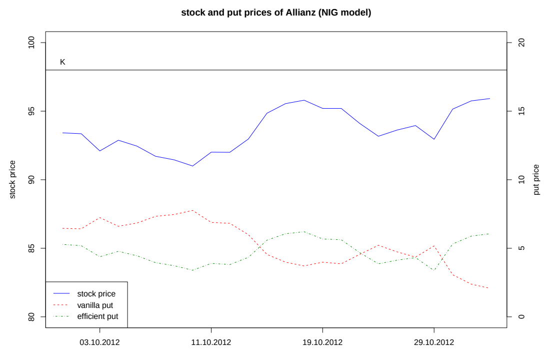

Figure 6: Stock and put prices along the period $[0, T]$ for Allianz strike $K = 98$, maturity $T = 23$ days.

Allianz	$c(X_T^{\text{Put}})$	$c(\underline{X}_T^{\text{Put}})$	Efficiency loss in %
NIG	6.4495	5.2825	18.09
VG	6.3681	5.2270	17.92
Normal	6.4324	5.2683	18.10
Volkswagen	$c(X_T^{\text{Put}})$	$c(\underline{X}_T^{\text{Put}})$	**Efficiency loss in %**
NIG	8.0064	4.0871	48.95
VG	7.9765	4.0603	49.10
Normal	7.9909	4.0749	49.01

Table 3: Comparison of the cost of a long put option on Allianz and Volkswagen, resp., and the corresponding cost-efficient payoffs in different Lévy models. $S_0 = 93.42$, $K = 98$, $T = 23$ for Allianz and $S_0 = 130.55$, $K = 135$, $T = 23$ for Volkswagen.

Figure 5 displays the payoff X_T^{Put} of a long put option on one Allianz stock with strike $K = 98$ and maturity $T = 23$ days, and its cost-efficient counterparts $\underline{X}_T^{\text{Put}}$ for the three Lévy models under consideration. Although the payoff profiles look quite similar, a closer look reveals that the optimal payoff is model-dependent and slightly varies between the different models.

Note that the cost-efficient long put payoff is increasing and is bounded by K as is its vanilla counterpart, as follows from

$$\lim_{S_T\to\infty} \underline{X}_T^{\text{Put}} = \lim_{S_T\to\infty} \big(K - S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(\ln(\frac{S_T}{S_0})))}\big)_+ = K.$$

In Table 3 we compare the cost of a long put option on Allianz and Volkswagen with their cost-efficient counterparts for the Lévy models discussed in Section 2. All computations are based on the estimated parameters given in Table 1 above. The initial stock prices S_0 of Allianz resp. Volkswagen are the closing prices at October 1, 2012, and the time to maturity is chosen to be $T = 23$ trading days, meaning that the put options mature on November 1, 2012. According to Proposition 1.2, the cost of the efficient put can be calculated by

$$c\big(\underline{X}_T^{\text{Put}}\big) = E\big[e^{-rT} Z_T^{\bar\theta} \underline{X}_T^{\text{Put}}\big] = \frac{1}{M_{dist}(\bar\theta)} \int_0^1 e^{\bar\theta F_{dist}^{-1}(1-y) - rT} \big(K - S_0 e^{F_{dist}^{-1}(y)}\big)_+ \,\mathrm{d}y \tag{3.4}$$

where $dist$ is $NIG(\alpha, \beta, \delta T, \mu T)$, $VG(\lambda T, \alpha, \beta, \mu T)$, or $N((\mu - \frac{\sigma^2}{2})T, \sigma^2 T)$.

The cost $c(X_T^{\text{Put}})$ of the vanilla put in the NIG model is given by

$$\begin{aligned} c(X_T^{\text{Put}}) &= E_{\bar\theta}\big[e^{-rT}(K - S_T)_+\big] \\ &= e^{-rT} \int_{-\infty}^{\ln(K/S_0)} (K - S_0 e^x) Z_T^{\bar\theta,x}\, d_{NIG(\alpha,\beta,\delta T,\mu T)}(x) \,\mathrm{d}x \\ &= e^{-rT} K F_{NIG(\alpha,\beta+\bar\theta,\delta T,\mu T)}\big(\ln\big(\tfrac{K}{S_0}\big)\big) - S_0 F_{NIG(\alpha,\beta+\bar\theta+1,\delta T,\mu T)}\big(\ln\big(\tfrac{K}{S_0}\big)\big), \end{aligned} \tag{3.5}$$

where $Z_T^{\bar\theta,x} = \frac{e^{\bar\theta x}}{M_{NIG(\alpha,\beta,\delta T,\mu T)}(\bar\theta)}$. For the VG model one analogously obtains

$$c(X_T^{\text{Put}}) = e^{-rT} K F_{VG(\lambda T,\alpha,\beta+\bar\theta,\mu T)}\big(\ln\big(\tfrac{K}{S_0}\big)\big) - S_0 F_{VG(\lambda T,\alpha,\beta+\bar\theta+1,\mu T)}\big(\ln\big(\tfrac{K}{S_0}\big)\big). \tag{3.6}$$

In the Samuelson model, $c(X_T^{\text{Put}})$ is calculated by the well-known Black–Scholes put price formula.

In an exponential Lévy model with $L_T \overset{d}{=} NIG(\alpha, \beta, \delta T, \mu T)$ or $L_T \overset{d}{=} VG(\lambda T, \alpha, \beta, \mu T)$ and with Esscher parameter $\overline{\theta}$ we have

$$c(X_T^{\text{Put}}) = e^{-rT} K F_{L_T^{\bar\theta}}\Big(\ln\Big(\frac{K}{S_0}\Big)\Big) - S_0 F_{L_T^{\bar\theta+1}}\Big(\ln\Big(\frac{K}{S_0}\Big)\Big), \tag{3.7}$$

where $L_T^{\bar\theta+k} \overset{d}{=} NIG(\alpha, \beta + \bar\theta + k, \delta T, \mu T)$ or $VG(\lambda T, \alpha, \beta + \bar\theta + k, \mu T)$, $k = 0, 1$. If $L_T \overset{d}{=} N((\mu - \frac{\sigma^2}{2})T, \sigma^2 T)$, then we have

$$c(X_T^{\text{Put}}) = e^{-rT} K \Phi(h) - S_0 \Phi(h - \sigma\sqrt{T}), \tag{3.8}$$

where $h = \frac{1}{\sigma\sqrt{T}}(\ln(\frac{K}{S_0}) - (r - \frac{\sigma^2}{2})T)$.

For symmetric Lévy processes which fulfill $L_T \overset{d}{=} vT - L_T$, where $v \in \mathbb{R}$, the cost of the cost-efficient put (3.4) can be evaluated without any integration. Numerical computations of prices, then, become a lot easier. The price formula for the cost-efficient put in the Samuelson model is given in Bernard et al. (2014, Section 5.2). Some similar calculations yield in the Lévy case the following result (for details see Wolf (2014)).

Proposition 3.1 (Price of efficient puts in symmetric Lévy models) *Let X_T^{Put} be the payoff of a vanilla put option with strike K, maturity $T > 0$. Suppose $(L_t)_{t\geq 0}$ is a*

Lévy process such that $L_T \overset{d}{=} vT - L_T$. If $\bar{\theta}$ is an Esscher parameter, then the cost of the cost-efficient put $\underline{X}_T^{\text{Put}}$ w.r.t. the Esscher measure is given by $c(X_T^{\text{Put}})$ if $\bar{\theta} > 0$ and by

$$\begin{aligned} c(\underline{X}_T^{\text{Put}}) &= e^{-rT} K\Big(1 - F_{L_T^{\bar{\theta}}}\big(\ln(\frac{S_0}{K}) + vT\big)\Big) \\ &\quad - e^{-(r-v)T} S_0 \frac{M_{L_T}(\bar{\theta}-1)}{M_{L_T}(\bar{\theta})}\Big(1 - F_{L_T^{\bar{\theta}-1}}\big(\ln(\frac{S_0}{K}) + vT\big)\Big) \end{aligned} \tag{3.9}$$

if $\bar{\theta} < 0$. Here $L_T^{\bar{\theta}}$ denotes the Lévy process at maturity under the Esscher measure $Q^{\bar{\theta}}$. In particular, in the Samuelson model we have that $L_T \overset{d}{=} 2(\mu - \frac{\sigma^2}{2})T - L_T$. Thus for $\bar{\theta} < 0$

$$c(\underline{X}_T^{\text{Put}}) = e^{-rT} K\Phi(\underline{h}) - e^{2(\mu-r)T} S_0 \Phi(\underline{h} - \sigma\sqrt{T}) \tag{3.10}$$

where $\underline{h} = \frac{1}{\sigma\sqrt{T}}\big(\ln(\frac{K}{S_0}) - (\mu - \frac{\sigma^2}{2})T + (r-\mu)T\big)$.

The results from Table 3 show that the savings from choosing the cost-efficient strategies can be quite large: For Allianz, the cost of the efficient put is less than 83% of the price of the plain vanilla put, and in case of Volkswagen the vanilla put is almost twice as expensive as the efficient put. The great differences in the efficiency losses of the Allianz and Volkswagen puts may seem somewhat surprising at first glance because the stock price to strike ratio $\frac{S_0}{K}$ is roughly the same in both cases (0.953 for Allianz and 0.967 for Volkswagen). But the difference is induced by the greater magnitude of positive trend in the VW data compared to Allianz as seen from Table 2. The value of $|\bar{\theta}|$ for Volkswagen is more than 2.5 times as large as that of Allianz. For each stock itself the efficiency losses obtained under the different Lévy models are of almost the same size and thus seem to be widely model-independent.

In contrast to the latter static formulas and results we also consider time dynamic behavior of the cost of the cost-efficient payoff. Therefore, we keep the payoff function $\underline{\omega}^{\text{Put}}(y) = \big(K - S_0 e^{F_{L_T}^{-1}(1 - F_{L_T}(\ln(\frac{y}{S_0})))}\big)_+$ of the cost-efficient long put fixed within the trading period $[0, T]$. The payoff function $\underline{\omega}^{\text{Put}}$ depends on S_0 which becomes a location parameter in this context. In consequence, the price at time $t < T$ of a cost-efficient long put with maturity T is given by

$$c_t(\underline{X}_T^{\text{Put}}) = e^{-r(T-t)} E\Big[Z_{T-t}^{\bar{\theta}}\big(K - S_0 e^{F_{L_T}^{-1}(1 - F_{L_T}\left(\ln(\frac{y}{S_0}) + L_{T-t}\right))}\big)_+\Big]\Big|_{y=S_t}. \tag{3.11}$$

In Figure 6 we notice that during some time in the trading period $[0, T]$ the cost of the cost-efficient long put exceeds the cost of the plain vanilla long put. For the case of the Allianz stock this is even beneficial for writers of the cost-efficient long put $\underline{X}_T^{\text{Put}}$ since at maturity, November 1, 2012, the higher price corresponds to a higher payout. However, we could also have the reverse situation. In other words, an initially optimal strategy may become less profitable as its vanilla counterpart if the market scenario significantly changes in between.

Although, the cost-efficient put behaves like a modified call, i.e. it is increasing in L_T, both X_T^{Put} and $\underline{X}_T^{\text{Put}}$ end up in the money, whereas a plain vanilla call would expire worthless. But besides this abnormal behavior the progression of the cost of the long put and its cost-efficient counterpart exhibit similar price behavior as one would expect from vanilla long call and put options.

Remark 3.2 (Short put option) *Similarly for the short put* $X_T^{-\text{Put}} = -(K - S_0 e^{L_T})_+$ *which is inefficient for* $\underline{\theta} > 0$ *the cost-efficient version is given by*

$$\underline{X}_T^{-\text{Put}} = G_{-\text{Put}}^{-1}(1 - F_{L_T}(L_T)) = \left(S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))} - K\right)_- . \tag{3.12}$$

and we obtain by simple arguments the following duality relation.
For $\bar{\theta} < 0$ *holds*

$$\underline{X}_T^{\text{Put}} = -\overline{X}_T^{-\text{Put}} \text{ and } c\big(\underline{X}_T^{\text{Put}}\big) = -c\big(\overline{X}_T^{-\text{Put}}\big). \tag{3.13}$$

Similarly, if $\bar{\theta} > 0$ *we have*

$$\underline{X}_T^{-\text{Put}} = -\overline{X}_T^{\text{Put}} \quad \text{as well as} \quad c\big(\underline{X}_T^{-\text{Put}}\big) = -c\big(\overline{X}_T^{\text{Put}}\big) .$$

3.2 Call options

Call options are inefficient in decreasing markets i.e. when $\bar{\theta} > 0$. For the call $X_T^{\text{Call}} = (S_T - K)_+ = (S_0 e^{L_T} - K)_+$ with payoff function $\omega^{\text{Call}}(y) := (y - K)_+$ we obtain the payoff distribution function $G_{\text{Call}} = F_{X_T^{\text{Call}}}$ by

$$G_{\text{Call}}(x) = P\big(X_T^{\text{Call}} \le x\big) = \begin{cases} F_{L_T}\big(\ln\big(\frac{K+x}{S_0}\big)\big), & \text{if } x \ge 0, \\ 0, & \text{if } x < 0. \end{cases} \tag{3.14}$$

Applying Proposition 1.2 for $\bar{\theta} > 0$ the cost-efficient payoff that generates the same distribution G_{Call} as the long call option is given by

$$\underline{X}_T^{\text{Call}} = G_{\text{Call}}^{-1}(1 - F_{L_T}(L_T)) = \left(S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))} - K\right)_+ \tag{3.15}$$

with payoff function $\underline{\omega}^{\text{Call}}(y) := (S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(\ln(\frac{y}{S_0})))} - K)_+$.

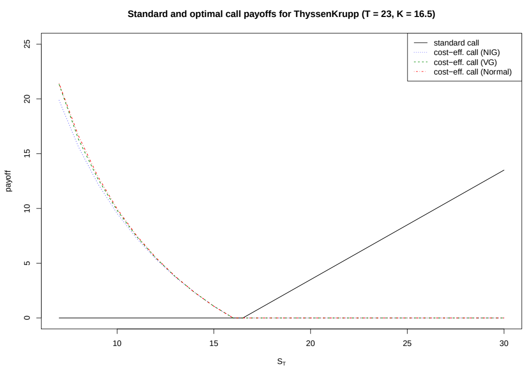

Figure 7: Classical call and its cost-efficient counterparts for ThyssenKrupp. $S_0 = 16.73$, closing price October 1, 2012.

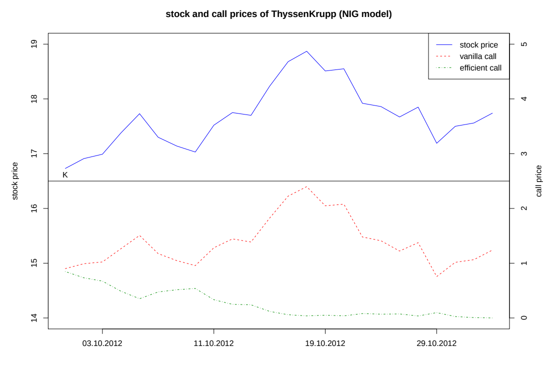

Figure 8: Stock and call prices along the period $[0, T]$ for ThyssenKrupp, strike $K = 16.5$, maturity $T = 23$ days.

