Approximative solutions of best choice problems

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Abstract

We consider the full information best choice problem from a sequence $X_1, \dots, X_n$ of independent random variables. Under the basic assumption of convergence of the corresponding imbedded point processes in the plane to a Poisson process we establish that the optimal choice problem can be approximated by the optimal choice problem in the limiting Poisson process. This allows to derive approximations to the optimal choice probability and also to determine approximatively optimal stopping times. An extension of this result to the best m-choice problem is also given.

Keywords: best choice problem, optimal stopping, Poisson process

Mathematics Subject Classification 2000: 60 G40, 62 L15

1 Introduction

The best choice problem of a random sequence $X_1, \dots, X_n$ is to find a stopping time $\tau \leqslant n$ to maximize the best choice probability $P(X_\tau = M_n)$, where $M_n = \max_{1 \leqslant i \leqslant n} X_i$, under all stopping times $\tau \leqslant n$. Thus we aim to find an optimal stopping time $T_n \leqslant n$ such that

$$P(X_{T_n} = M_n) = \max P(X_\tau = M_n) \tag{1.1}$$

over all stopping times $\tau \leqslant n$.

Gilbert and Mosteller (1966) found the solution of the full information best choice problem, where $X_1, \dots, X_n$ are iid with known continuous distribution function F. In this case the optimal stopping time is given by

$$T_n = \min\{k \leqslant n \mid X_k = M_k, \quad F(X_k) \geqslant b_{n-k}\} \tag{1.2}$$

*On June 30, 2011 Andreas Faller died completely unexpected. He was an extraordinary talented young researcher.

where $b_0 = 0$, $\sum_{j=1}^{i} \binom{i}{j}(b_i^{-1} - 1)^j j^{-1} = 1, \quad i = 1, 2, \ldots$

The asymptotic behaviour of b_i is described by $b_i \uparrow 1$, $i(1-b_i) \to c = 0.8043\ldots$ The optimal probability $v_n = P(X_{T_n} = M_n)$ does not depend on F, is strictly decreasing in n and has limiting value v_∞

$$v_n \to v_\infty = 0.580164\ldots \tag{1.3}$$

A continuous time version of the problem with random number of points given by a homogeneous Poisson process with intensity λ was studied in Sakaguchi (1976). As $\lambda \to \infty$ the same limit v_∞ appears as limiting optimal choice probability as observed in Berezovskiĭ and Gnedin (1984) and Gnedin and Sakaguchi (1992). In Gnedin (1996) this limiting optimal choice probability v_∞ was identified as optimal choice probability in an associated plane Poisson process on $[0,1] \times (-\infty, 0]$ with intensity measure $\lambda_{[0,1]} \otimes \lambda_{(-\infty,0]}$. The link with the original problem was established by an explicit imbedding of finite iid sequences in the Poisson process. Multistop best choice problems for Poisson processes were considered in Sakaguchi (1991) and Saario and Sakaguchi (1992).

The approach in Gnedin (1996) was extended in Kühne and Rüschendorf (2000c) to the best choice problem for (inhomogeneous) discounted sequences $X_i = c_i Y_i$, where (Y_i) are iid and c_i are constants which imply convergence of imbedded normalized point processes $N_n = \sum_{i=1}^{n} \mathcal{E}_{(\frac{i}{n}, \frac{X_i - b_n}{a_n})}$ to some Poisson process N in the plane. The proofs in that paper make use of Gnedin's (1996) result as well as of some general approximation results in [KR][1] (2000a). The aim of this paper is to extend this approach to general inhomogeneous best choice problems for independent sequences under the basic assumption of convergence of the imbedded point processes N_n to some Poisson process N in the plane. Subsequently we also consider an extension to the m-choice problem, where m choices described by stopping times $1 \leqslant T_1 < \cdots < T_m \leqslant n$ are allowed and the aim is to find m stopping times $T_1^m < \cdots < T_m^m \leqslant n$ with

$$P(X_{T_1^m} \vee \cdots \vee X_{T_m^m} = M_n) = \sup P(\bigvee_{i=1}^{m} X_{T_i} = M_n) \tag{1.4}$$

the sup being over all stopping times $T_1 < \cdots < T_m \leqslant n$.

For the corresponding generalized Moser problem of maximizing EX_τ resp. $E \bigvee_{i=1}^{m} X_{\tau_i}$ a general approximation approach has been developed in [KR] (2000a), [FR] (2009) for $m = 1$, resp. in [KR] (2002) and [FR] (2010) for $m \geqslant 1$; see also Goldstein and Samuel-Cahn (2006). For a detailed history of this problem we refer to Ferguson (2007) for $m = 1$ resp. to [FR] (2010) in case $m \geqslant 1$. Our results for (1.3) are in particular applicable to sequences $X_i = c_i Z_i + d_i$ with iid random sequences (Z_i) and with discount and observation factors c_i, d_i. The corresponding

[1]Kühne and Rüschendorf is abbreviated within this paper with [KR], Faller and Rüschendorf with [FR].

results for the Moser type problems for these sequences can be found in [FR] (2009; 2010).

There are further interesting types of choice problems as e.g. the expected rank choicee problem (see Bruss and Ferguson, 1993; Saario and Sakaguchi, 1995), multiple buying selling problems (Gnedin, 1981; Bruss and Ferguson, 1997; Bruss, 2010) or multicriteria extensions (Gnedin, 1992) which are not dealt with in this paper.

