Optimal multiple stopping with sum-payoff

Andreas Faller and Ludger Rüschendorf
University of Freiburg

Abstract

We consider the optimal stopping of independent, discrete time sequences $X_1, \ldots, X_n$ where m stops are allowed. The payoff is the sum of the stopped values. Under the assumption of convergence of related imbedded point processes to a Poisson process in the plane we derive approximatively optimal stopping times and stopping values. The solutions are obtained via a system of m differential equations of first order. As application we consider the case that $X_i = c_i Z_i + d_i$ with (Z_i) i.i.d. in the domain of attraction of an extreme value distribution. We obtain explicit results for stopping values and approximative optimal stopping rules.

Keywords: optimal multiple stopping, best choice problem, extreme values, imbedded Poisson process

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1 Introduction

For discrete time sequences $X_1, \ldots, X_n$ we consider the optimal stopping problem with m allowed stops $1 \leq T_1 < \cdots < T_m \leq n$. The aim is to choose the stopping times T_i such that the expectation of the sum of the stopped values is maximized, i.e.

$$E \sum_{i=1}^{m} X_{T_i} = \sup E \sum_{i=1}^{m} X_{\tau_i} \tag{1.1}$$

over all stopping times $1 \leq \tau_i < \cdots < \tau_m \leq n$.

It is essential to assume that $\tau_1 < \tau_2 < \cdots < \tau_m$. The case where inequality constraints $\tau_1 \leq \cdots \leq \tau_m$ are assumed, reduces this problem to the one-stopping problem. The optimal stopping times are then given by $T_1 = \cdots = T_m$, where T_1 is optimal for the one stopping problem.

In some instances problem (1.1) has been solved in Saario (1992); Stadje (1985, 1990); Saario and Sakaguchi (1992) while an extension of the problem to vector offers is considered in Sakaguchi (1973, 1978); Stadje (1985); Bruss and Ferguson (1997); Bruss (2010). A solution of problem (1.1) as proposed in our paper is so far not available in the literature. Applications of multiple stopping problems as considered in this paper and some variations of them with additional constraints are of interest in recent work on multi-exercise options as e.g. for swing options in energy markets (see, e.g., Bender (2010)).

Our aim in this paper is to obtain approximative optimal solutions for general independent sequences. We shall make use of the approach developed in [KR] (2000a) and extended in [FR] (2011a).[1] The basic assumption in this approach is the convergence of the imbedded 2-dimensional point processes

$$N_n = \sum_{i=1}^{n} \delta_{(\frac{i}{n}, X_i^n)} \xrightarrow{d} N \tag{1.2}$$

to some Poisson process N in the plane, where $X_i^n = \frac{X_i - a_n}{b_n}$ is a scaled version of X_i. The scalings typically arise from the central limit theorem for maxima or related point process convergence results. It is shown in these papers that the optimal one-stopping problem of the (X_i) can be approximatively obtained from the optimal solution in the limit case of a Poisson process. The solution for the limit case is given by a differential equation of first order.

This approach has been extended to m-stopping problems with max-payoff where the aim is to maximize the expected value of the max of the stopped values, i.e.

$$E \max(X_{T_1}, \dots, X_{T_m}) = \sup \tag{1.3}$$

over all stopping sequences $1 \le T_n < \cdots < T_m$ in [FR] (2011b). Note that for the max stopping problem we could also admit stopping times with inequality constraints of the form $T_i \le T_{i+1}$.

For the max case an approximation result has been stated even for dependent sequences. The present paper is concerned with a development of this method for the sum case. Due to some technical problems with an extension of the discretization technique – which was used in the max case – to the sum case for dependent variables, we restrict in this paper to independent sequences.

In Section 2 we start with a necessary recursive formulation (optimality principle) for the optimal multiple stopping in the sum case. In Section 3 we describe the solution of the m-stopping problem in the Poisson case. In Section 4 we establish convergence of the discrete time m-stopping problem to the optimal m-stopping of the continuous time Poisson process. As application we consider in Section 5 the case where $X_i = c_i Z_i + d_i$ with (Z_i) an i.i.d. sequence and with discount and observation cost factors c_i, d_i. We obtain explicit solutions in the case that the distribution F of Z_i is in the domain of attraction of a max-stable law.

[1] [KR] is the abbreviation for Kühne and Rüschendorf, [FR] the one for Faller and Rüschendorf.

2 Multiple stopping of finite sequences

For a finite discrete time sequence $X_1, \dots, X_n$ of not necessarily independent random variables a version of the optimality equation can be stated as follows for the m-stopping problem with payoff given by the sum.

Define by backwards induction for $m \in \mathbb{N}$ the sequence of *optimal thresholds* inductively w.r.t. the underlying filtration $(\mathcal{F}_i)_{1 \le i \le n-m+1}$ by

$$\begin{aligned} W^m_{n-m+1} &:= -\infty \\ W^m_i &:= E[(X_{i+1} + W^{m-1}_{i+1}) \vee W^m_{i+1} \mid \mathcal{F}_i], \qquad i = n-m, \dots, 0 \end{aligned} \tag{2.1}$$

and define the corresponding threshold stopping times $T^m_\ell(k)$ for $1 \le m \le n$, $0 \le k \le n-m$ by

$$\begin{aligned} T^m_1(k) &:= \min\{k < i \le n-m+1 \mid X_i + W^{m-1}_i > W^m_i\} \\ T^m_\ell(k) &:= \min\{T^m_{\ell-1}(k) < i \le n-m+\ell \mid X_i + W^{m-\ell}_i > W^{m-\ell+1}_i\}, \quad 2 \le \ell \le m. \end{aligned} \tag{2.2}$$

Then, as in the one-stopping problem it holds that $(T^m_i(k))$ are optimal stopping times.