Figure 7 displays the payoff X_T^{Call} of a long call option on one ThyssenKrupp stock with strike $K = 16.5$ and maturity $T = 23$ days, and its cost-efficient counterparts $\underline{X}_T^{\text{Call}}$ for the three Lévy models under consideration. As seen before the optimal payoff is model-dependent and slightly varies between the different models.

Next, we state formulas for the cost of the standard call in the three Lévy models. Let $(L_t)_{t\geq 0}$ be a Lévy process, with $\mathcal{L}(L_T) = NIG(\alpha, \beta, \delta T, \mu T)$ or $VG(\lambda T, \alpha, \beta, \mu T)$. If $\bar{\theta}$ is a Esscher parameter for L, then we have

$$c(X_T^{\text{Call}}) = S_0\Big(1 - F_{L_T^{\bar{\theta}+1}}\Big(\ln\Big(\frac{K}{S_0}\Big)\Big)\Big) - e^{-rT}K\Big(1 - F_{L_T^{\bar{\theta}}}\Big(\ln\Big(\frac{K}{S_0}\Big)\Big)\Big), \tag{3.16}$$

where $\mathcal{L}(L_T^{\bar{\theta}+k})$ is $NIG(\alpha, \beta + \bar{\theta} + k, \delta T, \mu T)$ or $VG(\lambda T, \alpha, \beta + \bar{\theta} + k, \mu T)$, $k = 0, 1$. If $\mathcal{L}(L_T) = N((\mu - \frac{\sigma^2}{2})T, \sigma^2 T)$, then we have

$$c(X_T^{\text{Call}}) = S_0\Phi(-h + \sigma\sqrt{T}) - e^{-rT}K\Phi(-h), \tag{3.17}$$

where $h = \frac{1}{\sigma\sqrt{T}}(\ln(\frac{K}{S_0}) - (r - \frac{\sigma^2}{2})T)$.

Similarly to the case of a put in (3.1) also for a call a simple formula can be given for cost-efficient calls for symmetric Lévy models which fulfill $L_T \stackrel{d}{=} vT - L_T$, $v \in \mathbb{R}$

Proposition 3.3 (Price of efficient calls in symmetric Lévy models)
Suppose $(L_t)_{t\geq 0}$ is a Lévy process such that $\mathcal{L}(L_T) = \mathcal{L}(vT - L_T)$. If $\bar{\theta}$ is a Esscher parameter, then the cost of the cost-efficient call $\underline{X}_T^{\text{Call}}$ equals

$$\begin{aligned} c(\underline{X}_T^{\text{Call}}) = e^{-(r-v)T} S_0 \frac{M_{L_T}(\bar{\theta} - 1)}{M_{L_T}(\bar{\theta})} F_{L_T^{\bar{\theta}-1}}\Big(\ln\Big(\frac{S_0}{K}\Big) + vT\Big) \\ - e^{-rT} K F_{L_T^{\bar{\theta}}}\Big(\ln\Big(\frac{S_0}{K}\Big) + vT\Big) \end{aligned} \tag{3.18}$$

if $\bar{\theta} > 0$, where $L_T^{\bar{\theta}}$ denotes the Lévy process at maturity under the Esscher measure $Q^{\bar{\theta}}$. In particular, in the Samuelson model we have that $L_T \stackrel{d}{=} 2(\mu - \frac{\sigma^2}{2})T - L_T$, thus

$$c(\underline{X}_T^{\text{Call}}) = e^{2(\mu-r)T} S_0 \Phi(-\underline{h} + \sigma\sqrt{T}\,) - e^{-rT} K\Phi(-\underline{h}) \tag{3.19}$$

if $\bar{\theta} > 0$, where $\underline{h} = \frac{1}{\sigma\sqrt{T}}\big(\ln(\frac{K}{S_0}) - (\mu - \frac{\sigma^2}{2})T + (r - \mu)T\big)$.

E.ON	$c(X_T^{\text{Call}})$	$c(\underline{X}_T^{\text{Call}})$	**Efficiency loss in %**
NIG	0.7502	0.7018	6.45
VG	0.7398	0.6893	6.83
Normal	0.7550	0.7073	6.32
Thyssen	$c(X_T^{\text{Call}})$	$c(\underline{X}_T^{\text{Call}})$	**Efficiency loss in %**
NIG	0.9016	0.8484	5.90
VG	0.8989	0.8443	6.07
Normal	0.8987	0.8418	6.33

Table 4: Comparison of the cost of a long call option on E.ON and ThyssenKrupp, resp., and the corresponding cost-efficient payoffs in different Lévy models. $S_0 = 17.48$, $K = 17.24$, $T = 23$ for E.ON and $S_0 = 16.73$, $K = 16.5$, $T = 23$ for ThyssenKrupp.

In Table 4 above we compare the cost of a long call option on E.ON and ThyssenKrupp with their cost-efficient counterparts for the Lévy models discussed in Section 2. The results from Table 4 show that the savings from choosing the cost-efficient strategies are close to each other as is the magnitude of $|\bar{\theta}|$ for ThyssenKrupp and E.ON (compare Table 2). The time dynamic behavior of the cost-efficient call compared to call is displayed in Figure 8. There we notice that the formerly bearish market setting of the ThyssenKrupp changed to a bullish market setting, since the drift of the stock price altered its direction from negative to positive (cf. Figure 2). Moreover, the stock price remains above the strike during the entire trading period $[0, T]$. As the cost-efficient long call behaves like an modified put, it decreases over $[0, T]$ and expires worthless. Indeed, this is an unpropitious example for writers of the cost-efficient long call $\underline{X}_T^{\text{Call}}$.

Again as in the put case we get the following symmetry relation between long and short calls: If $\bar{\theta} > 0$, it holds that

$$\underline{X}_T^{\text{Call}} = -\overline{X}_T^{-\text{Call}} \text{ and } c\big(\underline{X}_T^{\text{Call}}\big) = -c\big(\overline{X}_T^{-\text{Call}}\big).$$

Similarly, if $\bar{\theta} < 0$ we have $\underline{X}_T^{-\text{Call}} = -\overline{X}_T^{\text{Call}}$ as well as $c\big(\underline{X}_T^{-\text{Call}}\big) = -c\big(\overline{X}_T^{\text{Call}}\big)$.

3.3 Self-quanto calls and puts

A quanto option is a (typically European) option whose payoff is converted into a different currency or numeraire at maturity at a pre-specified rate, called the quanto-factor. Such products are attractive for speculators and investors who wish to have exposure to a foreign asset, but without the corresponding exchange rate risk. Quanto options are attractive because they shield the purchaser from exchange rate fluctuations. In the special case of a self-quanto option the numeraire is the underlying asset price at maturity itself. The payoff of a long self-quanto call with maturity T and strike price K is

$$X_T^{\text{sqC}} = S_T \cdot (S_T - K)_+ = S_0 e^{L_T} (S_0 e^{L_T} - K)_+$$

which is monotonically increasing in L_T and thus not cost-efficient if $\bar{\theta} > 0$. Its payoff function is then given by $\omega^{\text{sqC}}(y) := y(y - K)_+$. To derive the corresponding distribution function $G_{\text{sqC}} = F_{X_T^{\text{sqC}}}$, observe that the positive solution S_T^* of the quadratic equation $S_T^2 - K S_T = x$, $x > 0$, is given by

$$S_T^* = \frac{K}{2} + \sqrt{\frac{K^2}{4} + x},$$

then $\{S_T^2 - K S_T - x \leq 0\} = \{S_T \leq S_T^*\}$, hence

$$G_{\text{sqC}}(x) = P\big(X_T^{\text{sqC}} \leq x\big) = \begin{cases} F_{L_T}\left(\ln\left(\frac{\frac{K}{2} + \sqrt{\frac{K^2}{4} + x}}{S_0}\right)\right), & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

The generalized inverse is given by

$$G_{\text{sqC}}^{-1}(y) = S_0 e^{F_{L_T}^{-1}(y)} \big(S_0 e^{F_{L_T}^{-1}(y)} - K\big)_+, \quad y \in (0, 1)\,. \tag{3.20}$$

Consequently according to Proposition 1.2 the cost-efficient strategy for a long self-quanto call in the case $\bar{\theta} > 0$ is,

$$\underline{X}_T^{\text{sqC}} = G_{\text{sqC}}^{-1}(1 - F_{L_T}(L_T)) = S_0 e^{F_{L_T}^{-1}(1 - F_{L_T}(L_T))} \big(S_0 e^{F_{L_T}^{-1}(1 - F_{L_T}(L_T))} - K\big)_+ \tag{3.21}$$

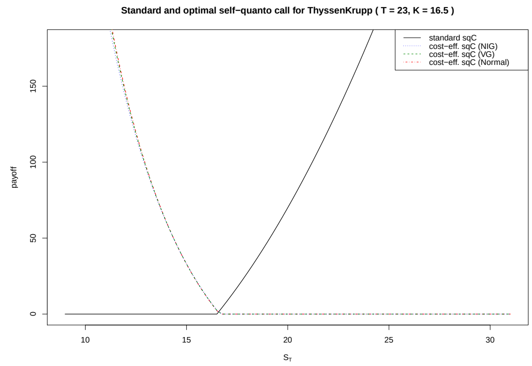

Figure 9: Classical self-quanto call and its cost-efficient counterparts for ThyssenKrupp. $S_0 = 16.73$

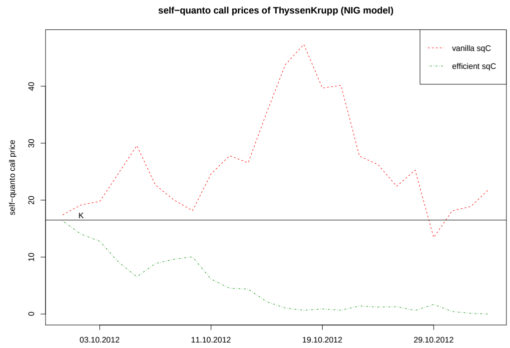

Figure 10: Cost of a classical self-quanto call and its cost-efficient counterpart for ThyssenKrupp.

with payoff function $\underline{\omega}^{\text{sqC}}(y) := S_0 e^{F_{L_T}^{-1}\left(1-F_{L_T}(\ln(\frac{y}{S_0}))\right)}\left(S_0 e^{F_{L_T}^{-1}\left(1-F_{L_T}(\ln(\frac{y}{S_0}))\right)} - K\right)_+$.

Figure 9 displays the payoff X_T^{sqC} of a long self-quanto call option on one ThyssenKrupp stock with strike $K = 16.5$ and maturity $T = 23$ days, and its cost-efficient counterparts $\underline{X}_T^{\text{sqC}}$ for the three Lévy models under consideration.

The cost of the efficient self-quanto call can be calculated using (3.20),

$$c(\underline{X}_T^{\text{sqC}}) = \frac{1}{M_{dist}(\bar{\theta})} \int_0^1 e^{\bar{\theta} F_{dist}^{-1}(y) - rT} S_0 e^{F_{dist}^{-1}(1-y)} \left(S_0 e^{F_{dist}^{-1}(1-y)} - K\right)_+ \mathrm{d}y \tag{3.22}$$

where *dist* is $NIG(\alpha, \beta, \delta T, \mu T)$, $VG(\lambda T, \alpha, \beta, \mu T)$, or $N((\mu - \frac{\sigma^2}{2})T, \sigma^2 T)$.

If $\bar{\theta}$ is an Esscher parameter and $M_{L_T}(\bar{\theta} + 2) < \infty$, then

$$c(X_T^{\text{sqC}}) \leq e^{-rT} E_{\bar{\theta}}[S_T^2] = e^{-rT} S_0^2 \frac{M_{L_T}(\bar{\theta} + 2)}{M_{L_T}(\bar{\theta})} < \infty. \tag{3.23}$$

In general this holds true for the Samuelson model, since the moment generating function of L_1, $M_{N((\mu - \frac{\sigma^2}{2}), \sigma^2)}(u)$ is defined for all $u \in \mathbb{R}$. For the *NIG* resp. *VG* model

$$\bar{\theta} + 2 \in (-\alpha - \beta, \alpha - \beta) \text{ and } \bar{\theta} \in (-\alpha - \beta, \alpha - \beta) \tag{3.24}$$

which implies that $\bar{\theta} \in (-\alpha - \beta, \alpha - \beta - 2)$. All estimated parameters from the daily log returns of E.ON and ThyssenKrupp from Table 2 fulfill this condition as well as equation (3.24). We get the following pricing formula.

Proposition 3.4 (Price of a vanilla self-quanto call) *Let $(L_t)_{t \geq 0}$ be a Lévy process, such that $L_T \overset{d}{=} NIG(\alpha, \beta, \delta T, \mu T)$ or $VG(\lambda T, \alpha, \beta, \mu T)$. If $M_{L_T}(\bar{\theta} + 2) < \infty$, where $\bar{\theta}$ is a Esscher parameter, then we have*

$$c(X_T^{\text{sqC}}) = \frac{M_{L_T}(\bar{\theta} + 2)}{M_{L_T}(\bar{\theta} + 1)} S_0^2 \left(1 - F_{L_T^{\bar{\theta}+2}}\left(\ln\left(\frac{K}{S_0}\right)\right)\right) - S_0 K \left(1 - F_{L_T^{\bar{\theta}+1}}\left(\ln\left(\frac{K}{S_0}\right)\right)\right) \tag{3.25}$$

where $L_T^{\bar{\theta}+k} \overset{d}{=} NIG(\alpha, \beta + \bar{\theta} + k, \delta T, \mu T)$ or $VG(\lambda T, \alpha, \beta + \bar{\theta} + k, \mu T)$, $k = 0, 1, 2$.

If $L_T \overset{d}{=} N((\mu - \frac{\sigma^2}{2})T, \sigma^2 T)$, *then we have*

$$c(X_T^{\mathrm{sqC}}) = e^{(r+\sigma^2)T} S_0^2 \Phi\big(-h' + \sigma\sqrt{T}\big) - S_0 K \Phi\big(-h'\big)\,, \tag{3.26}$$

where $h' = \frac{1}{\sigma\sqrt{T}}(\ln(\frac{K}{S_0}) - (r + \frac{\sigma^2}{2})T)$.

For details of the proof we refer to Wolf (2014).

Remark 3.5 (symmetric Lévy case) *In the case of symmetric Lévy processes where* $L_T \overset{d}{=} \theta T - L_T$ *for some* $\theta \in \mathbb{R}$ *the formulas for the cost of efficient self-quanto calls simplify to*

$$\begin{aligned} c(\underline{X}_T^{\mathrm{sqC}}) = e^{vT} S_0 \Big(& e^{-(r-v)T} S_0 \frac{M_{L_T}(\bar\theta - 2)}{M_{L_T}(\bar\theta)} F_{L_T^{\bar\theta - 2}} \Big(\ln\Big(\frac{S_0}{K}\Big) + vT\Big) \\ & - e^{-rT} K \frac{M_{L_T}(\bar\theta - 1)}{M_{L_T}(\bar\theta)} F_{L_T^{\bar\theta - 1}} \Big(\ln\Big(\frac{S_0}{K}\Big) + vT\Big)\Big) \end{aligned} \tag{3.27}$$

where $L_T^{\bar\theta}$ *denotes the Lévy process at maturity under the Esscher measure* $Q^{\bar\theta}$.