2 Approximative optimal best choice solution

We consider the optimal best choice problem (1.1) for a sequence (X_i) of independent random variables, i.e. to find optimal stopping times $T_n \leqslant n$ such that

$$P(X_{T_n} = M_n) = \sup_{\tau \leqslant n} P(X_\tau = M_n), \tag{2.1}$$

over all stopping times $\tau \leqslant n$. The basic assumption in our approach is convergence of the imbedded planar point process to a Poisson point process N,

$$N_n = \sum_{i=1}^{n} \delta_{(\frac{i}{n}, X_i^n)} \xrightarrow{d} N \tag{2.2}$$

on $M_c = [0,1] \times (c, \infty)$. Here $X_i^n = \frac{X_i - b_n}{a_n}$ is a normalization of X_i typically induced from a form of the central limit theorem for maxima $a_n > 0$, $b_n \in \mathbb{R}$ and $c \in \mathbb{R} \cup \{-\infty\}$. For some general conditions to imply this convergence and examples see [KR] (2000a,b) or [FR] (2009). We consider Poisson processes N restricted on M_c which may have infinite intensity along the lower boundary $[0,1] \times \{c\}$. We assume that the intensity measure μ of N is a Radon measure on M_c with the topology induced by the usual topology on $[0,1] \times \overline{\mathbb{R}}$. Thus any compact set $A \subset M_c$ has only finitely many points of N. Convergence in distribution '$N_n \xrightarrow{d} N$ on M_c' means convergence in distribution of the restricted point processes. We generally assume that the intensity measure μ is Lebesgue-continuous with density denoted as $g(t,x)$. Thus the Poisson process

$$N = \sum_{k \geqslant 1} \delta_{(\tau_k, y_k)} \text{ on } M_c \tag{2.3}$$

does not have multiple points.

We consider also the best choice problem for the continuous time Poisson process N. An N-stopping time $T : \Omega \to [0,1]$ is a stopping time w.r.t. the filtration $\mathcal{A}_t := \sigma(N(\cdot \cap [0,t] \times (c,\infty]))$, $t \in [0,1$. An N-stopping time T is called 'optimal best choice stopping time' for N if

$$P(Y_T = \sup_k Y_k) = \sup P(Y_S = \sup_k Y_k) \tag{2.4}$$

the supremum being taken over all N-stopping times S.

In the following theorem we derive the optimal stopping time T for the continuous time best choice problem for the Poisson process N. Further we show that the best choice problem for $X_1, \dots, X_n$ is approximated by the best choice problem for N. This allows us to get approximations of the best choice probabilities $v_n = P(X_{T_n} = M_n)$ and to construct asymptotically optimal best choice stopping sequences $\hat{T}_n$. Our approximation result needs the following intensity condition, which is not necessary, when dealing with the limiting best choice problem (see Section 3).

(I) Intensity Condition: For all $t \in [0,1)$ let $\mu((t,1] \times (c,\infty]) = \infty$.

Theorem 2.1 *Let the imbedded point process N_n converge in distribution to the Poisson process N on M_c and let the intensity condition (I) hold true. Then we get:*

(a) The optimal best choice stopping time T for N is given by

$$T = \inf\{\tau_k \mid Y_k = \sup_{\tau_j \in [0,\tau_k]} Y_j, Y_k > v(\tau_k)\}, \tag{2.5}$$

where the threshold $v : [0,1] \to [c,\infty)$ is a solution of the integral equation

$$\begin{aligned} &\int_t^1 \int_{v(t)}^{\infty} e^{\mu((r,1]\times(v(t),y])} \mu(dr,dy) = 1 \quad \forall t \in [0,1), \\ &v(1) = c \end{aligned} \tag{2.6}$$

v is monotonically nonincreasing and can be chosen right continuous. The optimal probability for the best choice problem for N is given by

$$\begin{aligned} s &:= P(Y_T = \sup_k Y_k) \\ &= \int_0^1 e^{-\mu([0,r]\times(c,\infty])} \int_{v(r)}^{\infty} e^{-\mu((r,1]\times(y,\infty])} \mu(dr,dy) \\ &\quad + \int_0^1 \int_c^{v(r')} \int_{r'}^1 e^{-\mu([0,r]\times(y',\infty])} \int_{y'\vee v(r)}^{\infty} e^{-\mu((r,1]\times(y,\infty])} \mu(dr,dy)\mu(dr',dy'). \end{aligned} \tag{2.7}$$

(b) Approximation of the optimal best choice probabilities holds true:

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} P(X_{T_n} = M_n) = s.$$

(c) $\hat{T}_n := \min\{1 \leqslant i \leqslant n \mid X_i = M_i, X_i > a_n v(\frac{i}{n}) + b_n\}$ defines an asymptotically optimal sequence of stopping times, i.e.

$$\lim_{n\to\infty} P(X_{\hat{T}_n} = M_n) = s.$$

Proof: For $(t,x) \in [0,1) \times [c,\infty)$ we want to determine optimal stopping times $T_n(t,x) > tn$ for X_j^n, $1 \leqslant j \leqslant n$, which maximize

$$P(X_T^n = x \vee \max_{tn<j\leqslant n} X_j^n)$$

under all stopping times $\tau > tn$. Define for $i > tn$

$$Z_i^n(t,x) = P\Big(X_i^n = x \vee \max_{tn<j\leqslant n} X_j^n \mid X_1^n, \dots, X_i^n\Big).$$

Then we have to maximize $EZ_\tau^n(t,x) = P\big(X_\tau^n = x \vee \max_{tn<j\leqslant n} X_j^n\big)$. By the classical recursive equation for optimal stopping of finite sequences the optimal stopping times $T_n(t,x)$ are given by (see e.g. Proposition 2.1 in [FR] (2010)) $T_n(t,x) := T_n^{>tn}(t,x)$ where $T_n^{>n}(t,x) := n$ and

$$\begin{aligned}
T_n^{>k}(t,x) = \min\big\{k < i \leqslant n \mid\ & P(X_i^n = x \vee \max_{tn<j\leqslant n} X_j^n \mid X_1^n, \dots, X_i^n) \\
& > P(X^n_{T_n^{>i}(t,x)} = x \vee \max_{tn<j\leqslant n} X_j^n \mid X_1^n, \dots, X_i^n), X_i^n = x \vee \max_{tn<j\leqslant i} X_j^n\big\} \\
= \min\big\{k < i \leqslant n \mid\ & P(X_i^n \geqslant \max_{i<j\leqslant n} X_j^n \mid X_i^n) \\
& > P(X^n_{T_n^{>i}(t,x)} = X_i^n \vee \max_{i<j\leqslant n} X_j^n \mid X_1^n, \dots, X_i^n), X_i^n = x \vee \max_{tn<j\leqslant i} X_j^n\big\}.
\end{aligned}$$