Proposition 2.1 (Multiple stopping, sum case) *$(T^m_\ell(k))$ are optimal m-stopping times in the sum case in the sense that for all stopping times $k < T_1 < \dots < T_m \le n$ it holds that*

$$\begin{aligned} E\left[\sum_{\ell=1}^m X_{T^m_\ell(k)} \mid \mathcal{F}_k\right] &= E\left[X_{T^m_1(k)} + W^{m-1}_{T^m_1(k)} \mid \mathcal{F}_k\right] \\ &= W^m_k \ge E\left[\sum_{\ell=1}^m X_{T_\ell} \mid \mathcal{F}_k\right] P \ a.s. \end{aligned} \tag{2.3}$$

Proof: The proof is by induction in m. Our extended induction hypothesis is that for any $\mathcal{F}$-stopping time $S \le n-m$ it holds that

$$\begin{aligned} E\left[\sum_{\ell=1}^m X_{T^m_\ell(k)} \mid \mathcal{F}_S\right] &= E\left[X_{T^m_1(S)} + W^{m-1}_{T^m_1(S)} \mid \mathcal{F}_S\right] \\ &= W^m_S \ge E\left[\sum_{\ell=1}^m X_{T_\ell} \mid \mathcal{F}_S\right] P a.s. \end{aligned} \tag{2.4}$$

for all stopping times $S < T_1 < \dots < T_m \le n$.

In the xase $m = 1$ (2.4) is proved in [FR] (2011b, Remark 2.2). For the induction step $m \to m+1$ let $S < T_1 < \dots < T_{m+1} \le n$. By induction hypothesis we obtain

$$E[X_{T_1} + \dots + X_{T_{m+1}} \mid \mathcal{F}_S] = E[X_{T_1} \mid \mathcal{F}_S] + E[E[X_{T_2} + \dots + X_{T_{m+1}} \mid \mathcal{F}_{T_1}] \mid \mathcal{F}_S]$$

$$\leq E[X_{T_1} \mid \mathcal{F}_S] + E[X_{T_1^m(T_1)} + \cdots + X_{T_m^m(T_1)} \mid \mathcal{F}_S]$$
$$= E[X_{T_1} + W_{T_1}^m \mid \mathcal{F}_S].$$

This is maximized by choosing T_1 to be the optimal one-stopping time of the process $(X_i + W_i^m)$. This optimal stopping time is given from the case $m = 1$ by $T_1 = T_1^{m+1}(S)$. In consequence the optimal stopping times for $1 \leq \ell \leq m$ are obtained by $T_\ell^m(T_1^{m+1}(S)) = T_{\ell+1}^{m+1}(S)$ and the maximizing value is given by W_S^{m+1}. □

3 Optimal m-stopping of Poisson processes

We consider optimal m-stopping of a Poisson process $N = \sum \delta_{(\tau_k, y_k)}$ in the plane restricted to some set $M_c = \{(t, x) \in [0, 1] \times \overline{\mathbb{R}}; x > c\}$. For the applications in mind it is essential to consider the case where the intensity of N may be infinite along the lower boundary of M_c.

As in [KR] (2000a) resp. [FR] (2011a) who consider the case $m = 1$ we assume that the intensity measure μ of N is a Radon measure on M_c with the topology on M_c induced by the usual topology on $[0, 1] \times \overline{\mathbb{R}}$. Thus any compact set $A \subset M_c$ has only finitely many points. By convergence in distribution '$N_n \xrightarrow{d} N$ on M_c' we mean convergence in distribution of the restricted point processes.

We generally assume the *boundedness condition*

$$\text{(B)} \qquad E[(\sup_k Y_k)^+] < \infty. \tag{3.1}$$

Let $\mathcal{A}_t = \sigma(N(\cdot \cap [0, t] \times \overline{\mathbb{R}} \cap M_f))$, $t \in [0, 1]$, denote the relevant filtration of the point process N. A stopping time for N or N-stopping time is a mapping $T : \Omega \to [0, 1]$ with $\{T \leq t\} \in \mathcal{A}_t$ for each $t \in [0, 1]$. Denote by

$$\overline{Y}_T := \sup\{Y_k : 1 \leq k \leq N(M_f), T = \tau_k\}, \quad \sup \emptyset := -\infty,$$

the reward w.r.t. stopping time T.

Analogously to the multiple stopping in the max case (see [FR] (2011b)) we define the optimal m-stopping curves with guarantee c by

$$u^m(t) := \sup \left\{ E\left[\overline{Y}_{T_1} \vee c + \cdots + \overline{Y}_{T_m} \vee c\right] \mid t < T_1 < \cdots < T_m \leq 1 \right\}, \quad t \in [0, 1) \tag{3.2}$$
$$u^m(1) := mc$$

the supremum being over all stopping sequences. The order restrictions are interpreted as $T_{i-1} < T_i$ on $\{T_{i-1} < 1\}$ and $T_i = 1$ on $\{T_{i-1} = 1\}$.

As in the previous work we also need the following conditions:

$$\text{(S) } \textit{Separability condition:} \qquad u^m(t) > c \text{ for } t \in [0, 1). \tag{3.3}$$

(D) *Differentiability condition*:
There exists a version of the density g of μ on M_c, such that the intensity function

$$G(t,y) := \int_y^\infty g(t,z)dz \tag{3.4}$$

is continuous on $M_c \cap [0,1] \times \mathbb{R}$.