In particular, in the Samuelson model we have that $L_T \overset{d}{=} 2(\mu - \frac{\sigma^2}{2})T - L_T$, *thus*

$$c(\underline{X}_T^{\mathrm{sqC}}) = e^{2(\mu-r)T} S_0 \Big(e^{-rT} e^{2(\mu + \frac{\sigma^2}{2})T} S_0 \Phi(-h + \sigma\sqrt{T}) - K\Phi(-h)\Big) \tag{3.28}$$

where $h = \frac{1}{\sigma\sqrt{T}}\big(\ln(\frac{K}{S_0}) - (\mu + \frac{\sigma^2}{2})T + (r-\mu)T\big)$.

We display the cost of a long self-quanto call option on E.ON and ThyssenKrupp with

E.ON	$c(X_T^{\mathrm{sqC}})$	$c(\underline{X}_T^{\mathrm{sqC}})$	**Efficiency loss in %**
NIG	14.6161	13.6394	6.68
VG	14.3741	13.3613	7.05
Normal	14.6988	13.7397	6.53
Thyssen	$c(X_T^{\mathrm{sqC}})$	$c(\underline{X}_T^{\mathrm{sqC}})$	**Efficiency loss in %**
NIG	17.4182	16.3441	6.17
VG	17.3619	16.2628	6.50
Normal	17.3394	16.1980	6.58

Table 5: Comparison of the cost of a long self-quanto call option on E.ON and ThyssenKrupp, resp., and the corresponding cost-efficient payoffs. $S_0 = 17.48$, $K = 17.24$, $T = 23$ for E.ON and $S_0 = 16.73$, $K = 16.5$, $T = 23$ for ThyssenKrupp.

their cost-efficient counterparts for the three Lévy models under consideration in Table 5. Again, we emphasize that the relative efficiency loss of the self-quanto option on E.ON and ThyssenKrupp has almost the same size. The same is true for the corresponding Esscher parameter $\bar\theta$.

Utilizing Proposition 1.2 and the explicit formula of the payoff function $\underline{\omega}^{\mathrm{sqC}}$ the price at time $t < T$ of a cost-efficient long call with maturity T can be computed as

$$\begin{aligned} c_t(\underline{X}_T^{\mathrm{sqC}}) = e^{-r(T-t)} E\Big[& Z_{T-t}^{\bar\theta}\, S_0 e^{F_{L_T}^{-1}(1 - F_{L_T}\big(\ln(\frac{y}{S_0}) + L_{T-t})\big)} \cdot \\ & \big(S_0 e^{F_{L_T}^{-1}(1 - F_{L_T}\big(\ln(\frac{y}{S_0}) + L_{T-t})\big)} - K\big)_+\Big]\,\Big|_{y=S_t}. \end{aligned}$$

From Figure 10 the leverage effect of the self-quanto call payoff is clearly recognizable in comparison to the standard long call payoff. The peaks and lows are more pronounced then in the vanilla call case (compare Figure 8).

Again, we have the symmetry relation $G^{-1}_{\text{sqC}}(y) = -G^{-1}_{-\text{sqC}}(1-y)$ and conclude:
If $\bar{\theta} > 0$, then $\underline{X}_T^{\text{sqC}} = -\overline{X}_T^{-\text{sqC}}$ and $c\big(\underline{X}_T^{\text{sqC}}\big) = -c\big(\overline{X}_T^{-\text{sqC}}\big)$.
If $\bar{\theta} < 0$ then $\underline{X}_T^{-\text{sqC}} = -\overline{X}_T^{\text{sqC}}$ as well as $c\big(\underline{X}_T^{-\text{sqC}}\big) = -c\big(\overline{X}_T^{\text{sqC}}\big)$.

3.4 Self-quanto put options

A long self-quanto put $X_T^{\text{sqP}} = S_T(K - S_T)_+$ with payoff function $\omega^{\text{sqP}}(y) := y(K-y)_+$ and strike $K > 0$ is designed to profit from moderate decreasing prices of the underlying security. It provides highest outcomes when the price is at $\frac{K}{2}$. Its distribution function can be calculated and is presented in the following lemma. Since the payoff function ω^{sqP} is not monotone self-quanto puts are not efficient for $\bar{\theta} \neq 0$. The payoff distribution function G_{sqP} can be calculated as

$$G_{\text{sqP}}(x) = \begin{cases} 1, & x \geq \frac{K^2}{4}, \\ 1 - \Bigg(F_{L_T}\bigg(\ln\Big(\frac{\frac{K}{2}+\sqrt{\frac{K^2}{4}-x}}{S_0}\Big)\bigg) - F_{L_T}\bigg(\ln\Big(\frac{\frac{K}{2}-\sqrt{\frac{K^2}{4}-x}}{S_0}\Big)\bigg)\Bigg), & 0 < x < \frac{K^2}{4}, \\ 1 - F_{L_T}\big(\ln(\frac{K}{S_0})\big), & x = 0, \\ 0, & x < 0\,. \end{cases} \tag{3.29}$$

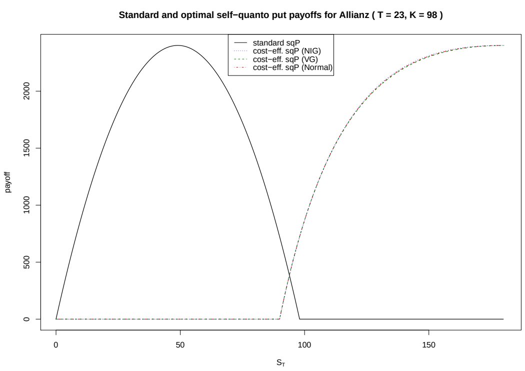

Figure 11: Classical self-quanto put and its cost-efficient counterparts for Allianz. $S_0 = 93.42$

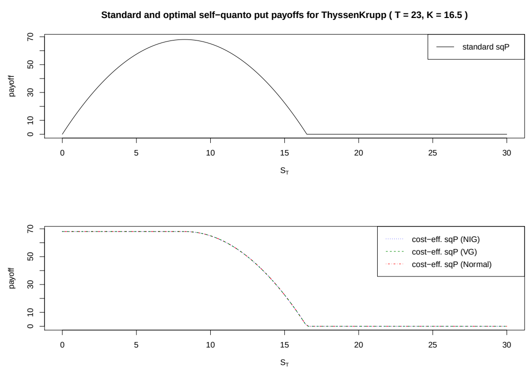

Figure 12: Classical self-quanto put and its cost-efficient counterparts for ThyssenKrupp. $S_0 = 16.73$

Figure 11 displays the payoff X_T^{sqP} of a long self-quanto put option on one Allianz stock with strike $K = 98$ and maturity $T = 23$ days and its cost-efficient counterparts $\underline{X}_T^{\text{sqP}}$ for the three Lévy models under consideration. The bearish market counterpart is illustrated in Figure 12 which shows the payoff $\underline{X}_T^{\text{sqP}}$ of a cost-efficient long self-quanto put option on one ThyssenKrupp stock with strike $K = 16.5$ and maturity $T = 23$ days. Note, that all three Lévy models generate fairly equal plots.

For the price of a standard and optimal self-quanto put we have similarly as in the call case the following simplified formula.

Proposition 3.6 (Price of a vanilla self-quanto put) *Let* $(L_t)_{t\geq 0}$ *be a Lévy process, such that* $L_T \overset{d}{=} NIG(\alpha,\beta,\delta T,\mu T)$ *or* $VG(\lambda T,\alpha,\beta,\mu T)$. *If* $\bar\theta$ *is a Esscher parameter, then we have*

$$c(X_T^{\text{sqP}}) = S_0 K F_{L_T^{\bar\theta+1}}\Big(\ln\Big(\frac{K}{S_0}\Big)\Big) - S_0^2 E\Big[\frac{e^{(\bar\theta+2)L_T}}{M_{L_T}(\bar\theta+1)}\mathbb{1}_{\{L_T<\ln(\frac{K}{S_0})\}}\Big]$$

where $L_T^{\bar\theta+k} \overset{d}{=} NIG(\alpha,\beta+\bar\theta+k,\delta T,\mu T)$ *or* $VG(\lambda T,\alpha,\beta+\bar\theta+k,\mu T)$, $k=0,1,2$. *If in addition* $M_{L_T}(\bar\theta+2)<\infty$, *then*

$$c(X_T^{\text{sqP}}) = S_0 K F_{L_T^{\bar\theta+1}}\Big(\ln\Big(\frac{K}{S_0}\Big)\Big) - \frac{M_{L_T}(\bar\theta+2)}{M_{L_T}(\bar\theta+1)} S_0^2 F_{L_T^{\bar\theta+2}}\Big(\ln\Big(\frac{K}{S_0}\Big)\Big) \tag{3.30}$$

For $L_T \overset{d}{=} N((\mu-\frac{\sigma^2}{2})T,\sigma^2 T)$ *we have*

$$c(X_T^{\text{sqP}}) = S_0 K\Phi\big(h\big) - e^{(r+\sigma^2)T}S_0^2\Phi\big(h-\sigma\sqrt{T}\big) \tag{3.31}$$

where $h=\frac{1}{\sigma\sqrt{T}}(\ln(\frac{K}{S_0})-(r+\frac{\sigma^2}{2})T)$.

Allianz	$c(X_T^{\text{sqP}})$	$c(\underline{X}_T^{\text{sqP}})$	**Efficiency loss in %**
NIG	547.2179	452.8534	17.24
VG	542.4431	449.5875	17.12
Normal	546.3491	452.2157	17.23
ThyssenKrupp	$c(X_T^{\text{sqP}})$	$c(\underline{X}_T^{\text{sqP}})$	**Efficiency loss in %**
NIG	9.5988	9.5987	0.001041
VG	9.5737	9.5736	0.001044
Normal	9.5826	9.5825	0.001043

Table 6: Comparison of the cost of a long self-quanto put option on Allianz and ThyssenKrupp resp., and the corresponding cost-efficient payoffs. $S_0=93.42$, $K=98$, $T=23$ for Allianz and $S_0=16.73$, $K=16.5$ for ThyssenKrupp.

Table 6 gives the cost of a long self-quanto put option on Allianz with their cost-efficient counterparts for the three Lévy models under consideration. To cover the bearish markets the analogous results for the cost of a long self-quanto put option on ThyssenKrupp are included in Table 6. One observes that for the ThyssenKrupp stock the efficiency loss is insignificantly small. This is due to the fact that equality of the payoff functions on sets with high probability (under the Esscher martingale measure) plus boundedness on the complementary set implies nearly equal cost and, thus small efficiency loss. This is quantified in the next remark in the *NIG* case.

Remark 3.7 *Assume that* $L_T \overset{d}{=} NIG(\alpha,\beta,\delta T,\mu T)$, *then the payoff functions of the long self-quanto put* X_T^{sqP} *and of the cost-efficient counterpart* $\underline{X}_T^{\text{sqP}}$ *are both bounded by* $C=(\frac{K}{2})^2=68.0625$ *and are identical on the interval* $I=[\ln(\frac{8.25}{S_0}),\ln(\frac{30}{S_0})]$ *with probability (w.r.t. Esscher martingale measure) nearly one,* $F_{L_T^{\bar\theta}}(I)=F_{L_T^{\bar\theta}}(\ln(30/S_0)) - F_{L_T^{\bar\theta}}(\ln(8.25/S_0)) = 99.9999\%$. *Hence*

$$\begin{aligned}\ell(X_T^{\mathrm{sqP}}) &= e^{-rT}E_{Q^{\bar\theta}}[X_T^{\mathrm{sqP}} - \underline{X}_T^{\mathrm{sqP}}]\\ &= \int_I (X_T^{\mathrm{sqP},x} - \underline{X}_T^{\mathrm{sqP},x})dF_{L_T^{\bar\theta}}(x) + \int_{I^{\mathsf{c}}} (X_T^{\mathrm{sqP},x} - \underline{X}_T^{\mathrm{sqP},x})dF_{L_T^{\bar\theta}}(x)\\ &= \int_{I^{\mathsf{c}}} (X_T^{\mathrm{sqP},x} - \underline{X}_T^{\mathrm{sqP},x})dF_{L_T^{\bar\theta}}(x)\\ &\le \sup_{x\in I^{\mathsf{c}}}(X_T^{\mathrm{sqP},x} - \underline{X}_T^{\mathrm{sqP},x}) \cdot F_{L_T^{\bar\theta}}(I^{\mathsf{c}}) \le C \cdot 0.00001 = 0.0001089\,,\end{aligned}$$

where $X_T^{\mathrm{sqP},x} = \omega^{\mathrm{sqP}}(S_0e^x) = S_0e^x(K - S_0e^x)_+$ *and* $\underline{X}_T^{\mathrm{sqP},x} = \underline{\omega}^{\mathrm{sqP}}(x) = G_{\mathrm{sqP}}^{-1}(1 - F_{L_T}(\ln(\frac{x}{S_0})))$.

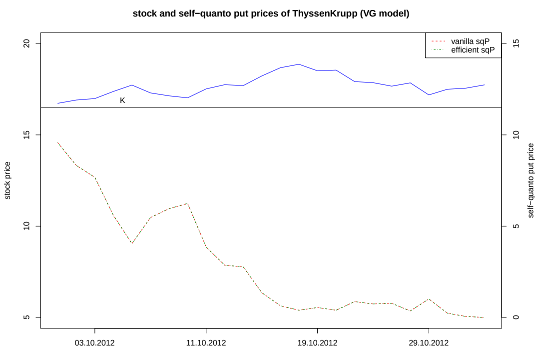

Figure 13: Evolution of prices of standard and cost-efficient self-quanto put with strike $K = 16.5$ for ThyssenKrupp in the VG model.

Figure 14: Evolution of prices of standard and cost-efficient self-quanto put with strike $K = 98$ for Allianz in the NIG model.

Figure 13 illustrates the nearly equal payoff function of the standard and optimal self-quanto put in the bearish market situation. This leads to almost identical prices during the entire trading period $[0, T]$ in October 2012. Figure 14, shows distinctive similarities to Figure 6 with more pronounced peaks and lows which is due to the design of the long self-quanto put option. The prices of the efficient options always roughly move in the direction opposite to that of the standard options which reflects the reversed monotonicity properties of the underlying payoff profiles in Figure 11.

3.5 Long straddle options

A long straddle investment strategy allows the holder to profit based on how much the price of the underlying security moves, regardless of the direction of price movement. A long straddle option X_T^{strdl} is realized by going long in both a call option and a put option on some stock, index or other underlying. i.e. $X_T^{\mathrm{strdl}} = X_T^{\mathrm{Call}} + X_T^{\mathrm{Put}}$. It involves buying the put and call options at the same strike $K > 0$ with the same maturity T. A profit is gained if the underlying price moves a long way from the strike price, either above or below. The payoff distribution function is given by

Lemma 3.8 *Let $(L_t)_{t\ge0}$ be a Lévy process with continuous distribution function F_{L_T} at maturity $T > 0$. The distribution function G_{strdl} of the payoff of the long straddle X_T^{strdl}*

with strike $K > 0$ *at maturity* T *is given by*

$$G_{\text{strdl}}(x) = \begin{cases} F_{L_T}(\ln(\frac{K+x}{S_0})), & x \geq K, \\ F_{L_T}(\ln(\frac{K+x}{S_0})) - F_{L_T}(\ln(\frac{K-x}{S_0})), & 0 \leq x < K, \\ 0, & x < 0\,. \end{cases} \tag{3.32}$$

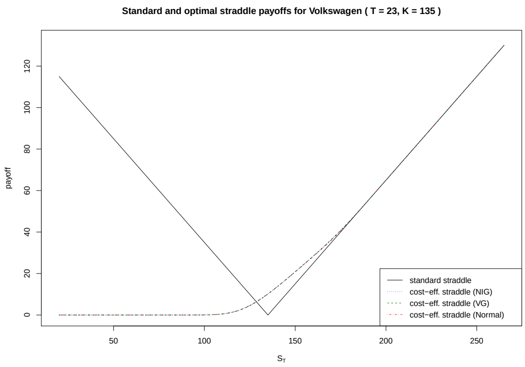

Figure 15: Payoff functions of a classical straddle option and its cost-efficient counterparts for Volkswagen.