By backward induction in $l = n-1, \dots, \lfloor tn \rfloor$ we obtain for $tn < i \leqslant l$ that $T_n^{>l}(t,x) = T_n^{>l}(\frac{i}{n}, X_i^n)$ on $\{X_i^n = x \vee \max_{tn<j\leqslant i} X_j^n\}$ and further that $T_n^{>l}(t,x)$ is independent of $\sigma(X_1^n, \dots, X_{\lfloor tn \rfloor}^n$. Thus

$$\begin{aligned}
T_n(t,x) = \min\big\{tn < i \leqslant n \mid\ & P(X_i^n \geqslant \max_{i<j\leqslant n} X_j^n \mid X_i^n) \\
& > P(X^n_{T_n(\frac{i}{n}, X_i^n)} = X_i^n \vee \max_{i<j\leqslant n} X_j^n \mid X_i^n), X_i^n = x \vee \max_{tn<j\leqslant i} X_j^n\big\} \\
= \min\big\{tn < i \leqslant n \mid\ & h_n(\tfrac{i}{n}, X_i^n) > g_n(\tfrac{i}{n}, X_i^n), X_i^n = x \vee \max_{tn<j\leqslant i} X_j^n\big\}
\end{aligned}$$

where

$$\begin{aligned}
g_n(t,x) &:= P(X^n_{T_n(t,x)} = x \vee \max_{tn<j\leqslant n} X_j^n) \\
h_n(t,x) &:= P(\max_{tn<j\leqslant n} X_j^n \leqslant x).
\end{aligned} \tag{2.8}$$

h_n is monotonically nondecreasing in (s,t) and converges uniformly in compact sets in $[0,1] \times (c,\infty]$ to

$$h_\infty(t,x) := P(N((t,1] \times (x,\infty]) = 0) = e^{-\mu((t,1]\times(x,\infty])}.$$

To prove that $g_n(t,x)$ also converges uniformly on compact sets in $[0,1] \times (c,\infty]$ we decompose g_n into two monotone components.

$$\begin{aligned}
g_n(t,x) &= \sup_{\substack{\tau>tn \\ \text{stopping times}}} P(X_\tau^n = x \vee \max_{tn<j\leqslant n} X_j^n) \\
&= \sup_{\substack{\tau>tn \\ \text{stopping times}}} P(X_\tau^n = x \vee \max_{tn<j\leqslant n} X_j^n, \max_{tn<j\leqslant n} X_j^n > x)
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{\substack{\tau > tn \\ \text{stopping times}}} P(X^n_\tau \vee x = \max_{tn<j\leqslant n}(X^n_j \vee x), \max_{tn<j\leqslant n} X^n_j > x) \\
&= \sup_{\substack{\tau > tn \\ \text{stopping times}}} P(X^n_\tau \vee x = \max_{tn<j\leqslant n}(X^n_j \vee x)) - P(\max_{tn<j\leqslant n} X^n_j \leqslant x) \\
&=: \tilde{g}_n(t,x) - h_n(t,x). \qquad (2.9)
\end{aligned}
$$

For the second equality we use that by assumption $P(X^n_i = x) = 0$. $\tilde{g}_n(t,x)$ is monotonically nondecreasing in (t,x) and thus converges pointwise to some function $\tilde{g}_\infty(t,x)$. We next prove that $\tilde{g}_\infty$ is continuous which then implies uniform convergence. On one hand side we have for $s < t$ and $x > c$

$$
\begin{aligned}
\tilde{g}_n(s,x) &\leqslant \sup_{\substack{\tau > sn \\ \text{stopping times}}} P(X^n_\tau \vee x \geqslant \max_{tn<j\leqslant n}(X^n_j \vee x)) \\
&\leqslant \sup_{\substack{\tau > sn \\ \text{stopping times}}} P(X^n_\tau \vee x \geqslant \max_{tn<j\leqslant n}(X^n_j \vee x), \max_{sn<j\leqslant tn}(X^n_j \vee x) \leqslant \max_{tn<j\leqslant n}(X^n_j \vee x)) \\
&\qquad + P(\max_{sn<j\leqslant tn}(X^n_j \vee x) > \max_{tn<j\leqslant n}(X^n_j \vee x)) \\
&\leqslant \tilde{g}_n(t,x) + P(\max_{sn<j\leqslant tn} X^n_j > x).
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\tilde{g}_n(s,x) &\geqslant \sup_{\substack{\tau > sn \\ \text{stopping times}}} P(X^n_\tau \vee x \geqslant \max_{sn<j\leqslant n}(X^n_j \vee x)) \\
&\geqslant \sup_{\substack{\tau > sn \\ \text{stopping times}}} P(X^n_\tau \vee x \geqslant \max_{tn<j\leqslant n}(X^n_j \vee x)) \\
&\qquad - P(\max_{sn<j\leqslant tn}(X^n_j \vee x) > \max_{tn<j\leqslant n}(X^n_j \vee x)) \\
&\geqslant \tilde{g}_n(t,x) - P(\max_{sn<j\leqslant tn} X^n_j > x).
\end{aligned}
$$

This implies

$$
|\tilde{g}_n(s,x) - \tilde{g}_n(t,x)| \leqslant P(\max_{sn<j\leqslant tn} X^n_j > x).
$$

and thus continuity of $\tilde{g}_\infty(t,x)$ in t.