Theorem 3.1 *Let the Poisson process satisfy the boundedness condition (B) and the separation condition (S).*

a) If $c \in \mathbb{R}$, then it holds for $m \in \mathbb{N}$ and $t \in [0,1)$ that $c < u^m(t) - u^{m-1}(t) \leq u^1(t)$. Further

$$\begin{aligned} u^m(t) &= E\left[Y_{T_1^m(t)} \vee c + \cdots + Y_{T_m^m(t)} \vee c\right] \\ &= E\left[Y_{T_1^m(t)} \vee c + u^{m-1}(T_1^m(t))\right] \end{aligned} \tag{3.5}$$

with optimal stopping times given by

$$\begin{aligned} T_1^m(t) &:= \inf\{\tau_k > t \mid Y_k + u^{m-1}(\tau_k) > u^m(\tau_k)\}, \\ T_l^m(t) &:= \inf\{\tau_k > T_{l-1}^m(t) \mid Y_k + u^{m-l}(\tau_k) > u^{m-l+1}(\tau_k)\}, \quad \inf\emptyset := 1, \end{aligned}$$

for $2 \leq l \leq m$. u^m is the optimal stopping curve of the point process

$$N^m := \sum_k \delta_{(\tau_k, Y_k + u^{m-1}(\tau_k))} \qquad \text{in } M_{c+u^{m-1}}$$

with guarantee value mc. Under the differentiability condition (D) *for N, u^m solves the differential equation*

$$\begin{aligned} \frac{\partial}{\partial t} u^m(t) &= -\int_{u^m(t)-u^{m-1}(t)}^\infty G(t,y)dy \qquad \text{for } t \in [0,1), \\ u^m(1) &= mc. \end{aligned} \tag{3.6}$$

b) If $c = -\infty$ and if (S) *also holds for N^k, $0 \leq k \leq m$, then the statements in a) extend to this case.*

Proof: The proof is by induction in m. The induction hypothesis is extended to

$$E[\overline{Y}_{T_1^m(S)} \vee c + \cdots + \overline{Y}_{T_m^m(S)} \vee c] = Eu^m(S) \geq E[\overline{Y}_{T_1} \vee c + \cdots + \overline{Y}_{T_m} \vee c]$$

for all $S < T_1 < \cdots < T_m$. The case $m = 1$ is proved in [FR] (2011b, Prop. 3.1). For the induction step $m \to m+1$ let S be an N stopping time. Then for $S < T_1 < \cdots < T_{m+1} \leq 1$ holds by the induction hypotheses

$$\begin{aligned} &E[\overline{Y}_{T_1} \vee c + \ldots + \overline{Y}_{T_{m+1}} \vee c] \\ &\quad = E[\overline{Y}_{T_1} \vee c] + E[\overline{Y}_{T_2} \vee c + \cdots + \overline{Y}_{T_{m+1}} \vee c] \end{aligned}$$

$$\leq E[\overline{Y}_{T_1} \vee c] + E\overline{Y}_{T_1^m(T_1)} \vee c + \cdots + \overline{Y}_{T_m^m(T_1)} \vee c]$$
$$= E[\overline{Y}_{T_1} \vee c + u^m(T_1)]. \tag{3.7}$$

To solve the one-stopping problem in (3.7) we apply Proposition 3.1 from [FR] (2011a) with $v(t,z) := z + u(t)$, $Z := c$. We have to establish for $t \in [0,1)$ the separation condition i.e. the existence of a stopping time $T > t$ with

$$E[\overline{Y}_T \vee c + u^m(T)] > c + u^m(t). \tag{3.8}$$

For $c = -\infty$ this is fulfilled by the separation condition for N^m.

For $c \in \mathbb{R}^1$ we choose $T := T_1^m(t)$ and obtain by induction hypothesis

$$\begin{aligned} E[\overline{Y}_T \vee c + u^m(T)] &= E[\overline{Y}_T \vee c + u^{m-1}(T)] + E[u^m(T) - u^{m-1}(T)] \\ &= u^m(t) + E[u^m(T) - u^{m-1}(T)]. \end{aligned}$$

Since by induction hypothesis $u^m(s) - u^{m-1}(s) > c$ for $s \in [0,1)$ we just have to show that $P(T < 1) > 0$. To this aim note that

$$\begin{aligned} P(T<1) &= P(\exists \tau_k > t : Y_k > u^m(\tau_k) - u^{m-1}(\tau_k)) \\ &= P(N(M_{u^m - u^{m-1}} \cap (t,1] \times \mathbb{R}) \geq 1) \\ &= 1 - \exp(-\mu(M_{u^m-u^{m-1}} \cap (t,1] \times \mathbb{R})) > 0 \end{aligned}$$

if and only if $\mu(M_{u^m-u^{m-1}} \cap (t,1] \times) > 0$. Since $u^m - u^{m-1} \leq u$ and $\mu(M_u \cap (t,1] \times \mathbb{R}) > 0$ this is the case. From $\mu(M_u \cap (t,1] \times \mathbb{R}) = \int_t^1 \int_{u(s)}^\infty g(s,y)dyds = 0$ we could conclude that $u'(s) = 0$ for all $s \in [t,1]$ in contradiction to (S).