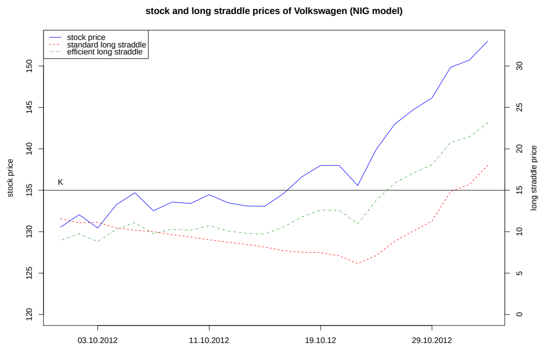

Figure 16: Evolution of prices of standard and cost-efficient long straddle with strike $K = 135$ for Volkswagen in the NIG model.

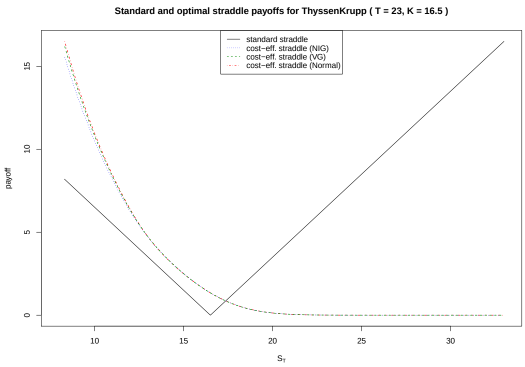

Figure 17: Payoff functions of a classical straddle option and its cost-efficient counterparts for ThyssenKrupp.

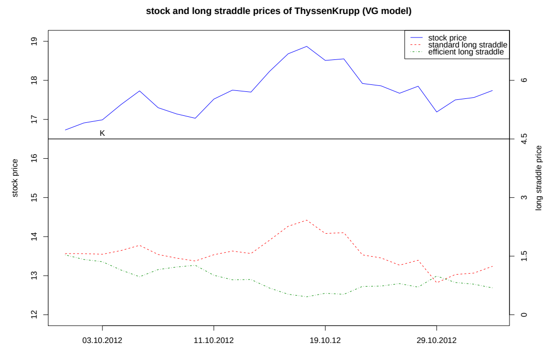

Figure 18: Evolution of prices of standard and cost-efficient long straddle with strike $K = 16.5$ for ThyssenKrupp in the VG model.

The cost of the cost-efficient long straddle is compared with its vanilla counterpart in Table 7, while in Figure 15 we contrast the payoffs X_T^{strdl} of a long straddle option on one Volkswagen stock with strike $K = 135$ and maturity $T = 23$ days, and its cost-efficient counterparts $\underline{X}_T^{\text{strdl}}$ for the three Lévy models under consideration. For the bearish markets we present in Figure 17, the payoff $\underline{X}_T^{\text{strdl}}$ of a cost-efficient long straddle option on one ThyssenKrupp stock with strike $K = 16.5$ and maturity $T = 23$ days.

Regarding Figure 15 we notice that with increasing stock price at maturity of the Volkswagen the payoff of the cost-efficient long straddle dominates that of the standard long straddle. This difference becomes more and more irrelevant with increasing stock price and takes the greatest value if the stock prices moves around the exercise price. A reverse pattern is depicted in Figure 17. Thus, the evolution of the prices of standard and cost-efficient long straddle for Volkswagen, Figure 16, shows that close to maturity, where

Volkswagen	$c(X_T^{\text{strdl}})$	$c(\underline{X}_T^{\text{strdl}})$	Efficiency loss in %
NIG	11.5759	8.9844	22.39
VG	11.5161	8.9239	22.51
Normal	11.5448	8.9722	22.28
ThyssenKrupp	$c(X_T^{\text{strdl}})$	$c(\underline{X}_T^{\text{strdl}})$	**Efficiency loss in %**
NIG	1.5717	1.5377	2.17
VG	1.5662	1.5312	2.23
Normal	1.5657	1.5293	2.32

Table 7: Comparison of the cost of a long straddle option on Volkswagen and ThyssenKrupp resp., and the corresponding cost-efficient payoffs. $S_0 = 130.55$, $K = 135$, $T = 23$ for Volkswagen and $S_0 = 16.73$, $K = 16.5$ for ThyssenKrupp.

the stock price rapidly increases, the costs are increasing too, and the cost-efficient long straddle is more expensive than the standard long straddle option. For the ThyssenKrupp stock we have a more complex situation. Here, the trend alters from bearish to bullish within the trading period, thus the stock price increases close to maturity. Hence, the payoff of the cost-efficient long straddle becomes less worthy which is illustrated in Figure 18.

3.6 Long call butterfly spread options

A long (call) butterfly option strategy is created to earn substantial but limited profits with great probability. It is a limited risk and non-directional financial investment strategy, and due to its design it is a suitable neutral option strategy for low volatility markets. A long butterfly spread is the combination of two long calls C_3 and C_1 with strikes $K_3 > K_1 > 0$, and two short calls $-C_2$ with strike $K_2 = \frac{K_1+K_3}{2}$. The payoff X_T^{bfly} of a butterfly spread is given by

$$X_T^{\text{bfly}} = (S_T - K_1)_+ + (S_T - K_3)_+ - 2(S_T - K_2)_+.$$

An investor may take a long butterfly position if he expects that the market is mildly volatile, thus profiting the most if the stock price is at K_2. The payoff distribution function can be calculated and is given by:

$$G_{\text{bfly}}(x) = \begin{cases} 1, & x > K_2 - K_1, \\ 1 - F_{L_T}(\ln(\frac{K_3 - x}{S_0})) + F_{L_T}(\ln(\frac{K_1 + x}{S_0})), & 0 \le x \le K_2 - K_1, \\ 0, & x < 0\,. \end{cases} \tag{3.33}$$

In Figure 19 the payoffs of a butterfly spread and its efficient counterpart of one Allianz stock with strikes $K_1 = 94$ and $K_3 = 104$ are presented. For the bearish markets, Figure 20 shows the payoff $\underline{X}_T^{\text{bfly}}$ of a cost-efficient long butterfly spread option on one ThyssenKrupp stock with strikes $K_1 = 12$ and $K_3 = 20$. The cost of a long butterfly spread option on Allianz and ThyssenKrupp with their cost-efficient counterparts for the three Lévy models under consideration are considered in Table 8. Again, in case of the ThyssenKrupp stock we notice that the more the payoffs functions resemble on sets with greater mass the smaller becomes the efficiency loss (cf Figure 20 and also Remark 3.7). We see from Figure 20 that the payoff of the cost-efficient version of the long butterfly spread is dominated if the stock price of the underlying is greater than approximately 15.5. Since the stock price of the ThyssenKrupp is above 15.5 during the entire trading period $0 < t \le T$ the cost of

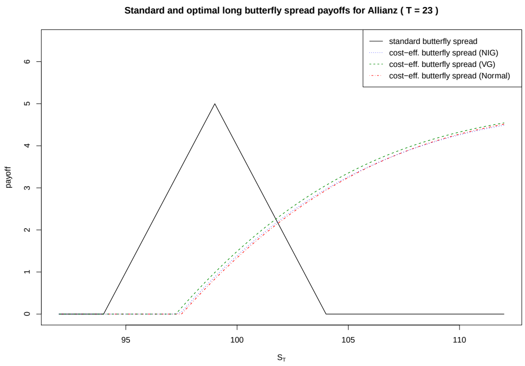

Figure 19: Payoff functions of a classical long butterfly option and its cost-efficient counterparts for Allianz. $S_0 = 93.42$, $K_1 = 94$ and $K_3 = 104$

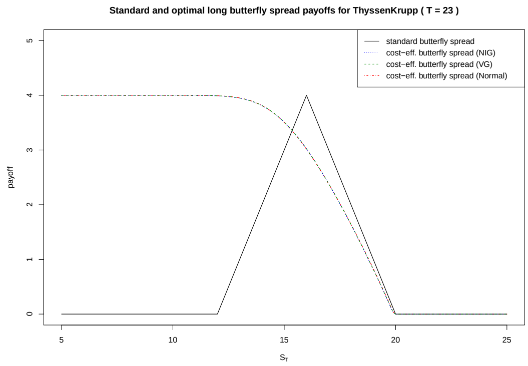

Figure 20: Payoff functions of a classical long butterfly option and its cost-efficient counterparts for ThyssenKrupp. $S_0 = 16.73$, $K_1 = 12$ and $K_3 = 20$

Allianz	$c(X_T^{\text{bfly}})$	$c(\underline{X}_T^{\text{bfly}})$	**Efficiency loss in %**
NIG	0.8398	0.7739	7.84
VG	0.8475	0.7825	7.67
Normal	0.8349	0.7691	7.87
ThyssenKrupp	$c(X_T^{\text{bfly}})$	$c(\underline{X}_T^{\text{bfly}})$	**Efficiency loss in %**
NIG	2.4153	2.4008	0.60
VG	2.4210	2.4064	0.60
Normal	2.4196	2.4044	0.63

Table 8: Comparison of the cost of a long butterfly spread option on Allianz and ThyssenKrupp resp., and the corresponding cost-efficient payoffs. $S_0 = 93.42$, $K_1 = 94$, $K_3 = 104$, for Volkswagen and $S_0 = 16.73$, $K_1 = 12$, $K_3 = 20$ for ThyssenKrupp.

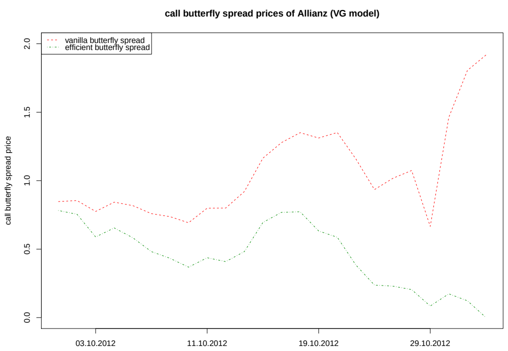

Figure 21: Evolution of prices of standard and cost-efficient long call butterfly spread with strikes $K_1 = 94$ and $K_3 = 104$ for Allianz in the VG model in October 2012, $T = 23$ days.

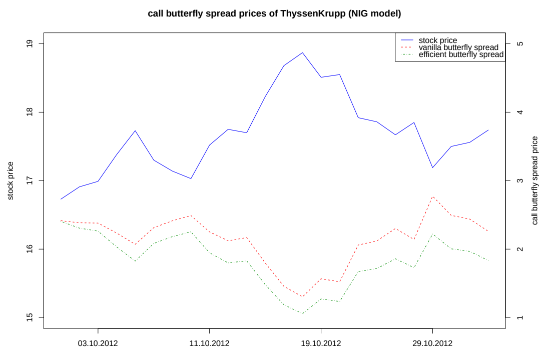

Figure 22: Evolution of prices of standard and cost-efficient long call butterfly spread with strikes $K_1 = 12$ and $K_3 = 20$ for ThyssenKrupp in the NIG model in October 2012, $T = 23$ days.

the standard butterfly spread dominates the cost of the efficient counterpart which can be seen in Figure 22.

4 Efficiency loss for monotone payoff functions

The efficiency loss $\ell(\bar{\theta}) = \ell(\bar{\theta}, \eta) = e^{-rT} E_{\bar{\theta}}(X_T - \underline{X}_T)$ depends on the Escher parameter $\bar{\theta} = \bar{\theta}(\eta)$ and in particular on the model parameter $\eta = (\eta_1, \ldots, \eta_k)$,. In Hammerstein et al. (2014) it has been shown that $\ell(\bar{\theta}, \eta)$ is an increasing function in $|\bar{\theta}|$ which leads in particular in the examples of put and call options to the result that the magnitude $|\bar{\theta}|$ of market trend determines the magnitude of the efficiency loss.

In the previous sections we typically reported the relative efficiency loss $\frac{\ell(\bar{\theta})}{c(X_T)} = \ell_r(\bar{\theta})$ which might be more relevant for applications. In this section we study the efficiency loss for plain vanilla puts and calls as well as for self-quanto calls. For vanilla puts as for its cost-efficient counterparts the cost rises with increasing strike price. Hence, it would be interesting to know how the efficiency loss resp. relative efficiency loss behaves when changing the strike. The next theorem confirms that the efficiency loss $\ell(K)$ in case of the put option is increasing in the strike while the relative efficiency loss $\ell_r(K)$ shows an opposite behavior. This has noticeable consequences for trading put options, when investors are seeking to maximize their (relative) efficiency loss. Related results for the call resp. self-quanto call option are given too.