To prove continuity of $\tilde{g}_\infty$ in x let $x < y$. Then

$$
\begin{aligned}
0 &\leqslant \tilde{g}_n(t,y) - \tilde{g}_n(t,x) \\
&\leqslant \sup_{\substack{\tau > sn \\ \text{stopping times}}} P(X^n_T \vee y = \max_{tn<j\leqslant n}(X^n_j \vee y), X^n_T \vee x < \max_{tn<j\leqslant n}(X^n_j \vee x)) \\
&\leqslant P(x < \max_{tn<j\leqslant n} X^n_j \leqslant y).
\end{aligned}
$$

In consequence $\tilde{g}_\infty(t,x)$ is continuous and $g_n \to g_\infty$ uniformly on compact subsets of $[0,1] \times (c,\infty)$. Point process convergence and the representation in (2.8) imply that

$$
g_\infty(t,x) = P\big(Y_{T(t,x)} = x \vee \sup_{t<\tau_k\leqslant 1} Y_k\big)
$$

with the N-stopping time

$$T(t,x) = \inf\big\{\tau_k > t \mid Y_k = x \vee \sup_{t<\tau_j\leqslant\tau_k} Y_j, h_\infty(\tau_k, Y_k) > g_\infty(\tau_k, Y_k)\big\}.$$

The argument above also applies to the modified random variables

$$\tilde{X}_i^n := \sup_{\frac{i-1}{n}<\tau_k\leqslant\frac{i}{n}} Y_k.$$

Defining g_∞, h_∞ as above we obtain in this case $g_n(t,x) \downarrow g_\infty(t,x)$ since the discrete stopping problems majorize the continuous time stopping problem. In consequence $T(t,x)$ are the optimal stopping times for N, i.e.

$$g_\infty(t,x) = \sup_{\substack{\tau>t \\ N\text{-stopping time}}} P\big(Y_\tau = x \vee \sup_{t<\tau_k\leqslant 1} Y_k\big),$$

As a result we get the following estimate

$$g_n(t,x) \geqslant P(X^n_{\hat{T}_n(t,x)} = x \vee \max_{tn<j\leqslant n} X_j^n) \to g_\infty(t,x),$$

with the stopping times

$$\hat{T}_n(t,x) := \min\big\{tn < i \leqslant n \mid X_i^n = x \vee \max_{tn<j\leqslant i} X_j^n, h_\infty(\tfrac{i}{n}, X_i^n) > g_\infty(\tfrac{i}{n}, X_i^n)\big\}.$$

Thus the limit $g_\infty(t,x)$ is the same for (X_i^n) and for $(\tilde{X}_i^n)$.

The arguments above in the case $x > c$ can also be extended to the case $x = c$. Note that for $x > c$

$$\begin{aligned} g_\infty(t,x) &\leqslant P\big(Y_{T(t,x)} = \sup_{t<\tau_k\leqslant 1} Y_k\big) \\ &\leqslant \sup_{\substack{T>t \\ N\text{-stopping time}}} P\big(Y_T = \sup_{t<\tau_k\leqslant 1} Y_k\big) \leqslant g_\infty(t,x) + h_\infty(t,x). \end{aligned}$$

From the intensity assumption (I) and since $h_\infty(t,c) = 0$ we obtain

$$\begin{aligned} g_\infty(t,c) := \lim_{x\downarrow c} g_\infty(t,x) &= P\big(Y_{T(t,c)} = \sup_{t<\tau_k\leqslant 1} Y_k\big) \\ &= \sup_{\substack{\tau>t \\ N\text{-stopping time}}} P\big(Y_\tau = \sup_{t<\tau_k\leqslant 1} Y_k\big). \end{aligned}$$

Since for $x > 0$

$$g_n(t,x) \leqslant g_n(t,c) \leqslant g_n(t,x) + h_n(t,x)$$

it follows that

$$\lim_{n\to\infty} g_n(t,c) = g_\infty(t,c).$$

By assumption (I) $P(N((t,1] \times (c,\infty]) \geqslant 1) > 0$ for $t \in [0,1)$. Thus $g_\infty(t,x)$ is monotonically nonincreasing in x with $g_\infty(t,c) > 0$ and $g_\infty(t,\infty) = 0$. Also by (I)

$h_\infty(t,x)$ is monotonically nondecreasing in x with $h_\infty(t,c)=0$ and $h_\infty(t,\infty)=1$. In consequence there exists $v(t)\in(c,\infty)$ with

$$h_\infty(t,v(t)) = g_\infty(t,v(t)). \tag{2.10}$$

This implies the representation

$$T(t,x) = \inf\big\{\tau_k > t \mid Y_k = x \vee \sup_{t<\tau_j\leqslant\tau_k} Y_j, Y_k > v(\tau_k)\big\}.$$

We next prove that v is monotonically nonincreasing. Since $g_\infty(t,x)-h_\infty(t,x)$ is monotonically nonincreasing in x for any t it is sufficient to show that for $s<t$

$$g_\infty(s,v(t)) - h_\infty(s,v(t)) \geqslant 0. \tag{2.11}$$

To that aim we get for $s<t$

$$\begin{aligned} g_\infty(t,x)-g_\infty(s,x) &\leqslant P(Y_{T(t,x)} = x\vee \sup_{\tau_k>t} Y_k) - P(Y_{T(t,x)} = x \vee \sup_{\tau_k>s} Y_k)\\ &\leqslant P(x\vee \sup_{s<\tau_k\leqslant t} Y_k > Y_{T(t,x)} = x\vee\sup_{\tau_k>t} Y_k)\\ &\leqslant P(\sup_{s<\tau_k\leqslant t} Y_k > x) g_\infty(t,x). \end{aligned}$$

On the other hand

$$h_\infty(t,x) - h_\infty(s,x) = P(\sup_{s<\tau_k\leqslant t} Y_k > x) h_\infty(t,x),$$

which yields with $x := v(t)$ the claim in (2.11).