Thus we can apply Proposition 3.1 in [FR] (2011a) for the case $m = 1$ and obtain that the optimal stopping time in (3.7) is given by $T_1^{m+1}(S)$. The optimality of the stopping times $T_\ell^{m+1}(S)$ then follows from the recursive equation $T_\ell^m(T_1^{m+1})(S)) = T_{\ell+1}^{m+1}(S)$, see Proposition 2.1. □

Remark 3.2 *The differential equations in* (3.6) *for the optimal stopping curves have been derived in the special case that* $G(t,y) = \lambda(1 - F(y))$ *in Saario (1992). This concerns the case where an i.i.d.-sequence of random variables with d.f.* F *is observed at random time points from a homogeneous Poisson process with intensity* λ*. In our general case we have to deal with the problem that we have infinitely many points along the lower boundary points.*

Remark 3.3 (Calculation of optimal m-stopping curves) *In the case that the intensity function* G *is separable i.e. of the form* $G(t,y) = a(t)H(y)$ *explicit solutions of the optimality differential equations in* (3.6) *can be obtained from classical results on differential equations in separate variables (see [KR] (2000a, Prop. 2.6)). In [FR] (2011a) explicit solutions of* (3.6) *in the case* $m = 1$ *have been found for intensity functions of the form*

$$G(t,y) = H\left(\frac{y}{v(t)}\right)\frac{v'(t)}{v(t)} \tag{3.9}$$

$$\text{and} \quad G(t,y) = H(y - v(t))v'(t). \tag{3.10}$$

This observation extends also to m-stopping problems and yields the following explicit results:

a) *Let G satisfy* (3.9) *and consider the case C1) in [FR] (2011b) with v monotonically nonincreasing,* $v(1) = 0$ *and with* $R(x) := x - \int_x^\infty H(y)dy = 0$ *for some* $x \geq 0$. *Then it holds that* $c = 0$ *and we obtain for* $m \in \mathbb{N}$

$$u^m(t) = r_m v(t), \tag{3.11}$$

where $r_m > r_{m-1} \geq 0$ *solve the equation*

$$r_m = \int_{r_m - r_{m-1}}^\infty H(y)dy, \quad \text{with } r_0 := 0. \tag{3.12}$$

Thus the system of differential equations in (3.6) *reduces to the much simpler equations in* (3.12).

b) *If G satisfies* (3.9) *and consider the case C2) in [FR] (2011b) with v monotonically nondecreasing,* $v(1) = \infty$ *and assume that* $R(r) = 0$ *for some* $-\infty < r < 0$, *where*

$$R(x) := x + \int_x^\infty H(y)dy, \quad x \in (-\infty, \infty)$$
$$R(-\infty) := \lim_{x \downarrow -\infty} R(x).$$

Then it holds that $c = -\infty$ *and*

$$u^m(t) = r_m v(t), \quad m \in \mathbb{N}.$$

Here $r_m < r_{m-1} \leq 0$ *are solutions of the equations*

$$r_m = -\int_{r_m - r_{m-1}}^0 H(y)dy, \quad r_0 := 0. \tag{3.13}$$

c) *Let G satisfy* (3.10) *and consider the case C3) in [FR] (2011b) with* $a = -\infty$. *Then* $c = -\infty$ *and*

$$u^m(t) = r_m v(t), \tag{3.14}$$

where $r_m < r_{m-1} \leq 0$ *solve the equations*

$$r_m = \int_{r_m - r_{m-1}}^\infty H(y)dy, \quad r_0 := 0. \tag{3.15}$$

d) *Also the case of intensity functions of the form*

$$G_{c,d}(t,y) = \begin{cases} 0 & \text{if } \frac{y}{v(t)} \geq t, \\ \frac{1}{t}\left(-\frac{y}{v(t)} + d\right)^2 & \text{if } \frac{y}{v(t)} < t, \end{cases} \tag{3.16}$$

with $v(t) := t^{c-\frac{1}{\alpha}}$ *as treated in Example 3.5 in [FR] (2011a) allows a solution in the m-stopping case since the processes* $N^m = N^m_{c,d}$ *satisfy the separation condition. This case will also appear in our examples in Section 5. For details see Faller (2009).*

4 Approximation of m-stopping problems

The aim of this section is to prove that the discrete time m-stopping problem can be approximated by the m-stopping problem of the limit model given by the Poisson process N. The general assumption in this section is that $N_n = \sum_{i=1}^n \delta_{(\frac{i}{n}, X_i^n)} \xrightarrow{d} N$ on M_c and also the separation condition (S) for the transformed Poisson processes N^k, $0 \le k \le m-1$, as defined in Theorem 3.1. The proof is based on a discretization technique. Due to technical difficulties we assume in this section that $(X_i^n)_{1\le i\le n}$ are independent – an extension to non independent sequences however seems to be possible. Let $\mathcal{F}^n$ denote the canonical discrete filtrations.

Define $W_i^{n,0} := 0$ and inductively for $m \in \mathbb{N}$, $W_1^{n,m}, \dots, W_{n-m+1}^{n,m}$ the sequence of optimal thresholds of the processes $(X_i^n + W_i^{n,m-1})_{1\le i\le n-m+1}$. For $t \in [0,1]$ define

$$u_n^m(t) := \begin{cases} W_{\lfloor tn\rfloor}^{n,m}, & \text{if } t \in [0, \frac{n+m-1}{n}), \\ 0, & \text{if } t \in [\frac{n+m-1}{n}, 1]. \end{cases}$$

Thus for $m \in \mathbb{N}$ and $t \in [0, \frac{n+m-1}{n})$

$$\begin{aligned} u_n^m(t) &= \sup\left\{ E[X_{T_1}^n + \dots + X_{T_m}^n] \,\middle|\, \begin{array}{c} tn < T_1 < \ldots < T_m \le n \\ \mathcal{F}^n \text{ stopping times} \end{array} \right\} \\ &= E[X^n_{T_1^{n,m}(t)} + \dots + X^n_{T_m^{n,m}(t)}] \end{aligned}$$

with optimal m-stopping times given by

$$\begin{aligned} T_1^{n,m}(t) &:= \min\{tn < i \le n-m+1 \mid X_i^n + u_n^{m-1}(\tfrac{i}{n}) > u_n^m(\tfrac{i}{n})\}, \\ T_l^{n,m}(t) &:= \min\{T_{l-1}^{n,m}(t) < i \le n-m+l \mid X_i^n + u_n^{m-l}(\tfrac{i}{n}) > u_n^{m-l+1}(\tfrac{i}{n})\} \end{aligned}$$

for $2 \le l \le m$. Then as in (2.3) the following recursive representation holds:

$$u_n^m(t) = E\left[X^n_{T_1^{n,m}(t)} + u_n^{m-1}\left(\frac{T_1^{n,m}(t)}{n} \right) \right], \quad t \in [0,1), \tag{4.1}$$

and u_n^m is decreasing in m.