Theorem 4.1 (Efficiency loss vs. relative efficiency loss for X_T^{Put}, influence of strike) *Let $(L_t)_{t\geq 0}$ be a Lévy process with continuous and strictly increasing distribution function F_{L_T} at maturity $T > 0$. Suppose X_T^{Put} is the payoff of a long put option with strike $K > 0$ and let $\bar{\theta}$ be an Esscher parameter.*

1. *The efficiency loss, $\ell(K) := c(X_T^{\text{Put}}) - c(\underline{X}_T^{\text{Put}})$ is increasing in K.*
2. *$\frac{\partial}{\partial K} c(\underline{X}_T^{\text{Put}}) \leq \frac{\partial}{\partial K} c(X_T^{\text{Put}})$ and the costs of the standard and efficient put are increasing in K.*
3. *The relative efficiency loss $\ell_r(K) := \frac{\ell(K)}{c(X_T^{\text{Put}})}$ decreases in K.*

PROOF:

1. If $\bar{\theta} > 0$ then $\underline{X}_T^{\text{Put}} = X_T^{\text{Put}}$ and $\ell(K) \equiv 0$, thus w.l.g., let $\bar{\theta} < 0$. By definition $\ell(K) \geq 0$ for all $K \in \mathbb{R}_+$, thus $\ell : \mathbb{R}_+ \to \mathbb{R}_+$. Define $C := e^{-rT} E_{\bar{\theta}}[S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(L_T)} - S_T]$ and observe that the pair $(X_1^+, X_2^+) := (Z_T^{\bar{\theta}}, S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))})$ is comonotonic whereas the pair $(X_1^-, X_2^-) := (Z_T^{\bar{\theta}}, S_T)$ is countermonotonic. The marginals $F_{X_1^+} = F_{X_1^-}$ and $F_{X_2^+} = F_{X_2^-}$ are equal, thus, Hoeffdings inequality implies

$$e^{rT} C = \text{Cov}(Z_T^{\bar{\theta}}, S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))}) - \text{Cov}(Z_T^{\bar{\theta}}, S_T) \geq 0$$

and thus $C \geq 0$. Since the first term is non-negative and the second is non-positive it holds that $C = 0$ if and only if $\text{Cov}(Z_T^{\bar{\theta}}, S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))}) = \text{Cov}(Z_T^{\bar{\theta}}, S_T) = 0$. Since $Z_T^{\bar{\theta}} = h(S_T)$ for some decreasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$, this can only be true if and only if L_T respectively L_1 is degenerate. By our general assumption this case is excluded and thus $C > 0$. Further, we easily obtain that $\ell(0) = 0$ and we also have

$$\begin{aligned} E_{\bar\theta}[(K-a)_+ - (K-b)_+] &= E_{\bar\theta}[\min(K,b) - \min(K,a)] \\ &\le E_{\bar\theta}[\min(K,b)] \\ &\le E_{\bar\theta}[b]\,. \end{aligned}$$

Hence, with the identity $(K-a)_+ = K - \min(K,a)$ for all $K, a \in \mathbb{R}_+$, the dominated convergence yields

$$\begin{aligned} \lim_{K\to\infty} E_{\bar\theta}[\min(K,b) - \min(K,a)] &= E_{\bar\theta}[\lim_{K\to\infty} \min(K,b)] - E_{\bar\theta}[\lim_{K\to\infty} \min(K,a)] \\ &= E_{\bar\theta}[b-a] \end{aligned}$$

for all $a, b \in \mathbb{R}_+$, such that $E_{\bar\theta}[b], E_{\bar\theta}[a] < \infty$. Putting $a = S_T$ and $b = S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))}$ we have shown that

$$\lim_{K\to\infty} \ell(K) = C > 0.$$

For the proof of 1) it is sufficient to show the existence of a $K^* \in \mathbb{R}_+$ such that ℓ is convex on $[0, K^*)$ and concave on $[K^*, \infty)$. For $\bar\theta < 0$ the plain vanilla put is most-expensive. By Proposition 1.2 the efficiency loss as is given by

$$\ell(K) = \frac{1}{M_{L_T}(\bar\theta)} \int_0^1 e^{\bar\theta F_{L_T}^{-1}(1-y) - rT} \big((K - S_0 e^{F_{L_T}^{-1}(1-y)})_+ - (K - S_0 e^{F_{L_T}^{-1}(y)})_+\big)\,\mathrm{d}y.$$

Note that $\ell(K)$ is bounded from above by the price $c(X_T^{\mathrm{Put}})$ of the original long put which obviously is finite for all $K \in \mathbb{R}_+$. Moreover, the functions

$$\begin{aligned} f_1(K,y) &= e^{\bar\theta F_{L_T}^{-1}(1-y) - rT} (K - S_0 e^{F_{L_T}^{-1}(1-y)})_+ \quad \text{and} \\ f_2(K,y) &= e^{\bar\theta F_{L_T}^{-1}(1-y) - rT} (K - S_0 e^{F_{L_T}^{-1}(y)})_+ \end{aligned}$$

are differentiable in K for all $y \in [0,1]$. The points $K = S_0 e^{F_{L_T}^{-1}(1-y)}$, $K = S_0 e^{F_{L_T}^{-1}(y)}$ can be neglected since the left- and right-hand derivatives are bounded. The partial derivatives are

$$\begin{aligned} \frac{\partial}{\partial K} f_1(K,y) &= e^{\bar\theta F_{L_T}^{-1}(1-y) - rT} \mathbb{1}_{[0, F_{L_T}(\ln(\frac{K}{S_0})))}(1-y) \qquad \text{and} \\ \frac{\partial}{\partial K} f_2(K,y) &= e^{\bar\theta F_{L_T}^{-1}(1-y) - rT} \mathbb{1}_{[0, F_{L_T}(\ln(\frac{K}{S_0})))}(y). \end{aligned}$$

It holds that $|\frac{\partial}{\partial K} f_i(K,y)| \le e^{\bar\theta F_{L_T}^{-1}(1-y) - rT}$, $i = 1, 2$. For the integrability of $e^{\bar\theta F_{L_T}^{-1}(1-y) - rT}$ observe that

$$\begin{aligned} \int_0^1 e^{\bar\theta F_{L_T}^{-1}(1-y) - rT}\,\mathrm{d}y &= \int_0^1 e^{\bar\theta F_{L_T}^{-1}(z) - rT}\,\mathrm{d}z \\ &= \int_{-\infty}^{\infty} e^{\bar\theta x - rT} f_{L_T}(x)\,\mathrm{d}x = e^{-rT} M_{L_T}(\bar\theta) < \infty, \end{aligned}$$

where f_{L_T} denotes the density of L_T which exists and is strictly positive on $\mathbb{R}$ due to our assumptions on F_{L_T}. Hence, we can interchange differentiation and integration and obtain

$$\frac{\partial \ell}{\partial K}(K) = \frac{e^{-rT}}{M_{L_T}(\bar{\theta})} \int_0^{F_{L_T}(\ln(\frac{K}{S_0}))} e^{\bar{\theta} F_{L_T}^{-1}(y)} - e^{\bar{\theta} F_{L_T}^{-1}(1-y)} \, \mathrm{d}y. \tag{4.1}$$

Differentiating w.r.t. K once again yields

$$\frac{\partial^2}{\partial^2 K}\ell(K) = \frac{e^{-rT}}{M_{L_T}(\bar{\theta})}\Big(f_{L_T}\Big(\ln\Big(\frac{K}{S_0}\Big)\Big)\Big)\frac{1}{K}\Big[e^{\bar{\theta} F_{L_T}^{-1}\left(F_{L_T}(\ln(\frac{K}{S_0}))\right)} - e^{\bar{\theta} F_{L_T}^{-1}\left(1-F_{L_T}(\ln(\frac{K}{S_0}))\right)}\Big].$$

Thus, $\frac{\partial^2}{\partial^2 K}\ell(K) > 0$ if and only if

$$\bar{\theta} F_{L_T}^{-1}\Big(F_{L_T}\Big(\ln\Big(\frac{K}{S_0}\Big)\Big)\Big) > \bar{\theta} F_{L_T}^{-1}\Big(1 - F_{L_T}\Big(\ln\Big(\frac{K}{S_0}\Big)\Big)\Big),$$

or equivalently, $F_{L_T}(\ln(\frac{K}{S_0})) < 1 - F_{L_T}(\ln(\frac{K}{S_0}))$, since $\bar{\theta} < 0$.

For $K^* := S_0 e^{F_{L_T}^{-1}(0.5)}$ we obtain $K < K^*$ if and only if $F_{L_T}(\ln(\frac{K}{S_0})) < 1 - F_{L_T}(\ln(\frac{K}{S_0}))$. Thus, ℓ is convex on $[0, K^*)$. Analogously, we get $\frac{\partial^2}{\partial^2 K}\ell(K) \leq 0$ if and only if $K \geq K^*$. Thus, ℓ is concave on $[K^*, \infty)$ as consequence we therefore get that ℓ is increasing.

2. This follows directly from the fact that $\frac{\partial}{\partial K}\ell(K) \geq 0$, thus $\frac{\partial}{\partial K}c(\underline{X}_T^{\text{Put}}) \leq \frac{\partial}{\partial K}c(X_T^{\text{Put}})$ and

$$\frac{\partial}{\partial K}c(\underline{X}_T^{\text{Put}}) = \frac{e^{-rT}}{M_{L_T}(\bar{\theta})} \int_0^{F_{L_T}(\ln(\frac{K}{S_0}))} e^{\bar{\theta} F_{L_T}^{-1}(1-y)} \, \mathrm{d}y \geq 0$$

as can be seen from equation (4.1).

3. Note that the function $\ell_{\text{r}}(K)$ is decreasing in K if and only if for all compact intervals $[K_1, K_2]$ with $K_1 < K_2 \in \mathbb{R}_+$ we have $\max_{K \in [K_1, K_2]} \ell_{\text{r}}(K) = \ell_{\text{r}}(K_1)$, or equivalently $K_1 \in \operatorname{argmax}_{K \in [K_1, K_2]} \ell_{\text{r}}(K)$. Since $c(\underline{X}_T^{\text{Put}})$ is an increasing function of K it holds that for all $K_1 < K_2 \in \mathbb{R}_+$ the cost of the efficient put $c(\underline{X}_T^{\text{Put}})$ is minimal at K_1, or equivalently for all $K_1 < K_2 \in \mathbb{R}_+$ we have

$$\begin{aligned} K_1 \in \operatorname*{argmin}_{K \in [K_1, K_2]} c(\underline{X}_T^{\text{Put}}) &= \operatorname*{argmin}_{K \in [K_1, K_2]} \big(c(X_T^{\text{Put}}) - \ell(K)\big) \\ &= \operatorname*{argmax}_{K \in [K_1, K_2]} \Big(\frac{\ell(K)}{c(X_T^{\text{Put}})}\Big) = \operatorname*{argmax}_{K \in [K_1, K_2]} \ell_{\text{r}}(K). \end{aligned}$$

Also, for $K_1 < K_2 \in \mathbb{R}_+$ it yields that $\max_{K \in [K_1, K_2]} \ell_{\text{r}}(K) = \ell_{\text{r}}(K_1)$, since ℓ_{r} is decreasing in K. Thus, the assertion is proven. □

The latter result reveals that potential greater savings in buying an efficient option with higher strike could be annihilated by the higher cost one has to pay for. The reverse holds true for the relative efficiency loss.

Remark 4.2

a) ***Total savings vs. higher distribution w.r.t $\leq_{st}$***
Consider an investor with budget $B \in \mathbb{R}_+$ who aims to buy a put option which generates a payoff X_T^{Put} at maturity $T > 0$. Further, consider put options $X_T^{\text{Put},i}$ with strike $K_i > 0$, $i = 1, 2$ and the efficient counterparts $\underline{X}_T^{\text{Put},i}$ with cost denoted by c_i respectively $\underline{c}_i$. We compute that

$$B = \frac{B}{c_i} \cdot c_i = \frac{B}{c_i}\big(\underline{c}_i + (c_i - \underline{c}_i)\big) = \frac{B}{c_i} \cdot \underline{c}_i + \frac{B}{c_i}(c_i - \underline{c}_i),\ i = 1, 2,$$

that is, $\frac{B}{c_i}(c_i - \underline{c}_i) = B \cdot \ell_r(K_i)$ denotes the total savings when buying $\frac{B}{c_i}$ shares of the efficient put option. By Theorem 4.1 we see that $B\ell_r(K_2) \leq B\ell_r(K_1)$ if $K_1 < K_2$, i.e. the total savings decrease in the strike, when buying the associated efficient put option. In other words, when choosing the put option with the higher strike K_2 which generates a stochastically larger distribution one has to pass on the amount of $B \cdot (\ell_r(K_1) - \ell_r(K_2)) \geq 0$ of potential savings.

b) ***Bounds for efficiency loss***
The proof of Theorem 4.1 also establishes the following bound for the efficiency loss of a long put option in bullish markets.

$$0 \leq \ell(X_T^{\text{Put}}) \leq S_0 E_{\bar{\theta}}[e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))-rT} - 1]. \tag{4.2}$$

Efficiency loss in %			
Allianz	$K_1 = 92$	$K_2 = 95$	$K_3 = 98$
NIG	24.01	20.89	18.09
VG	23.86	20.73	17.92
Normal	23.84	20.83	18.10
Volkswagen	$K_1 = 130$	$K_2 = 133$	$K_3 = 135$
NIG	54.94	51.33	48.95
VG	55.09	51.48	49.10
Normal	54.92	51.35	49.01

Table 9: Relative efficiency loss for a long put option on Allianz and Volkswagen $S_0 = 93.42$, $T = 23$ for Allianz and $S_0 = 130.55$, $T = 23$ for Volkswagen; from Table 1.

Some concrete results on the relative efficiency loss for put options with different strikes on the Allianz and Volkswagen stock can be found in Table 9. These show the decrease of the relative efficiency loss in the strike K for Allianz and Volkswagen. We give analogous results for the long call and the self-quanto call option. The results confirm that the efficiency loss in case of the plain call and self-quanto call option is decreasing and the relative efficiency loss is increasing in the strike, thus, has a reverse behavior as for the put option. For these examples comparison of the relative efficiency loss are given in Tables 10 and 11. Although the payoff profile of a plain vanilla call considerably differs from the profile of the self-quanto option, the monotonicity of the relative efficiency loss does not exhibit substantial differences.

Efficiency loss in %			
E.ON	$\boldsymbol{K_1 = 17.24}$	$\boldsymbol{K_2 = 19.48}$	$\boldsymbol{K_3 = 20.72}$
NIG	6.45	11.04	13.75
VG	6.83	11.66	14.53
Normal	6.32	10.61	13.15
ThyssenKrupp	$\boldsymbol{K_1 = 16.5}$	$\boldsymbol{K_2 = 18.5}$	$\boldsymbol{K_3 = 20.5}$
NIG	5.90	8.74	11.74
VG	6.07	8.96	11.99
Normal	6.33	9.27	12.35

Table 10: Relative efficiency loss for a long call option on E.ON and ThyssenKrupp $S_0 = 17.48$, $T = 23$ for E.ON and $S_0 = 16.73$, $T = 23$ for ThyssenKrupp.

Efficiency loss in %			
E.ON	$\boldsymbol{K_1 = 17.24}$	$\boldsymbol{K_2 = 19.48}$	$\boldsymbol{K_3 = 20.72}$
NIG	6.68	11.17	13.85
VG	7.05	11.80	14.67
Normal	6.53	10.72	13.20
ThyssenKrupp	$\boldsymbol{K_1 = 16.5}$	$\boldsymbol{K_2 = 18.5}$	$\boldsymbol{K_3 = 20.5}$
NIG	6.17	8.91	11.86
VG	6.33	9.13	12.11
Normal	6.58	9.43	12.46

Table 11: Relative efficiency loss for a long self-quanto call option on E.ON and ThyssenKrupp. $S_0 = 17.48$, $T = 23$ for E.ON and $S_0 = 16.73$, $T = 23$ for ThyssenKrupp.