Monotonicity of v allows us to determine $g_\infty(t,x)$. Note that for $x\geqslant v(t)$

$$\begin{aligned} g_\infty(t,x) &:= P\big(\exists n_0 : \forall n\geqslant n_0 : \exists \tfrac{i}{2^n}\in(t,1] : \exists \tfrac{j}{2^n} > x: \;\; N((\tfrac{i-1}{2^n},\tfrac{i}{2^n}]\times(\tfrac{j-1}{2^n},\tfrac{j}{2^n}])=1,\\ &\qquad N((t,\tfrac{i-1}{2^n}]\times(x,\infty])=0, \text{and } N((\tfrac{i}{2^n},1]\times(\tfrac{j}{2^n},\infty])=0\big)\\ &= \int_t^1 e^{-\mu((t,r]\times(x,\infty])}\int_x^\infty e^{-\mu((r,1]\times(y,\infty])}\mu(dr,dy), \end{aligned}$$

and in case $x\leqslant v(t)$

$$\begin{aligned} g_\infty(t,x) = {} & P(\exists n_0 : \forall n\geqslant n_0 : \exists\tfrac{i}{2^n}\in(t,1] : \exists \tfrac{j}{2^n} > x:\\ & N((\tfrac{i-1}{2^n},\tfrac{i}{2^n}]\times(\tfrac{j-1}{2^n},\tfrac{j}{2^n}])=1 \text{ for } \tfrac{j}{2^n} > v(\tfrac{i}{2^n}), N((t,\tfrac{i-1}{2^n}]\times(x,\infty])=0,\\ & \text{and } N((\tfrac{i}{2^n},1]\times(\tfrac{j}{2^n},\infty])=0)\\ & + P(\exists n_0 : \forall n\geqslant n_0 : \exists t<\tfrac{i'}{2^n}<\tfrac{i}{2^n}\leqslant 1, \exists x < \tfrac{j'}{2^n} < \tfrac{j}{2^n}:\\ & N((\tfrac{i-1}{2^n},\tfrac{i}{2^n}]\times(\tfrac{j-1}{2^n},\tfrac{j}{2^n}])=1 \text{ for } \tfrac{j}{2^n} > v(\tfrac{i}{2^n}),\\ & N((\tfrac{i'-1}{2^n},\tfrac{i'}{2^n}]\times(\tfrac{j'-1}{2^n},\tfrac{j'}{2^n}])=1 \text{ for } \tfrac{j'}{2^n} < v(\tfrac{i'}{2^n}),\\ & N((t,\tfrac{i-1}{2^n}]\times(\tfrac{j'}{2^n},\infty])=0, \text{and } N((\tfrac{i}{2^n},1]\times(\tfrac{j}{2^n},\infty])=0) \end{aligned}$$

$$= \int_t^1 e^{-\mu((t,r]\times(x,\infty])} \int_{x\vee v(r)}^\infty e^{-\mu((r,1]\times(y,\infty])}\mu(dr,dy)$$
$$+ \int_t^{v^{-1}(x)} \int_x^{v(r')} \int_{r'}^1 e^{-\mu((t,r]\times(y',\infty])} \int_{y'\vee v(r)}^\infty e^{-\mu((r,1]\times(y,\infty])}\mu(dr,dy)\mu(dr',dy').$$

This implies

$$g_\infty(t,v(t)) = \int_t^1 e^{-\mu((t,r]\times(v(t),\infty])} \int_{v(t)}^\infty e^{-\mu((r,1]\times(y,\infty])}\mu(dr,dy).$$

From condition (2.10) this is equal to $h_\infty(t,v(t)) = e^{-\mu((t,1]\times(v(t),\infty])}$, and as consequence we obtain equality from (2.6). □

In general the optimal best choice probability s in (2.7) has to be evaluated numerically. Certain classes of intensity functions however allow an explicit evaluation.

Example 2.1 *Consider densities of the intensity measure μ of the form*

$$g(t,y) = -a(t)F'(y), \tag{2.12}$$

where $a : [0,1] \to [0,\infty]$ is continuous and integrable, a is not identical zero in any neighbourhood of 1 and $F : [c,\infty] \to \mathbb{R}$ monotonically nonincreasing, continuous with $\lim_{x\downarrow c} F(x) = \infty$ and $F(\infty) = 0$. Defining $A(t) := \int_t^1 a(s)ds$, we obtain

$$\int_t^1 \int_x^\infty e^{\mu((r,1]\times(x,y])}\mu(dr,dy) = \int_0^{A(t)F(x)} \frac{e^y - 1}{y} dy.$$

In consequence v solves the equation

$$F(v(t)) = \frac{d}{A(t)}, \tag{2.13}$$

where $d = 0.8043522\ldots$ is the unique solution of

$$\int_0^d \frac{e^y-1}{y}dy = 1. \tag{2.14}$$

The asymptotic optimal choice probability can be obtained from (2.7) by some calculation as

$$s = e^{-d} + (e^d - 1 - d)\int_d^\infty \frac{e^{-y}}{y}dy = 0.5801642\ldots$$

This is identical to the asymptotic optimal choice probability in the iid case (see (1.3)). In particular in case of the three extreme value distribution types Λ, Φ_α, and Ψ_α, one gets for the limiting Poisson processes intensities with densities $g(t,x)$ of the form $g(t,x) = -F'(y)$ with $F(x) = e^{-x}$ in case Λ, $F(x) = x^{-\alpha}$, $x > 0$ in case Ψ_α, $\alpha > 0$, and $F(x) = (-x)^\alpha$ for $x \leqslant 0$, $F(x) = 0$, $x > 0$ in case Ψ_α, $\alpha > 0$. Thus, these cases fit the form in (2.12). Also the example of a best choice problem for $X_i = c_iY_i$ for some iid sequence dealt with in [KR] (2000c) fits this condition.