We need the following integrability conditions:

(U) *Uniform integrability condition*:
The (M_n^+) with $M_n := \max_{i\le n} X_i^n$ are uniformly integrable and $E \limsup_n M_n^+ < \infty$.

(L_+) *Uniform integrability from below*:
For any $m \in \mathbb{N}$ there exists a sequence $v_n^m : [0,1] \to \overline{\mathbb{R}}$ with $v_n^m \to u^m - u^{m-1}$ pointwise, $u^0 := 0$, such that the threshold stopping times

$$\begin{aligned} \hat{T}_1^{n,m}(t) &:= \min\{tn < i \le n-m+1 \mid X_i^n > v_n^m(\tfrac{i}{n})\}, \\ \hat{T}_l^{n,m}(t) &:= \min\{\hat{T}_{l-1}^{n,m}(t) < i \le n-m+l \mid X_i^n > v_n^{m-l+1}(\tfrac{i}{n})\}, \quad 2 \le l \le m, \end{aligned}$$

satisfy that

$$\lim_{s\uparrow 1} \limsup_{n\to\infty} E[(X^n_{\hat{T}_1^{n,m}(t)} + \dots + X^n_{\hat{T}_m^{n,m}(t)})\chi_{\{\hat{T}_m^{n,m}(t) > sn\}}] = 0.$$

For the m-stopping problem condition (L_+) is not needed for all $m \in \mathbb{N}$ but we need this uniform integrability condition for time points $k \leq m-1$. In fact condition (L_+) implies uniform integrability of $\left(X^n_{\hat{T}_1^{n,m}(t)} + \cdots + X^n_{\hat{T}_m^{n,m}(t)}\right)_{n\in\mathbb{N}}$ (see Faller (2009)). We denote by $T_\ell^{n,m} := T_\ell^{n,m}(0)$ and $T_\ell^m := T_\ell^m(0)$ the optimal m-stopping times of (X_i^n) resp.of N.

Theorem 4.1 (Convergence of m-stopping problems) *Assume point process convergence $N_n \xrightarrow{d} N$ on M_c and the uniform integrability condition* (U). *If $c \in \mathbb{R}$ then assume that $X^n_{n-i} \xrightarrow{L^1} c$, as $n \to \infty$ for $i = 0, \ldots, m-1$. If $c = \infty$ then condition* (L_+) *is assumed to hold. Then it holds:*

(a)
$$u_n^m(t) \to u^m(t), \quad t \in [0,1].$$

(b) Convergence of optimal stopping times and values holds
$$\left(\frac{T_l^{n,m}}{n}, X^n_{T_l^{n,m}}\right)_{1\leq l\leq m} \xrightarrow{d} (T_l^m, \overline{Y}_{T_l^m} \vee c)_{1\leq l\leq m}.$$

(c) If $c \in \mathbb{R}$, then
$$\hat{T}_1^{n,m} := \min\{1 \leq i \leq n-m+1 \mid X_i^n + u^{m-1}(\tfrac{i}{n}) > u^m(\tfrac{i}{n})\},$$
$$\hat{T}_l^{n,m} := \min\{\hat{T}_{l-1}^{n,m} < i \leq n-m+l \mid X_i^n + u^{m-l}(\tfrac{i}{n}) > u^{m-l+1}(\tfrac{i}{n})\}, \ \ 2 \leq l \leq m,$$

define an asymptotically optimal sequence of m-stopping times, i.e. $E[X^n_{\hat{T}_1^{n,m}} + \cdots + X^n_{\hat{T}_m^{n,m}}] \to u^m(0)$ for $n \to \infty$.

If $c = -\infty$ the same holds true for
$$\hat{T}_1^{n,m} := \min\{1 \leq i \leq n-m+1 \mid X_i^n > v_n^m(\tfrac{i}{n})\},$$
$$\hat{T}_l^{n,m} := \min\{\hat{T}_{l-1}^{n,m} < i \leq n-m+l \mid X_i^n > v_n^{m-l+1}(\tfrac{i}{n})\}, \quad 2 \leq l \leq m,$$

with v_n^m the functions appearing in condition (L_+).