Proposition 4.3 (Efficiency loss vs. relative efficiency loss for $\boldsymbol{X_T^{\text{Call}}}$) *Let $(L_t)_{t\geq 0}$ be a Lévy process with continuous and strictly increasing distribution function F_{L_T} at maturity $T > 0$. Let X_T^{Call} be the payoff of a long call option with strike $K > 0$ and let $\bar{\theta}$ be an Esscher parameter.*

1. *The efficiency loss, as a function of the strike, $\ell(K) := c(X_T^{\text{Call}}) - c(\underline{X}_T^{\text{Call}})$ is decreasing in K.*
2. *$\frac{\partial}{\partial K} c(X_T^{\text{Call}}) \leq \frac{\partial}{\partial K} c(\underline{X}_T^{\text{Call}})$ and the cost of the standard and efficient call are decreasing in K.*
3. *The relative efficiency loss $\ell_r(K) := \frac{\ell(K)}{c(X_T^{\text{Call}})}$ increases in K.*

As for the put option, the latter findings immediately implies the following bound for the efficiency loss of a long call option in bearish markets.

$$0 \leq \ell(X_T^{\text{Call}}) \leq S_0 E_{\bar{\theta}}[1 - e^{F_{L_T}^{-1}(1 - F_{L_T}(L_T)) - rT}] . \tag{4.3}$$

Remark 4.4 (monotonicity for self-quanto calls) *The monotonicity results in Theorem 4.1, Propositions 4.3 also hold true in the same way for the efficiency loss of self-quanto calls. For details see Wolf (2014).*

5 Delta hedging of cost-efficient strategies in Lévy models

In the following we discuss the *delta hedge*, i.e. the derivative of the cost of a strategy with respect to the underlying for the cost-efficient payoff. Furthermore, concrete hedging simulation schemes are provided for the standard and the cost-efficient put, long call butterfly spread and self-quanto put in the NIG model. Our delta hedging simulation schemes are inspired by the approach of Hull (1997, Section 14.5). Moreover, we demonstrate that delta hedging of cost-efficient puts can be efficiently applied in practice and that the obtained hedge errors are usually not greater, but often even smaller that those of the corresponding vanilla puts. Also an alternative delta hedging approach based on a rollover strategy is introduced. The delta hedging strategies obtained by this alternative hedging technique have the potential to outperform the classical ones in this context.

5.1 Introduction to Delta hedging

The Greek delta measures the exposure of a derivative to changes in the value of the underlying. By delta hedging we mean the process of keeping the delta of a portfolio which consists of related financial securities as close to zero as possible. Thus, by delta hedging investors attempt to make their portfolio immune to small changes in the price of the underlying asset in the next small interval of time. If the underlying asset is traded sufficiently liquid in the market, delta hedging is a simple, but nevertheless fairly effective way to cover a risky position and is therefore widely used in practice.

For puts $X_T^{\text{Put}} = (K - S_T)_+$ and calls $X_T^{\text{Call}} = (S_T - K)_+$ with strike K hedging of cost-efficient options with maturity T has already been investigated in Hammerstein et al. (2014). We first restate the main findings in this paper. Recall that the payoff function $\underline{\omega}^{\text{Put}}(y) = (K - S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(\ln(\frac{y}{S_0})))})_+$ of the cost-efficient long put (and call as well) is kept fixed within the trading period $[0,T]$. For $\bar{\theta} < 0$ the price at time $t < T$ of a cost-efficient long put with maturity T is given by

$$c_t(\underline{X}_T^{\text{Put}}) = e^{-r(T-t)} E\Big[Z_{T-t}^{\bar{\theta}} \left(K - S_0 e^{F_{L_T}^{-1}(1-F_{L_T}\left(\ln(\frac{y}{S_0})+L_{T-t}\right))}\right)_+\Big]\Big|_{y=S_t}. \tag{5.1}$$

For $\bar{\theta} > 0$ the price of the cost-efficient call option is given by

$$c_t(\underline{X}_T^{\text{Call}}) = e^{-r(T-t)} E\Big[Z_{T-t}^{\bar{\theta}} \left(S_0 e^{F_{L_T}^{-1}(1-F_{L_T}\left(\ln(\frac{y}{S_0})+L_{T-t}\right))} - K\right)_+\Big]\Big|_{y=S_t}. \tag{5.2}$$

Assuming strictly increasing distribution functions F_{L_t} the Greek delta of a cost-efficient payoff $\underline{X}_T$ with differentiable payoff-function $\underline{w}^X$ such that $\frac{\partial \underline{w}^X}{\partial S_t}\left(S_t e^{L_{T-t}}\right) \in \mathcal{L}^1(Z_{T-t}^{\bar{\theta}+1}P)$, where $\bar{\theta}$ is an Esscher parameter, then is given for $t < T$ by

$$\underline{\Delta}_t^X = \frac{\partial}{\partial S_t} c_t(\underline{\omega}^X(S_T)) = E_{\bar{\theta}+1}\left[\frac{\partial \underline{\omega}^X}{\partial S_t}(S_t e^{L_{T-t}})\right]. \tag{5.3}$$

For the basic vanilla payoffs the assumptions on $\underline{w}^{X_T}$ are fulfilled and lead to more concrete formulas in the Lévy models considered in this paper; for example for cost-efficient puts one gets

$$\underline{\Delta}_t^{\text{Put}} = S^* \int_{K^*}^{\infty} e^{\bar{\theta}x + F_{L_T}^{-1}(1-F_{L_T}(\ln(\frac{S_t}{S_0})+x))} \frac{d_{L_T}(\ln(\frac{S_t}{S_0}) + x)}{d_{L_T}(1 - F_{L_T}(\ln(\frac{S_t}{S_0}) + x))} d_{L_{T-t}}(x)\,\mathrm{d}x\,, \tag{5.4}$$

where $K^* := \ln(\frac{S_0}{S_t}) + F_{L_T}^{-1}(1 - F_{L_T}(\ln(\frac{K}{S_0})))$ and $S^* := \frac{S_0}{S_t \cdot M_{L_{T-t}}(\bar{\theta}+1)}$. In cases where the payoff function $\underline{\omega}^X$ of the cost-efficient payoff $\underline{X}_T$ is not explicitly given, as for all options X_T with non-monotone payoff function, the latter result becomes impractical. In such circumstances we utilize the standard approximation $\Delta^X = \frac{\Delta c}{\Delta S}$ for the Greek delta associated to a payoff X_T, where ΔS indicates a small change in the stock price and Δc expresses the corresponding change in the option price. In Figure 23 the relationship between the cost-efficient self-quanto put price and the underlying stock price is illustrated for Volkswagen in the NIG model at time $t = 0$. The Greek delta $\underline{\Delta}_0^{\text{sqP}}$ of the cost-efficient self-quanto put is the slope of the dotted line.

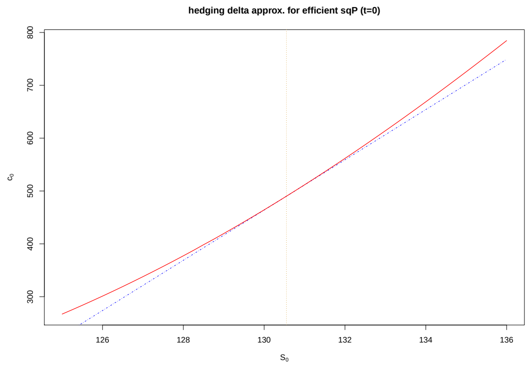

Figure 23: The relationship between the cost-efficient long self-quanto put price with strike $K = 135$, and the underlying stock price at initial time $t = 0$ for the Volkswagen stock in the NIG model, $T = 23$ days. The vertical dotted line marks the actual initial value $S_0 = 130.55$.

Using the representation of the NIG density we get from the formula for the cost of the vanilla put option in (3.5). we obtain

$$\begin{aligned}\Delta_t^{\text{Put}} = \frac{\partial c(X_{T-t}^{\text{Put}})}{\partial S_t} &= -\frac{Ke^{-r(T-t)}}{S_t}\, d_{NIG(\alpha,\beta+\bar{\theta},\delta(T-t),\mu(T-t))}\big(\ln\big(\tfrac{K}{S_t}\big)\big) \\ &\quad -F_{NIG(\alpha,\beta+\bar{\theta}+1,\delta(T-t),\mu(T-t))}\big(\ln\big(\tfrac{K}{S_t}\big)\big) \\ &\quad +d_{NIG(\alpha,\beta+\bar{\theta}+1,\delta(T-t),\mu(T-t))}\big(\ln\big(\tfrac{K}{S_t}\big)\big) \\ &= -\frac{Ke^{-r(T-t)}}{S_t}\, d_{NIG(\alpha,\beta+\bar{\theta},\delta(T-t),\mu(T-t))}\big(\ln\big(\tfrac{K}{S_t}\big)\big) \\ &\quad -F_{NIG(\alpha,\beta+\bar{\theta}+1,\delta(T-t),\mu(T-t))}\big(\ln\big(\tfrac{K}{S_t}\big)\big) \\ &\quad +\frac{Ke^{-r(T-t)}}{S_t}\, d_{NIG(\alpha,\beta+\bar{\theta},\delta(T-t),\mu(T-t))}\big(\ln\big(\tfrac{K}{S_t}\big)\big) \\ &= -F_{NIG(\alpha,\beta+\bar{\theta}+1,\delta(T-t),\mu(T-t))}\big(\ln\big(\tfrac{K}{S_t}\big)\big).\end{aligned} \tag{5.5}$$

Example 5.1 (Simulation of Delta hedging) We investigate the following example: A financial institution has sold for 40 871€ a cost-efficient long put option on 10 000 shares of Volkswagen (cf Table 3 for prices of a cost-efficient put). We assume that this is a non-dividend paying stock. The trading period is October 2012, thus $S_0 = 130.55$€ and $T = 23$ days. Further, the exercising price is $K = 135$€ and the riskless interest rate equals the continuously compounded daily Euribor rate at October 1, 2012, $r = 4.2027 \cdot 10^{-6}$. The hedge is supposed to be adjusted every three trading days, i.e. at October 1st, 4th, 8th, 11th and so on. Table 12 provide a delta hedging scheme for a cost-efficient long put. Initially, the delta equals $\underline{\Delta}_0^{\text{Put}} = 0.4082$. This means that as soon as the option is written, 532 905.10€ must be borrowed to buy 4 082 shares at a price of S_0. The financial institution encounters interest cost of 6.72€ for the first three trading days. If the delta declines, shares are sold to maintain the hedge implying a reduced cumulative and interest cost. Note, towards the end of the life of the option it is not necessary that the delta of a cost-efficient long put approaches 1.0 when it is apparent that the option will be exercised, since this is typically the case for a standard call only. The optimal long put behaves like an modified call and its corresponding payoff function strongly distinguishes from its vanilla counterpart (compare Figure 5).

Days	stock price	$\underline{\Delta}_t^{\text{Put}}$	shares purchased/ sold	cost of shares purchased/ sold	cumulative cost (interest cost)
$t = 1$	130.55	0.4082	(+) 4082	+532 905.10	+532 905.10 (0.00)
$t = 4$	133.55	0.4765	(+) 683	+ 91 009.75	+623 921.57 (6.72)
$t = 7$	133.58	0.4913	(+) 148	+ 19 769.25	+643 698.70 (7.87)
$t = 10$	133.50	0.4971	(+) 58	+ 7 743.00	+651 449.82 (8.21)
$t = 13$	134.60	0.545	(+) 479	+ 64 473.40	+715 931.41 (9.03)
$t = 16$	138.00	0.6829	(+) 1379	+190 302.00	+906 242.44 (11.43)
$t = 19$	143.00	0.8197	(+) 1368	+195 624.00	+1 101 877.87 (13.89)
$t = 22$	149.84	0.8020	(−) 177	−26 521.68	+1 075 370.08 (9.04)

Table 12: Simulation of delta hedging for a cost-efficient long put on 10 000 Volkswagen shares in the NIG model.

hedging cost:	total cost at maturity =	1 075 379.00€
	long position =	−1 227 060.00€
	payoff at maturity $\underline{\omega}^{\text{Put}}(S_T)$ =	181 069.50€
	cost of hedging =	29 388.50€

On one hand, at maturity the total cost for the hedger adds up to 1 075 379€ plus the payoff $\underline{\omega}^{\text{Put}}(S_T) = 181\,069.50$€ for the buyer of the optimal put. On the other hand, by selling the long position on the Volkswagen stock the hedger earns $8\,020 \times 153$€ = 1 227 060€, thus the cost of the option to the writer equals 29 388.50€ which is 11 482.50€ below the actual price of the option. The performance of the delta hedging gets steadily better as the hedge is monitored more frequently.

The analogous simulation of delta hedging of a standard long put on 10 000 Volkswagen stocks is presented in Table 13. Note that the option closes out of the money. The cost of hedging of the standard long put sums up to 65 108.74€ which is 14 955.26€ below the actual price (80 064€) of the option.

Days	stock price	Δ_t^{Put}	shares purchased/ sold	cost of shares purchased/ sold	cumulative cost (interest cost)
$t = 1$	130.55	−0.6050	(−) 6050	−789 827.50	−789 827.50 (0.00)
$t = 4$	133.55	−0.5320	(+) 730	+97 272.50	−692 545.04 (9.96)
$t = 7$	133.58	−0.5270	(+) 50	+ 6 678.80	−685 857.51 (8.73)
$t = 10$	133.50	−0.5363	(−) 93	−12 415.50	−698 264.36 (8.65)
$t = 13$	134.60	−0.4988	(+) 375	+50 475.00	−647 780.56 (8.80)
$t = 16$	138.00	−0.3411	(+) 1577	+217 626.00	−430 146.39 (8.17)
$t = 19$	143.00	−0.1081	(+) 2330	+333 190.00	−96 950.97 (5.42)
$t = 22$	149.84	−0.0026	(+) 1055	+158 081.20	+ 61 130.23 (1.22)

Table 13: Simulation of delta hedging for a long put on 10 000 Volkswagen shares in the NIG model.

hedging cost:	total cost at maturity =	61 130.74€
	short position =	3 978.00€
	payoff at maturity $\omega^{\text{Put}}(S_T)$ =	0.00€
	cost of hedging =	65 108.74€

We see that delta hedging of cost-efficient options is as complex as for standard options if the numerical techniques are present. From the hedgers point of view it is surely beneficial to provide several (differently priced) delta-hedgeable options with identical payoff distributions to its customers. Further hedging simulations of a long call butterfly spread and its cost-efficient counterpart for ThyssenKrupp and a delta hedging simulation of a self-quanto put and its cost-efficient counterpart for Volkswagen in the NIG model are given in Wolf (2014).

5.2 Alternative Delta hedging using cost-efficient strategies

While in Section 5.1 we used cost-efficient strategies at time $t = 0$ and kept the payoff profile fixed up to time T an alternative rollover strategy has been introduced in Hammerstein et al. (2014). In this strategy the cost-efficient payoffs $\underline{X}_{T-t}$ at time t are hedged by a rollover strategy, i.e. an Δ-hedge which reproduces the evolution of the efficient option prices $c(\underline{X}_{T-t})$. This can be regarded as an alternative way to hedge the final payoff X_T. We denote the corresponding hedging deltas by Δ_t^{ro}.