3 Poisson processes with finite intensities

Gnedin and Sakaguchi (1992) considered the best choice problem for iid sequence (Y_k) with distribution functions F arriving at Poisson distributed time points. For continuous F this can be described by a planar Poisson process N with density $g(t,y) = a(t)F'(y)$ where $a : [0,1] \to [0,\infty]$ is continuous integrable. N does not fulfill the infinite intensity condition (I) in Section 2. For the best choice problem in Poisson processes our method of proof can be modified to deal also with the case of finite intensity.

Let $N = \sum_k = \delta_{\tau_k, Y_k}$ be a Poisson process on $[0,1] \times (c,\infty]$ with finite Lebesgue continuous intensity measure, i.e. μ satisfies

($\mathbf{I_f}$) Finite Intensity Condition: $\mu([0,1) \times (c,\infty]) < \infty$.

Then the following modifications of the proof of Theorem 2.1 allow to solve this case. Note that under condition ($\mathrm{I_f}$) no longer $h_\infty(t,c) = 0$ and thus in general no longer one can find to any $t \in [0,1)$ an $x > c$ such that $h_\infty(t,x) < g_\infty(t,x)$. This property holds true only in $[0,t_0)$ with

$$t_0 := \sup\left\{ t \in [0,1] \mid \int_t^1 \int_c^\infty e^{\mu((r,1]\times(c,y])} \mu(dr,dy) > 1 \right\}. \tag{3.1}$$

This can be seen as follows: For $t \in [0,t_0)$ and for x close to c holds

$$g_\infty(t,x) \geqslant P(Y_{\tilde{T}(t,x)} = \sup_{\tau_k > t} Y_k \vee x) > h_\infty(t,x)$$

with stopping time $\tilde{T}(t,x) := \inf\{\tau_k > t \mid Y_k > x\}$. If for some $t \notin [0,t_0)$ there would exist some $v(t) \in (c,\infty)$ with $h_\infty(t,v(t)) = g_\infty(t,v(t))$, then $v(t) = c$ would give a contradiction.

So far we have obtained optimality of the stopping times $T(t,x)$ for $x > c$. We next consider the case $x = c$. Since N has only finitely many points in $[0,1] \times (c,\infty)$ it follows that

$$\begin{aligned} g_\infty(t,x) &= P(Y_{T(t,x)} = \sup_{\tau_k > t} Y_k > x) \\ &\xrightarrow{x \downarrow c} P(Y_{T(t,c)} = \sup_{\tau_k > t} Y_k > c) = P(Y_{T(t,c)} = \sup_{\tau_k > t} Y_k) - h_\infty(t,c). \end{aligned}$$

The inequality

$$\begin{aligned} &\sup_{\substack{T > t \\ N\text{-stopping time}}} P(Y_T = \sup_{\tau_k > t} Y_k) \\ &\leqslant \sup_{\substack{\tau > t \\ N\text{-stopping time}}} P(Y_T \vee x = \sup_{\tau_k > t} Y_k \vee x) = g_\infty(t,x) + h_\infty(t,x) \end{aligned}$$

then implies optimality of $T(t,c)$ and we obtain the following result.

Theorem 3.1 *Let the Poisson process satisfy the finite intensity condition* (I_f)*. Then the optimal choice stopping time for* N *is given by*

$$T = \inf\left\{\tau_k \mid Y_k = \sup_{\tau_j \in [0,\tau_k]} Y_j, Y_k > v(\tau_k)\right\},$$

where $v : [0,1] \to [c,\infty)$ *is a solution of the integral equation*

$$\begin{aligned}&\int_t^1 \int_{v(t)}^\infty e^{\mu((r,1]\times(v(t),y])}\mu(dr,dy) = 1 \quad \forall t \in [0,t_0),\\ &v(t) = c \quad \forall t \in [t_0, 1].\end{aligned}$$

v *is monotonically nonincreasing and can be chosen right continuous. The optimal choice probability is given by*

$$\begin{aligned}s &:= P(Y_T = \sup_k Y_k, T < 1)\\ &= \int_0^1 e^{-\mu([0,r]\times(c,\infty])} \int_{v(r)}^\infty e^{-\mu((r,1]\times(y,\infty])}\mu(dr,dy)\\ &\quad + \int_0^{t_0} \int_c^{v(r')} \int_{r'}^1 e^{-\mu([0,r]\times(y',\infty])} \int_{y'\vee v(r)}^\infty e^{-\mu((r,1]\times(y,\infty])}\mu(dr,dy)\mu(dr',dy').\end{aligned}$$

Example 3.1 *In case of the finite intensity measure* μ *with density* $g(t,y) = a(t)F'(y)$ *as in Gnedin and Sakaguchi (1992) let* $F(c) = 0$ *and* $A(t) := \int_t^1 a(s)ds$*. Then we obtain*

$$\int_t^1 \int_x^\infty e^{\mu((r,1]\times(x,y])}\mu(dr,dy) = \int_0^{A(t)(1-F(x))} \frac{e^y - 1}{y} dy.$$

Thus we get $t_0 = 0$ *and* $v \equiv c$ *if* $A(0) \leqslant d$ *where* d *is the constant given in* (2.14)*.*

If $A(0) > d$*, then* t_0 *is the smallest point satisfying* $A(t_0) = d$*, and for* $t \in [0, t_o)$ $v(t)$ *is a solution of the equation*

$$F(v(t)) = 1 - \frac{d}{A(t)},$$

Some detailed calculations yield in this case the optimal choice probability s *as*

$$s = \begin{cases} e^{-A(0)} \displaystyle\int_0^{A(0)} \frac{e^y - 1}{y} dy, & \text{if } A(0) \leqslant d,\\ e^{-d} + (e^d - 1 - d) \displaystyle\int_d^{A(0)} \frac{e^{-y}}{y} dy, & \text{if } A(0) > d.\end{cases}$$

This coincides with the results obtained in Gnedin and Sakaguchi (1992).