Proof: We consider first the case $c \in \mathbb{R}$. The proof is by induction in m. The case $m = 1$ follows from [FR] (2011a). For the induction step $m-1 \to m$ note that by (4.1)

$$u_n^m(t) = E\left[X^n_{T_1^{n,m}(t)} + u^{m-1}\left(\frac{T_1^{n,m}(t)}{n}\right)\right] + E\left[u_n^{m-1}\left(\frac{T_1^{n,m}(t)}{n}\right) - u^{m-1}\left(\frac{T_1^{n,m}(t)}{n}\right)\right].$$

Since by induction hypothesis $u_n^{m-1} \to u^{m-1}$ uniformly, the second term converges to zero. The first term is smaller than or equal to the optimal stopping curve of $\sum_{i=1}^n \delta_{(\frac{i}{n}, X_i^n + u^{m-1}(\frac{i}{n}))}$ in t. This however converges by the case $m = 1$ to the optimal stopping curve u^m of $N^m = \sum_k \delta_{(\tau_k, Y_k + u^{m-1}(\tau_k))}$ with guarantee value mc which is given in Theorem 3.1. In consequence we obtain

$$\limsup u_n^m(t) \le u(t).$$

For the converse direction note that

$$u_n^m(t) \ge E[X^n_{\hat{T}_1^{n,m}(t)} + \cdots + X^n_{\hat{T}_m^{n,m}(t)}]$$

with the stopping times

$$\hat{T}_1^{n,m}(t) := \min\{tn < i \le n-m \mid X_i^n + u^{m-1}(\tfrac{i}{n}) > u^m(\tfrac{i}{n})\},$$
$$\hat{T}_l^{n,m}(t) := \min\{T_{l-1}^{n,m}(t) < i \le n-m+l \mid X_i^n + u^{m-l}(\tfrac{i}{n}) > u^{m-l+1}(\tfrac{i}{n})\},\ \ 2 \le l \le m.$$

As in the convergence theorem for multiple threshold stopping times in the max case (see Proposition 5.1. of [FR] (2011b)) we obtain convergence of the threshold stopping times in the sum case

$$\begin{aligned} &E[X^n_{\hat{T}_1^{n,m}(t)} \vee c + \cdots + X^n_{\hat{T}_m^{n,m}(t)} \vee c] \\ &\to E[\overline{Y}_{T_1^m(t)} \vee c + \cdots + \overline{Y}_{T_m^m(t)} \vee c] = u^m(t). \end{aligned}$$

The assumption $X_i^n \stackrel{L^1}{\to} c$ for $i = n-m+1, \dots, n$ then implies that

$$E[X^n_{\hat{T}_1^{n,m}(t)} + \cdots + X^n_{\hat{T}_m^{n,m}(t)}] \ \to\ u^m(t).$$

This implies convergence

$$u_n^m(t) \to u^m(t).$$

In the case $c = -\infty$, we denote for $x > -\infty$ by

$$u_n^m(t,x) := \sup\{E[X^n_{T_1} \vee x + \cdots + X^n_{T_m} \vee x] \mid t < T_1 < \cdots < T_m \le 1\}$$

the optimal stopping curve with guarantee value x (instead of c). Analogously we define $u^m(t,x)$. Then from the first part of the proof

$$u_n^m(t) \le u_n^m(t,x) \stackrel{n\to\infty}{\to} u^m(t,x).$$

By the separation condition (S) holding true for N^m we obtain convergence of $u^m(t,x)$ to $u^m(t)$ for $x \to -\infty$ (compare the corresponding result in [FR] (2011a) in the case $m = 1$). For the converse direction note that

$$u_n^m(t) \ge E[X^n_{\hat{T}_1^{n,m}(t)} + \cdots + X^n_{\hat{T}_m^{n,m}(t)}]$$

with the stopping times based on the approximative threshold sequences v_n^m in condition (L_+). By uniform integrability the right hand side converges to $u^m(t)$. Thus we obtain $u_n^m(t) \to u^m(t)$.

b) and c) follow from an extension of the proof of Proposition 2.4 in [KR] (2000a). □

5 Optimal m-stopping of i.i.d. sequences with discount and observation costs

As application we study in this section the optimal m-stopping of i.i.d. sequences with discount and observation costs. In the case $m = 1$ this problem has been considered in various degree of generality in Kennedy and Kertz (1990, 1991), [KR] (2000b), and [FR] (2011a). m-stopping in the max case has been considered in [FR] (2011b).

Let $(Z_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence with d.f. F in the domain of attraction of an extreme value distribution G, thus for some constants $a_n > 0$, $b_n \in \mathbb{R}$

$$n(1 - F(a_n x + b_n)) \to -\log G(x), \quad x \in \mathbb{R}. \tag{5.1}$$

Consider $X_i = c_i Z_i + d_i$ the sequence with discount and observation factors, $c_i > 0$, $d_i \in \mathbb{R}$ and both sequences monotonically nondecreasing or nonincreasing. For convergence of the corresponding imbedded point processes

$$\hat{N}_n = \sum_{i=1}^{n} \delta_{\left(\frac{i}{n}, \frac{X_i - \hat{b}_n}{\hat{a}_n}\right)} \tag{5.2}$$

the following choices of $\hat{a}_n$, $\hat{b}_n$ turn out to be appropriate:

$$\begin{aligned} &\hat{a}_n := c_n a_n,\ \hat{b}_n := 0 && \text{for } F \in D(\Phi_\alpha) \text{ or } F \in D(\Psi_\alpha), \\ &\hat{a}_n := c_n a_n,\ \hat{b}_n := c_n b_n + d_n && \text{for } F \in D(\Lambda), \end{aligned} \tag{5.3}$$

where Φ_α, Ψ_α, Λ are the Fréchet, Weibull, and Gumbel distributions and a_n, b_n are the corresponding normalizations in (5.1). We give further conditions on c_i, d_i to establish point process convergence in (5.2). Related conditions are given in Haan and Verkaade (1987) in the treatment of i.i.d. sequences with trends resp. in [KR] (2000b).

In the following c denotes some general constant and not as before the guarantee value. The guarantee value of N is in case Φ_α given by 0 and in cases Ψ_α, Λ given generally by $-\infty$. We state the optimality results for all three cases. Based on the characterization of optimal solutions for the limiting Poisson case and on the approximation result we obtain *explicit* results for the optimal stopping curves (using Remark 3.3) and also *explicit* approximative stopping sequences. We first consider the case of Fréchet limits.