For a cost-efficient put at time t with time to maturity T-t we have

$$\underline{X}_{T-t}^{\text{Put}} = \left(K - S_t e^{F_{L_{T-t}}^{-1}(1-F_{L_{T-t}}(L_{T-t}))}\right)_+ . \tag{5.6}$$

We find

$$\underline{X}_{T-t}^{\text{Put}} \to X_T^{\text{Put}} \quad \text{and} \quad c\left(X_{T-t}^{\text{Put}}\right) - c\left(\underline{X}_{T-t}^{\text{Put}}\right) \to 0 \quad \text{as } t \to T.$$

For $\bar{\theta} < 0$ we have for the alternative delta Δ_t^{roP} of the long vanilla put X_T^{Put} at time t

$$\Delta_t^{\text{roP}} = -\frac{1}{M_{L_{T-t}}(\bar{\theta})} \int_0^{F_{L_{T-t}}\left(\ln\left(\frac{K}{S_t}\right)\right)} e^{\bar{\theta} F_{L_{T-t}}^{-1}(1-y) + F_{L_{T-t}}^{-1}(y) - r(T-t)} \, \mathrm{d}y. \tag{5.7}$$

For $\bar{\theta} > 0$, the alternative delta Δ_t^{roC} of the long vanilla call X_T^{Call} at time t is

$$\Delta_t^{\text{roC}} = \frac{1}{M_{L_{T-t}}(\bar{\theta})} \int_0^{1-F_{L_{T-t}}\left(\ln\left(\frac{K}{S_t}\right)\right)} e^{\bar{\theta} F_{L_{T-t}}^{-1}(y) + F_{L_{T-t}}^{-1}(1-y) - r(T-t)} \, \mathrm{d}y. \tag{5.8}$$

Equations (5.7) and (5.8) imply that the alternative deltas $\Delta_t^{\text{roP}}, \Delta_t^{\text{roC}}$ for the vanilla puts and calls have the same sign as their classical counterparts $\Delta_t^{\text{Put}}, \Delta_t^{\text{Call}}$, which is in line with the intuition. The absolute values of the rollover deltas for calls are smaller then the classical deltas of calls while this is also the case typically for puts.

Comparison of deltas:

1) For a vanilla call and if $\bar{\theta} > 0$, then for each $t \in [0, T)$: $0 \leq \Delta_t^{\text{roC}} \leq \Delta_t^{\text{Call}}$. (5.9)

2) In the put case if $\bar{\theta} < 0$ and $F_{L_{T-t}}\big(\ln\big(\frac{K}{S_t}\big)\big) \leq q^*$ where $q^* \in \left(\frac{1}{2}, 1\right]$ is the unique positive root of

$$\mathsf{D}_P(q) = \frac{1}{M_{L_{T-t}}(\bar{\theta})} \int_0^q e^{\bar{\theta} F_{L_{T-t}}^{-1}(y) + F_{L_{T-t}}^{-1}(y)} - e^{\bar{\theta} F_{L_{T-t}}^{-1}(1-y) + F_{L_{T-t}}^{-1}(y)} \, \mathrm{d}y \tag{5.10}$$

in $[0, 1]$, then $\Delta_t^{\text{Put}} \leq \Delta_t^{\text{roP}} \leq 0$.

For details see Hammerstein et al. (2014). The proof makes use of monotonicity properties of

$$\mathsf{D}_C(q) = \frac{1}{M_{L_{T-t}}(\bar{\theta})} \int_0^q e^{\bar{\theta} F_{L_{T-t}}^{-1}(1-y) + F_{L_{T-t}}^{-1}(1-y)} - e^{\bar{\theta} F_{L_{T-t}}^{-1}(y) + F_{L_{T-t}}^{-1}(1-y)} \, \mathrm{d}y$$

and

$$\mathsf{D}_P(q) = \frac{1}{M_{L_{T-t}}(\bar\theta)} \int_0^q e^{\bar\theta F^{-1}_{L_{T-t}}(y) + F^{-1}_{L_{T-t}}(y)} - e^{\bar\theta F^{-1}_{L_{T-t}}(1-y) + F^{-1}_{L_{T-t}}(y)}\, \mathrm{d}y\,.$$

More precisely it holds under the conditions specified above

1) For $\bar\theta > 0$, $\mathsf{D}_C \geq 0$ in $[0,1]$, D_C is increasing on $\left[0, \frac{1}{2}\right]$ and decreasing on $\left[\frac{1}{2}, 1\right]$.

2) For $\bar\theta < 0$, $\mathsf{D}_P \geq 0$ in $[0, q^*]$, D_P is increasing on $\left[0, \frac{1}{2}\right]$ and decreasing on $\left[\frac{1}{2}, 1\right]$.

As consequence we obtain for cost-efficient bull resp. bear spread options.

Corollary 5.2 *Under the assumptions above we have:*

a) For cost-efficient and vanilla bull spreads with strikes $0 < K_1 < K_2$, holds:
If $\bar\theta > 0$, then $0 \leq \Delta_t^{\text{ro-bull}} \leq \Delta_t^{\text{bull}}$ for $F_{L_{T-t}}(\ln(\frac{K_1}{S_t})) > \frac{1}{2}$ and
$0 \leq \Delta_t^{\text{bull}} \leq \Delta_t^{\text{ro-bull}}$ for $F_{L_{T-t}}(\ln(\frac{K_2}{S_t})) < \frac{1}{2}$.
For $\bar\theta < 0$ we have $\Delta_t^{\text{ro-bull}} = \Delta_t^{\text{bull}}$.

b) In the bear spread case, we have $\Delta_t^{\text{ro-bear}} = \Delta_t^{\text{bear}}$ for $\bar\theta > 0$.
If $\bar\theta < 0$, then $\Delta_t^{\text{ro-bear}} \leq \Delta_t^{\text{bear}} \leq 0$ for $F_{L_{T-t}}(\ln(\frac{K_1}{S_t})) > \frac{1}{2}$ and
$\Delta_t^{\text{bear}} \leq \Delta_t^{\text{ro-bear}} \leq 0$ for $F_{L_{T-t}}(\ln(\frac{K_2}{S_t})) < \frac{1}{2}$.

Proof:
a) Since the vanilla and cost-efficient bull spread coincide for $\bar\theta < 0$, the equation $\Delta_t^{\text{ro-bull}} = \Delta_t^{\text{bull}}$ is obvious. Let $\bar\theta > 0$ and denote by C_i a call option with strike K_i, $i = 1, 2$, then from the definition of a bull spread we easily arrive at

$$c(X_{T-t}^{\text{bull}}) = c(X_{T-t}^{C_1}) - c(X_{T-t}^{C_2}) \quad \text{and} \quad c(\underline{X}_{T-t}^{\text{bull}}) = c(\underline{X}_{T-t}^{C_1}) - c(\underline{X}_{T-t}^{C_2})\,.$$

The corresponding deltas are known and equal

$$\Delta_t^{\text{bull}} = \Delta_t^{C_1} - \Delta_t^{C_2} \quad \text{and} \quad \Delta_t^{\text{ro-bull}} = \Delta_t^{\text{roC}_1} - \Delta_t^{\text{roC}_2}\,.$$

From equations (5.1) and (5.2) for $T-t$ it is easily seen that both $\Delta_t^{C_i}$ and $\Delta_t^{\text{roC}_i}$ are decreasing functions in the strike K_i, i.e. $\Delta_t^{C_1} \geq \Delta_t^{C_2}$ and $\Delta_t^{\text{roC}_1} \geq \Delta_t^{\text{roC}_2}$. Thus, we have $\Delta_t^{\text{bull}} \geq 0$ and $\Delta_t^{\text{ro-bull}} \geq 0$. Now, consider the difference of the deltas for cost-efficient and vanilla bull spread which equals

$$\begin{aligned} \Delta_t^{\text{bull}} - \Delta_t^{\text{ro-bull}} &= (\Delta_t^{C_1} - \Delta_t^{C_2}) - (\Delta_t^{\text{roC}_1} - \Delta_t^{\text{roC}_2}) \\ &= (\Delta_t^{C_1} - \Delta_t^{\text{roC}_1}) - (\Delta_t^{C_2} - \Delta_t^{\text{roC}_2}) \\ &= \mathsf{D}_{C_1}(q_1) - \mathsf{D}_{C_2}(q_2) \end{aligned}$$

where $q_i = 1 - F_{L_{T-t}}(\ln(\frac{K_i}{S_t}))$. Since $q_1 > q_2$ we obtain, using the above stated monotonicity properties of D_C, D_P ,

$$\Delta_t^{\text{bull}} - \Delta_t^{\text{ro-bull}} \geq 0 \text{ for } q_1 < \frac{1}{2} \text{ and } \Delta_t^{\text{bull}} - \Delta_t^{\text{ro-bull}} \leq 0 \text{ for } q_2 > \frac{1}{2}.$$

This proves the assertion.

b) Again, the vanilla and cost-efficient bear spread coincide for $\bar{\theta} > 0$, the equation $\Delta_t^{\text{ro-bear}} = \Delta_t^{\text{bear}}$ is clear. Let $\bar{\theta} < 0$ and denote P_i a put option with strike K_i, $i = 1, 2$, then we obtain from the equations (5.1) and (5.2) that both $\Delta_t^{P_i}$ and $\Delta_t^{\text{roP}_i}$ are decreasing functions in the strike K_i. This implies completely analogous to the bull spread case that $\Delta_t^{\text{bear}} = \Delta_t^{P_2} - \Delta_t^{P_1} \leq 0$ and $\Delta_t^{\text{ro-bear}} = \Delta_t^{\text{roP}_2} - \Delta_t^{\text{roP}_1} \leq 0$.

Moreover, rearranging the difference yields

$$\begin{aligned}\Delta_t^{\text{ro-bear}} - \Delta_t^{\text{bear}} &= (\Delta_t^{\text{roP}_2} - \Delta_t^{\text{roP}_1}) - (\Delta_t^{P_2} - \Delta_t^{P_1}) \\ &= (\Delta_t^{\text{roP}_2} - \Delta_t^{P_2}) - (\Delta_t^{\text{roP}_1} - \Delta_t^{P_1}) \\ &= \mathsf{D}_{P_1}(q_2) - \mathsf{D}_{P_2}(q_1)\end{aligned}$$

where $q_i = F_{L_{T-t}}(\ln(\frac{K}{S_t})$. Since $q_2 > q_1$ we obtain,as above that,

$$\Delta_t^{\text{ro-bear}} - \Delta_t^{\text{bear}} \geq 0 \text{ for } q_2 < \tfrac{1}{2} \text{ and } \Delta_t^{\text{ro-bear}} - \Delta_t^{\text{bear}} \leq 0 \text{ for } q_1 > \tfrac{1}{2}\,.$$

Thus, the statement is proven. □

5.3 Application to real market data

In the following we illustrate the hedging results by some examples for the put case. We first consider the price evolution $(c(X_{T-t}^{\text{Put}}))_{0\leq t\leq T}$ of a vanilla put and a cost-efficient put $c_t(\underline{X}_T^{\text{Put}})_{0\leq t\leq T}$ on the Allianz and the Volkswagen stock which are assumed to be issued on October 1, 2012, and to mature on November 1, 2012. which are assumed to be issued

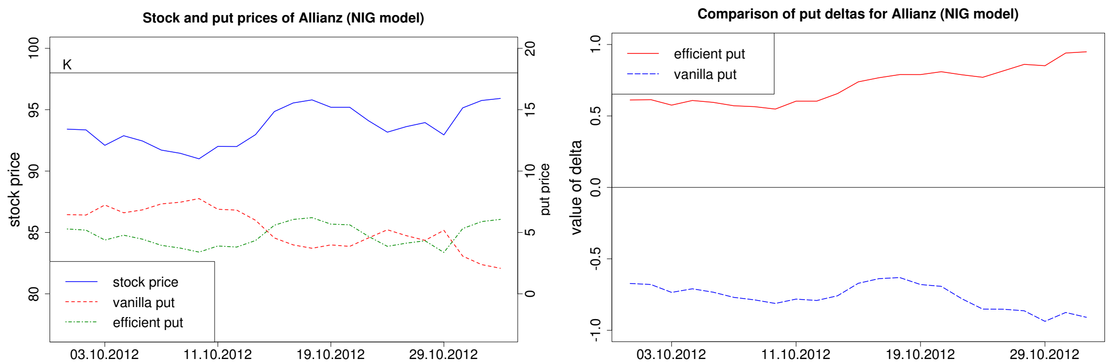

Figure 24: *Left:* Stock price of Allianz and the prices of the associated vanilla resp. efficient put. *Right:* Comparison of the deltas of the vanilla and the efficient put on Allianz.

on October 1, 2012, and to mature on November 1, 2012. Figure 24 shows the prices of the Allianz stock and the corresponding puts with strike $K = 98$ within the aforementioned time period, as well as the values of the deltas $(\Delta_t^{\text{Put}})_{0\leq t\leq T}$ resp. $(\underline{\Delta}_t^{\text{Put}})_{0\leq t\leq T}$ associated to both puts. Here, all calculations are based on the *NIG* model; the *NIG* parameters for Allianz can be found in Table 1. As is obvious from Figure 24, the price of the cost-efficient put evolves almost exactly in the opposite way as that of the vanilla put. This reflects the fact that the payoff profiles of both puts are, in some sense, reversed to each other (see Figure 5); the efficient put roughly behaves like a vanilla call. However, the efficient put ends in the money although the price of the Allianz stock remains below the strike price at maturity because its payoff function already takes positive values for some $S_T < K$. The opposite behavior of the efficient and the vanilla put is also mirrored in the values of the associated deltas. Because the values of the deltas at maturity are not relevant for hedging purposes anymore, Figure 24 only shows the deltas up to one day to maturity, that is, from October 1, 2012, to October 31, 2012. The results obtained for the other two Lévy models (normal and VG) look quite similar and therefore are not plotted here separately. Since the risk-neutral Esscher parameter roughly are of the same size for all three models (see Table 1) and also the put prices and efficiency losses in Table 3 are almost identical, one should not expect greater differences here.

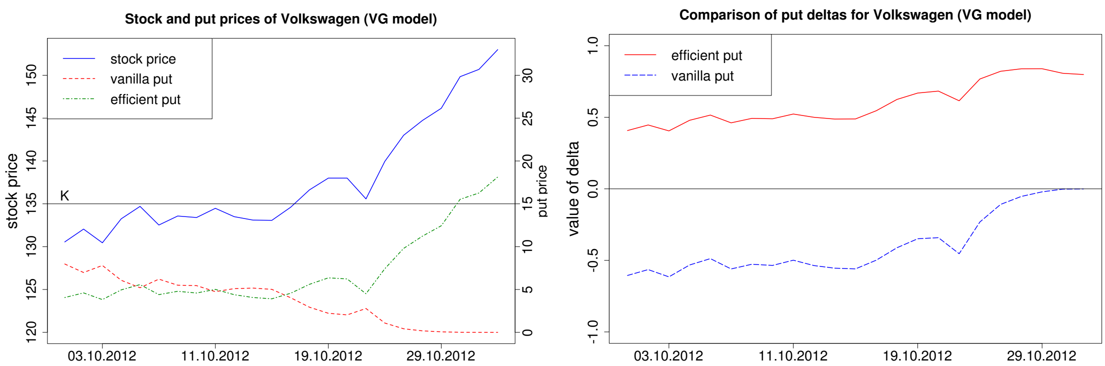

Figure 25: *Left:* Stock price of Volkswagen and the prices of the associated vanilla resp. efficient put. *Right:* Comparison of the deltas of the vanilla and the efficient put on Volkswagen.

Figure 25 shows the evolution of the prices of the Volkswagen stock and the cost-efficient and vanilla puts on it with strike $K = 135$ as well as the corresponding deltas. Again, the results do not differ much between all three Lévy models under consideration, thus we only show the plots for the VG case. The delta of the vanilla put in this model can be derived analogously as above to be

$$\Delta_t^{\text{Put}} = \frac{\partial c(X_{T-t}^{\text{Put}})}{\partial S_t} = -F_{VG(\lambda(T-t),\alpha,\beta+\bar{\theta}+1,\mu(T-t))}\big(\ln\big(\tfrac{K}{S_t}\big)\big).$$

Note that in this example we have $S_T > K$, therefore the vanilla put expires worthless, and the corresponding delta converges to zero, whereas the efficient put ends deep in the money.