4 The optimal m-choice problem

In this section we consider the optimal m-choice problem (1.4) for independent sequences (X_i). Let $X_i^n = \frac{X_i - b_n}{a_n}$ denote the normalized version as in Section 2. Similarly we consider the optimal m-choice problem for continuous time Poisson processes in the plane defined by

$$P\Big(\bigvee_{i=1}^{m} Y_{T_i^m} = \sup_k Y_k\Big) = \sup_{\substack{0 \leqslant T_1 < \cdots < T_m \leqslant 1 \\ T_i, N\text{-stopping times}}} P\Big(\bigvee_{i=1}^{m} Y_{T_i} = \sup_k Y_k\Big) \tag{4.1}$$

and call $(T_i^m) = (T_i^{n,m})$ optimal m-choice stopping times. The condition $t \leqslant T_1 < \cdots < T_m \leqslant 1$ means that $T_{i-1} < T_i$ on $\{T_{i-1} < 1\}$ and $T_i = 1$ on $\{T_{i-1} = 1\}$.

The following lemma gives a characterization of optimal m-choice stopping times in the discrete case.

Lemma 4.1 *Define for $(t, x) \in [0, 1) \times [c, \infty)$*

$$g_n^m(t, x) := \sup_{\substack{tn < T_1 < \cdots < T_m \leqslant n \\ \text{stopping times}}} P(X_{T_1}^n \vee \cdots \vee X_{T_m}^n = x \vee \max_{tn < j \leqslant n} X_j^n)$$

and

$$h_n^1(t, x) := P(\max_{tn < j \leqslant n} X_j^n \leqslant x),$$
$$h_n^m(t, x) := g_n^{m-1}(t, x) + h_n^1(t, x).$$

Then

$$g_n^m(t, x) = P(X^n_{T_1^{n,m}(t,x)} \vee \cdots \vee X^n_{T_m^{n,m}(t,x)} = x \vee \max_{tn < j \leqslant n} X_j^n)$$

with optimal stopping times given by

$$\begin{aligned} T_1^{n,m}(t, x) &:= \min\{tn < i \leqslant n - m + 1 \mid \\ &\qquad h_n^m(\tfrac{i}{n}, X_i^n) > g_n^m(\tfrac{i}{n}, X_i^n), X_i^n = x \vee \max_{tn < j \leqslant i} X_j^n\}, \\ T_l^{n,m}(t, x) &:= \min\{T_{l-1}^{n,m}(t, x) < i \leqslant n - m + l \mid \\ &\qquad h_n^{m-l+1}(\tfrac{i}{n}, X_i^n) > g_n^{m-l+1}(\tfrac{i}{n}, X_i^n), X_i^n = x \vee \max_{tn < j \leqslant i} X_j^n\} \end{aligned} \tag{4.2}$$

for $2 \leqslant l \leqslant m$.

Proof: Let $tn < S \leqslant n - m + 1$ be a stopping time and $Z \leqslant \max_{tn < j \leqslant S} X_j^n$ be a random variable. Furthermore let for $1 \leqslant i \leqslant n$, M_i^n be $\sigma(X_{i+1}^n, \ldots, X_n^n)$ measurable with $M_i^n \leqslant X_{i+1}^n \vee M_{i+1}^n$ and $P(M_i^n = x) = 0$ for all $x > c$. In order to maximize $P(Z \vee X_T^n \vee M_T^n = x \vee \max_{tn < j \leqslant n} X_j^n)$ w.r.t. all stopping times T, $S < T \leqslant n - m + 1$ we define

$$Y_i := P(Z \vee X_i^n \vee M_i^n = x \vee \max_{tn < j \leqslant n} X_j^n \mid X_1^n, \ldots, X_i^n).$$

Thus we have to maximize $EY_T = P(Z \vee X_T^n \vee M_T^n = x \vee \max_{tn<j\leqslant n} X_j^n)$. The optimal stopping times are given by $T_n(t,x) = T_n^{>tn}(t,x)$ with

$$\begin{aligned}
&T_n^{>k}(t,x)\\
&\quad = \min\{k < i \leqslant n-m+1 \mid P(Z \vee X_i^n \vee M_i^n = x \vee \max_{tn<j\leqslant n} X_j^n \mid X_1^n, \dots, X_i^n)\\
&\qquad\qquad > P(Z \vee X^n_{T_n^{>i}(t,x)} \vee M^n_{T_n^{>i}(t,x)} = x \vee \max_{tn<j\leqslant n} X_j^n \mid X_1^n, \dots, X_i^n),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad X_i^n = x \vee \max_{tn<j\leqslant i} X_j^n\}\\
&\quad = \min\{k < i \leqslant n-m+1 \mid P(X_i^n \vee M_i^n = X_i^n \vee \max_{i<j\leqslant n} X_j^n \mid X_i^n)\\
&\qquad\qquad > P(X^n_{T_n^{>i}(t,x)} \vee M^n_{T_n^{>i}(t,x)} = X_i^n \vee \max_{i<j\leqslant n} X_j^n \mid X_i^n),\\
&\qquad\qquad\qquad\qquad\qquad\qquad X_i^n = x \vee \max_{tn<j\leqslant i} X_j^n\}.
\end{aligned}$$