Theorem 5.1 *Let $F \in D(\Phi_\alpha)$ with $\alpha > 1$ and $F(0) = 0$ (i.e. $Z_i > 0$ P-a.s.). We assume that $b_n = 0$ and also assume*

$$\frac{d_n}{c_n a_n} \to d, \quad \frac{c_{\lfloor tn \rfloor}}{c_n} \to t^c \quad \forall t \in [0, 1]$$

with constants $c,\, d \in \mathbb{R}$*, and further that* c_n *does not converge to* 0*. Also let* $c > -\frac{1}{\alpha}$ *and assume that the function* $R : (d, \infty) \to \mathbb{R}$*,*

$$R(x) := x + \frac{\alpha}{\alpha - 1}\,\frac{1}{1 + c\alpha}\,(x - d)^{-\alpha+1}, \qquad x \in (d, \infty), \tag{5.4}$$

has no zero point.

a) Then

$$\frac{E[X_{T_1^{n,m}} + \cdots + X_{T_m^{n,m}}]}{\hat{a}_n} \to u^m(0) > 0, \tag{5.5}$$

where $u^m(t)$ *is the optimal* m*-stopping curve of the Poisson process* $\hat{N}$ *with intensity function*

$$\hat{G}(t, y) = t^{c\alpha}(y - dt^{c+\frac{1}{\alpha}})^{-\alpha} = H\left(\frac{y}{v(t)}\right)\frac{v'(t)}{v(t)} \qquad \text{on } M_{\hat{f}},$$

i.e. u^m *are solutions of the differential equations in* (3.6) *(c.f. Remark 3.3). Here* $v(t) := t^{c+\frac{1}{\alpha}}$*,* $H(x) := \frac{\alpha}{\alpha c+1}(x - d)^{-\alpha}$ *and* $\hat{f}(t) := dt^{c+\frac{1}{\alpha}}$*.*

b)
$$\begin{aligned}\hat{T}_1^{n,m} &:= \min\{\, 1 \le i \le n - m + 1 : X_i > \hat{a}_n(u^m(\tfrac{i}{n}) - u^{m-1}(\tfrac{i}{n}))\},\\ \hat{T}_\ell^{n,m} &:= \min\{\, \hat{T}_{\ell-1}^{n,m} < i \le n - m + \ell : X_i > \hat{a}_n(u^{m-l+1}(\tfrac{i}{n}) - u^{m-l}(\tfrac{i}{n}))\},\end{aligned}$$

$2 \le \ell \le m$*, are asmptotically optimal* m*-stopping times, i.e. the limit in* (5.5) *is attained also for these sequences.*

The next result concerns the Weibull limit case.

Theorem 5.2 *Let* $F \in D(\Psi_\alpha)$ *with* $\alpha > 0$ *and* $F(0) = 1$ *(i.e.* $Z_i \le 0$ *P-a.s.). Further let* $a_n \downarrow 0$ *and* $b_n = 0$*, and*

$$\frac{d_n}{c_n a_n} \to d, \quad \frac{c_{\lfloor tn \rfloor}}{c_n} \to t^c, \quad \forall t \in [0, 1]$$

for constants $c,\, d \in \mathbb{R}$*. If* $d_n > 0$*, then assume that either* $(d_n)_{n\in\mathbb{N}}$ *is monotonically nondecreasing or* $c_n a_n$ *does not converge to 0.*

a) If $c < \frac{1}{\alpha}$ *and* $d \le 0$*, then it holds*

$$\frac{E[X_{T_1^{n,m}} + \cdots + X_{T_m^{n,m}}]}{\hat{a}_n} \to u^m_{c,d}(0) < 0. \tag{5.6}$$

b) If $c > \frac{1}{\alpha}$ *and the function* $R : \mathbb{R} \to \mathbb{R}$

$$R(x) := \begin{cases} x, & \text{if } x \geq d, \\ x - \frac{\alpha}{\alpha+1}\frac{1}{1-c\alpha}(-x+d)^{\alpha+1}, & \text{if } x < d, \end{cases} \tag{5.7}$$

has no zero point then (5.6) *holds with* $u^m_{c,d}(0) > 0$. *Here* $u^m_{c,d}(t)$ *is the optimal* m-*stopping curve of the Poisson process* $\hat{N} = \hat{N}_{c,d}$ *with intensity function* $\hat{G} = G_{c,d}$ *given in* (3.16).

c) Let (w_n) *be an increasing sequence* $w_n < 0$ *such that* $n(1 - F(w_n)) \to \frac{\alpha+1}{\alpha}$ *(e.g.* $w_n = -\left(\frac{\alpha+1}{\alpha}\right)^{\frac{1}{\alpha}} a_n$*). Define functions* v^m_n *by*

$$v^m_n(t) := \frac{\gamma^m_{c,0}(t)}{u_{0,0}(t)}\frac{w_{\lfloor (1-t)n \rfloor}}{a_n} + \gamma^m_{c,d}(t) - \gamma^m_{c,0}(t),$$

with $\gamma^m_{c,d}(t) := u^m_{c,d}(t) - u^{m-1}_{c,d}(t)$. *Then the* m-*stopping times defined by*

$$\hat{T}^{n,m}_1 := \min\{\, 1 \leq i \leq n-m+1 : X_i > \hat{a}_n v^m_n(\tfrac{i}{n})\},$$
$$\hat{T}^{n,m}_\ell := \min\{\, \hat{T}^{n,m}_{\ell-1} < i \leq n-m+\ell : X_i > \hat{a}_n \gamma^{m-\ell+1}_{c,d}(\tfrac{i}{n}, \tfrac{1}{\hat{a}_n} X_{\hat{T}^{n,m}_{\ell-1}})\}$$

for $2 \leq \ell \leq m$, *are asymptotically optimal, i.e. convergence as in* (5.6) *does also hold for them.*

The final result concerns the Gumbel case.