However, computing the put deltas is only one side of the coin, market participants will surely be more interested in how well the hedging strategies based on them work in practice. The *NIG* and *VG* models are incomplete, so one cannot expect perfect hedging there, but also the Samuelson model is only complete in theory. Since in reality just discrete hedging is feasible, one will encounter hedge errors within this framework, too. The magnitude of these errors is, of course, relevant for practical applications. Therefore, we also calculate and compare the hedge errors that occur in delta hedging of the vanilla and efficient puts on Allianz and Volkswagen considered before.

Delta hedging strategy The hedge portfolios are rebalanced daily, hence the portfolio weights δ_t (amount of stock at time t) and b_t (amount of money on the savings account at t) just have to be calculated at the discrete times $t = 0, 1, \ldots, T-1$. For the vanilla puts

$\delta_t = \Delta_t^{\text{Put}}$, and in case of the efficient puts we have $\delta_t = \underline{\Delta}_t^{\text{Put}}$. Depending on the put type under consideration, we analogously set $c_t = c(X_{T-t}^{\text{Put}})$ or $c_t = c_t(\underline{X}_T^{\text{Put}})$, respectively. At the initial time $t = 0$, the hedge portfolio is set up with the weights δ_0 and $b_0 = -\delta_0 S_0 + c_0$ since the writer of the put obtains c_0 from the buyer, shorts $|\delta_0|$ stocks and deposits all income on his savings account. At time $t > 0$, the value of the portfolio *before* rebalancing is $\delta_{t-1} S_t + e^r b_{t-1}$, and we define the corresponding hedge error by

$$\mathrm{e}_t := c_t - \delta_{t-1} S_t - e^r b_{t-1},$$

so positive hedge errors mean losses and negative gains. At the end of the trading day, the new weights δ_t and $b_t = c_t - \delta_t S_t$ are chosen to ensure that the value of the portfolio again exactly coincides with the present put price. Using the above definition of e_t, we can alternatively represent b_t in the form

$$b_t = \mathrm{e}_t + e^r b_{t-1} + S_t(\delta_{t-1} - \delta_t).$$

This means that the hedge error is nothing but the amount of money one has to additionally inject in or withdraw from the savings account after adapting the stock position to make the value of the hedge portfolio congruent with the current put price.

Remark 5.3 *In general, the size of the hedge error also depends on the rebalancing frequency and the continuity properties of the payoff function. Our empirical results below show that for standard and efficient puts a daily rebalancing of the portfolio already is sufficient to get a fairly precise approximation to the current option prices. A thorough theoretical analysis of the behavior of hedge errors resulting from delta and quadratic hedging strategies in exponential Lévy models can be found in Brodén and Tankov (2011).*

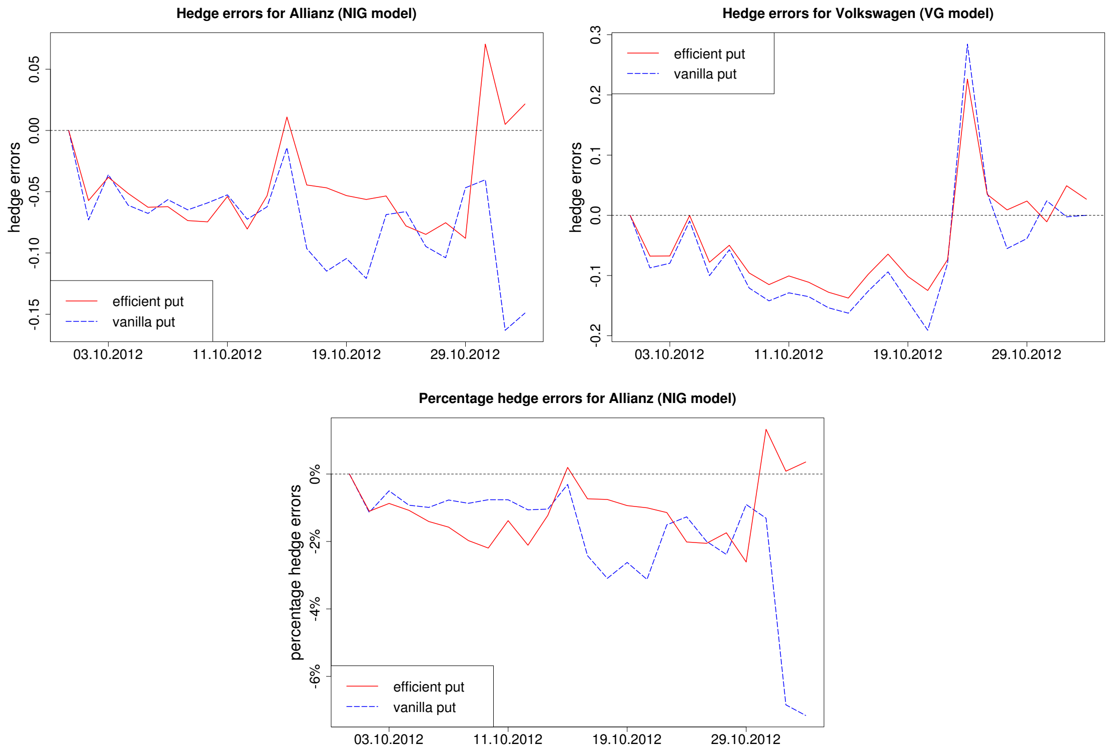

Figure 26: Delta hedge errors of the efficient and vanilla puts on Allianz with strike $K = 98$ and Volkswagen with strike $K = 135$.

The upper graphs of Figure 26 display the hedge errors obtained from delta hedging of the different puts on Allianz and Volkswagen. At the beginning, the hedge errors of the efficient and the vanilla puts behave fairly similarly, but with time passing the distinctions increase. This might again be explained by the different shapes of the payoff profiles and the different signs of the corresponding deltas which lead to more pronounced differences in the hedge errors as the time to maturity becomes smaller. The sums $\sum_{t=0}^{22} |\mathrm{e}_t|$ of the absolute hedge errors for Allianz are 1.296 (efficient put) and 1.798 (vanilla put), for Volkswagen we obtain 1.794 (efficient put) resp. 2.252 (vanilla put). This indicates that cost-efficient options can be hedged at least as efficiently as standard options. However, since the prices of vanilla and efficient puts can differ significantly over time, one should not only look at the absolute hedge errors to confirm this assertion, but also take the relative or percentage hedge errors $\tilde{\mathrm{e}}_t := \frac{\mathrm{e}_t}{c_t}$ into account. The values of $\tilde{\mathrm{e}}_t$ for the Allianz puts are shown in the lower graph of Figure 26 above. For the efficient put, we obtain $\sum_{t=0}^{22} |\tilde{\mathrm{e}}_t| = 0.299$, and the corresponding value for the vanilla put is 0.438. Analogous computations for the Volkswagen puts would not make much sense here because there the vanilla put ends up deep out of the money, therefore the $\tilde{\mathrm{e}}_t$ would tend to infinity as t approaches T.

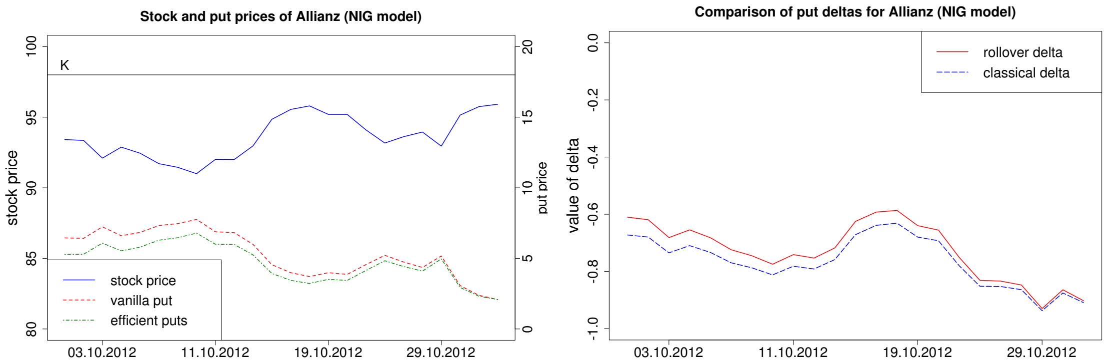

Figure 27: *Left:* Stock price of Allianz and the prices of the associated vanilla resp. efficient puts. *Right:* Comparison of the deltas Δ_t^{Put}, Δ_t^{roP} of the vanilla put on Allianz on the left.

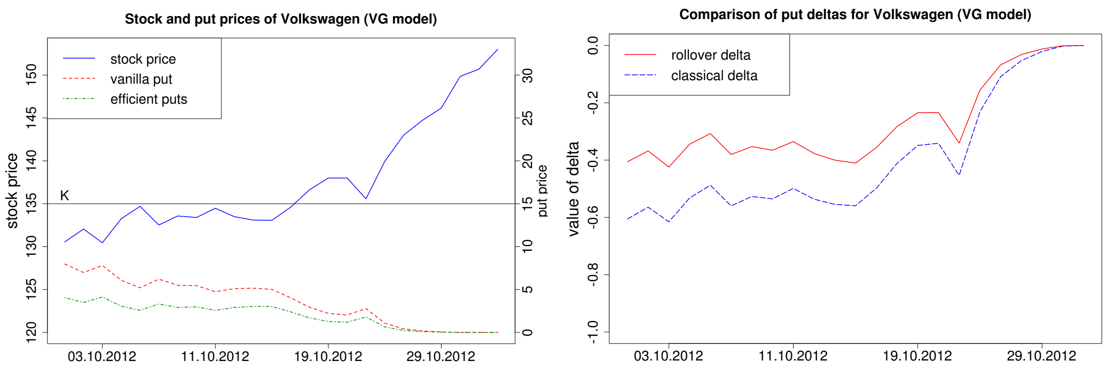

Figure 28: *Left:* Stock price of Volkswagen and the prices of the associated vanilla resp. efficient puts. *Right:* Comparison of the deltas Δ_t^{Put}, Δ_t^{roP} of the vanilla put on Volkswagen on the left.

In the last part of this section, we compare the alternative hedging strategy for vanilla puts based on the rollover-deltas Δ_t^{roP} with its classical counterpart and investigate if it can provide an efficient and more robust way to hedge the final put payoff $(K - S_T)_+$

as expected from our comparison result. For this purpose, we again consider the vanilla puts on Allianz and Volkswagen with the same strikes and maturity as before, but now contrast the corresponding price processes $(c(X^{\text{Put}}_{T-t}))_{0\le t\le T}$ with the series $(c(\underline{X}^{\text{Put}}_{T-t}))_{0\le t\le T}$ of prices of efficient puts which are newly initiated at each day t. Figures 27 and 28 show the stock and put price processes for Allianz in the NIG model and for Volkswagen in the VG model, respectively, as well as a graphical comparison of the associated classical put deltas Δ^{Put}_t and rollover- deltas Δ^{roP}_t. The condition of the comparison results in (5.9), (5.10) is fulfilled for all $0 \le t \le T$, the absolute values of the rollover-deltas are always smaller than those of the classical deltas for both stocks.

This indicates that the hedging strategies based on the rollover-deltas may indeed allow for a less expensive way to replicate the final put payoff. The advantage of lower hedging costs might be annihilated by larger hedging errors though. Therefore one has to take these into account before coming to a conclusion. Using some of the notations from above, we define the hedge error for the alternative hedging strategy by

$$\mathrm{e}_t := c(\underline{X}^{\text{Put}}_{T-t}) - \Delta^{\text{roP}}_t S_t - e^r b_{t-1}.$$

Observe that we do not use the price $c(X^{\text{Put}}_{T-t})$ of the vanilla put at time t in the above definition although we want to hedge its final payoff. Since the rollover-deltas Δ^{roP}_t are intended to replicate the prices $c(\underline{X}^{\text{Put}}_{T-t})$, and $c(\underline{X}^{\text{Put}}_{T-t}) < c(X^{\text{Put}}_{T-t})$ for all $0 \le t < T$ because $\bar{\theta} < 0$ here, a comparison of the value of the hedge portfolio at time t with $c(X^{\text{Put}}_{T-t})$ would lead to a systematic overestimation of the hedge error. Moreover, we only consider options of European type here. Therefore it is more important to look at the hedge error at maturity which tells us how precise the hedging strategies can reproduce the final obligation of the writer of the option. At time T, however, we have $c(\underline{X}^{\text{Put}}_{T-T}) = c(X^{\text{Put}}_{T-T}) = (K - S_T)_+$ as pointed out before, so there the hedge error is defined without ambiguity.

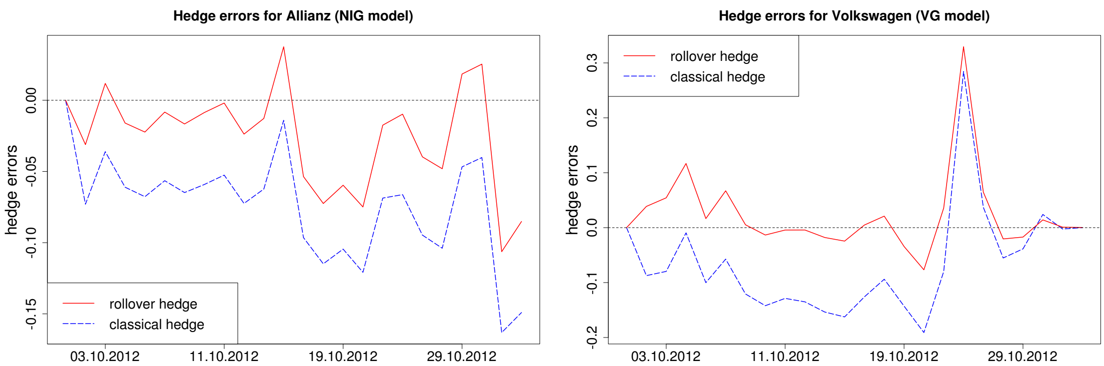

Figure 29: *Left:* Delta hedge errors of the vanilla put on Allianz with strike $K = 98$ in the NIG model. *Right:* Delta hedge errors of the vanilla put on Volkswagen with strike $K = 135$ in the VG model.

We finally take a look at the hedge errors obtained from the two delta hedging strategies for the vanilla puts on Allianz and Volkswagen which are visualized in Figure 29. For Allianz, the hedge errors e_T at maturity are -0.149 for the classical delta hedge and -0.085 for the alternative rollover-delta hedge, and the sum $\sum_{t=0}^{22} |\mathrm{e}_t|$ of the absolute hedge errors is 1.789 for the classical and 0.802 for the rollover hedge. The final hedge errors e_T for the Volkswagen put are zero for both hedging strategies (which is not so surprising because the vanilla put expires worthless here), and the sums of the absolute hedge errors are 2.252 for the classical and 0.983 for the rollover hedge. This shows that the latter can yield at least comparable and often even more accurate results than the classical delta hedging strategy. In case of the Allianz put, the classical delta hedge tends to superhedge the option, that is, the value of the hedge portfolio is always greater than the option price. The rollover hedge does the same on most days, but produces smaller absolute hedge errors. In view of the comparison results in (5.9), (5.10), we suppose that analogous assertions will also hold for calls and probably also for more complex options.

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