Theorem 5.3 *Let* $F \in D(\Lambda)$ *and assume*

$$\frac{b_n}{a_n}\left(1 - \frac{c_{\lfloor tn \rfloor}}{c_n}\right) \to c\log(t), \qquad \frac{d_n - d_{\lfloor tn \rfloor}}{c_n a_n} \to d\log(t) \quad \forall t \in [0,1]$$

for some constants c, $d \in \mathbb{R}$. *Assume also that* $(c_n)_{n\in\mathbb{N}}$ *and* $(d_n)_{n\in\mathbb{N}}$ *are monotonically nondecreasing and that* $c + d < 1$. *Then it holds:*

a)
$$\frac{E[X_{T^{n,m}_1} + \cdots + X_{T^{n,m}_m}] - \hat{b}_n}{\hat{a}_n} \to u^m(0), \tag{5.8}$$

where $u^m(t)$ *is the optimal* m-*stopping curve of the Poisson process* $\hat{N}$ *with intensity function*

$$\hat{G}(t,y) = e^{-y}t^{-(c+d)} \qquad \text{on } [0,1] \times \mathbb{R}.$$

(u^m) *are solutions of the differential equations in* (3.6) *(c.f. Remark 3.3).*

b) Let $(w_n)_{n\in\mathbb{N}}$ be an increasing sequence with $\lim_{n\to\infty} n(1-F(w_n)) = 1$ (e.g. $w_n := b_n$). Let (v_n^m) be defined as

$$v_n^m(t) := \frac{w_{\lfloor(1-t)n\rfloor} - b_n}{a_n} + (u^m(t) - u^{m-1}(t)) - \log(1-t).$$

Then

$$\hat{T}_1^{n,m} := \min\{\, 1 \le i \le n-m+1 : X_i > \hat{a}_n v_n^m(\tfrac{i}{n}) + \hat{b}_n\},$$
$$\hat{T}_\ell^{n,m} := \min\{\, \hat{T}_{\ell-1}^{n,m} < i \le n-m+\ell : X_i > \hat{a}_n v_n^{m-\ell+1}(\tfrac{i}{n}) + \hat{b}_n\}$$

define an asymptotically optimal sequence of m-stopping times, i.e. convergence as in (5.8) *holds for them.*

Proof: The proof can be given similarly to the proof of Theorems 3.1–3.3 in [FR] (2011a) in the case $m = 1$ using the approximation Theorem 4.1. For details of the proof of the uniform integrability condition we refer to Faller (2009, p. 101–107). □

References

C. Bender. Dual pricing of multi-exercise options under volume constraints. *Finance and Stochastics,* to appear, 2010.

F. T. Bruss. On a class of optimal stopping problems with mixed constraints. *Discrete Mathematics and Theoretical Computer Science*, 12:363–380, 2010.

F. T. Bruss and T. S. Ferguson. Multiple buying and selling with vector offers. *Journal of Applied Probability*, 34:959–973, 1997.

A. Faller. *Approximative Lösungen von Mehrfachstoppproblemen.* Dissertation, University of Freiburg, 2009.

A. Faller and L. Rüschendorf. On approximative solutions of optimal stopping problems. *Advances Applied Probability*, 43:1086–1108, 2011a.

A. Faller and L. Rüschendorf. On approximative solutions of multistopping problems. *Annals of Applied Probability*, 21:1965–1993, 2011b.

L. de Haan and E. Verkaade. On extreme-value theory in the presence of a trend. *Journal of Applied Probability*, 24:62–76, 1987.

D. P. Kennedy and R. P. Kertz. Limit theorems for threshold-stopped random variables with applications to optimal stopping. *Advances in Applied Probability*, 22:96–411, 1990.

D. P. Kennedy and R. P. Kertz. The asymptotic behavior of the reward sequence in the optimal stopping of i.i.d. random variables. *The Annals of Probability*, 19: 329–341, 1991.

R. Kühne and L. Rüschendorf. Approximation of optimal stopping problems. *Stochastic Processes and Their Applications*, 90:301–325, 2000a.

R. Kühne and L. Rüschendorf. Optimal stopping with discount and observation costs. *Journal of Applied Probability*, 37:64–72, 2000b.

V. Saario. Comparison of the discrete and continuous-time stochastic selling models. *Engineering Costs and Production Economics*, 12:15–20, 1992.

V. Saario and M. Sakaguchi. Some generalized house-selling problems. *Mathematica Japonica*, 35:861–873, 1992.

M. Sakaguchi. Optimal stopping in sampling from a bivariate distribution. *J. Oper. Res. Soc. Japan*, 16:186–201, 1973.

M. Sakaguchi. When to stop: randomly appearing bivariate target values. *J. Oper. Res. Soc. Japan*, 21:45–58, 1978.

W. Stadje. On multiple stopping rules. *Optimization*, 16:401–418, 1985.

W. Stadje. A full information pricing problem for the sale of several identical commodities. *Zeit. Oper. Res.*, 34:161–181, 1990.

Andreas Faller
Mathematical Stochastics
Eckerstr. 1
79104 Freiburg
Germany

Ludger Rüschendorf
Mathematical Stochastics
Eckerstr. 1
79104 Freiburg
Germany
ruschen@stochastik.uni-freiburg.de