For the second equality we use that $P(X_i^n = X_j^n \geqslant c) = 0$ for $i \neq j$ and thus $X_i^n = x \vee \max_{tn<j\leqslant i} X_j^n > x$ is strictly larger than Z. Thus Z can not be the maximum. In consequence we get for $k \geqslant S$

$$T_n^{>k}(t,x) = \min\{k < i \leqslant n-m+1 \mid \hat{h}_n(\tfrac{i}{n}, X_i^n) > \hat{g}_n(\tfrac{i}{n}, X_i^n), X_i^n = x \vee \max_{tn<j\leqslant i} X_j^n\} \tag{4.3}$$

where

$$\begin{aligned}
\hat{h}_n(t,x) &:= P(x \vee M^n_{\lfloor tn \rfloor} = x \vee \max_{tn<j\leqslant n} X_j^n)\\
&= P(M^n_{\lfloor tn \rfloor} = x \vee \max_{tn<j\leqslant n} X_j^n) + h_n^1(t,x),\\
\hat{g}_n(t,x) &:= P(X^n_{T_n(t,x)} \vee M^n_{T_n(t,x)} = x \vee \max_{tn<j\leqslant n} X_j^n).
\end{aligned}$$

By induction we obtain from (4.3) the representation in (4.2). Define for $m = 1$: $M_i^n := -\infty$, for $m = 2$: $M_i^n := X^n_{T_n^{>i}(t,x)}$, etc. □

Theorem 4.2 (Approximative solution of the best m-choice problem) *Let $N_n \xrightarrow{d} N$ on $M_c = [0,1] \times (c, \infty]$ and let N satisfy the intensity condition (I).*

a) The optimal m-choice stopping times for N are given by $T_1^m(0,c), \dots, T_m^m(0,c)$, where

$$\begin{aligned}
T_1^m(t,x) &= \inf\{\tau_k > t \mid Y_k = x \vee \sup_{\tau_j \in (t,\tau_k]} Y_j, Y_k > v^m(\tau_k)\},\\
T_l^m(t,x) &= \inf\{\tau_k > T_{l-1}^m(t,x) \mid Y_k = x \vee \sup_{\tau_j \in (t,\tau_k]} Y_j, Y_k > v^{m-l+1}(\tau_k)\},
\end{aligned}$$

for $2 \leqslant l \leqslant m$. The thresholds $v^m(t)$ are solutions of the equations

$$\begin{aligned}
g_\infty^m(t, v^m(t)) &= h_\infty^m(t, v^m(t)) \quad \text{for } t \in [0,1) \\
v^m(1) &= c,
\end{aligned} \tag{4.4}$$

with

$$g_\infty^m(t,x) := P(Y_{T_1^m(t,x)} \vee \cdots \vee Y_{T_m^m(t,x)} = x \vee \sup_{t<\tau_k\leqslant 1} Y_k)$$

and

$$\begin{aligned} h_\infty^1(t,x) &:= e^{-\mu((t,1]\times(x,\infty])} && \text{for } m = 1, \\ h_\infty^m(t,x) &:= g_\infty^{m-1}(t,x) + h_\infty^1(t,x) && \text{for } m \geqslant 2. \end{aligned}$$

$v^m : [0,1] \to [c,\infty)$ *is monotonically nonincreasing and can be chosen right continuous. Furthermore,* $v^m(t) \leqslant v^{m-1}(t)$ *for* $t \in [0,1]$ $(m \geqslant 2)$.

b)
$$\lim_{n\to\infty} P(X_{T_1^{n,m}} \vee \cdots \vee X_{T_m^{n,m}} = M_n) = s_m = g_\infty^m(0,c)$$

c) The stopping times $\hat{T}_l^m = \hat{T}_l^{n,m}$ *defined by*

$$\begin{aligned} \hat{T}_1^{n,m} &:= \min\{1 \leqslant i \leqslant n-m+1 \mid X_i = M_i, X_i > a_n v^m(\tfrac{i}{n}) + b_n\}, \\ \hat{T}_l^{n,m} &:= \min\{\hat{T}_{l-1}^{n,m} < i \leqslant n-m+l \mid X_i = M_i, X_i > a_n v^{m-l+1}(\tfrac{i}{n}) + b_n\} \end{aligned} \tag{4.5}$$

for $2 \leqslant l \leqslant m$ *are approximative optimal* m*-choice stopping times, i.e.*

$$\lim_{n\to\infty} P(X_{\hat{T}_1^{n,m}} \vee \cdots \vee X_{\hat{T}_m^{n,m}} = M_n) = s_m. \tag{4.6}$$

Proof: The proof of b), c) is similar to the corresponding part in Theorem 2.1. We therefore concentrate on the proof of a).

As in the proof of Theorem 2.1 we obtain

$$g_\infty^m(t,x) = \sup_{\substack{t<T_1,\ldots,T_m\leqslant 1 \\ N\text{-stopping times}}} P(Y_{T_1} \vee \cdots \vee Y_{T_m} = x \vee \sup_{t<\tau_k\leqslant 1} Y_k). \tag{4.7}$$

Furthermore we note that v^m is continuous in 1, $\lim_{t\uparrow 1} v^m(t) = c$. If not, then $v^m(t_n) \to d > c$ for some sequence $t_n \uparrow 1$ and thus $g_\infty^m(t_n, v^m(t_n)) \to 0$. The inequality $h_\infty^m(t_n, v^m(t_n)) \geqslant h_\infty^1(t_n, v^m(t_n)) \to e^{-0} = 1 > 0$ then yields a contradiction.

We have to show that for $m \geqslant 2$, $v^m \leqslant v^{m-1}$. Using the characterization of v^m in (4.4) this inequality follows from

$$g_\infty^m(t,x) - h_\infty^m(t,x) \leqslant g_\infty^{m-1}(t,x) - h_\infty^{m-1}(t,x). \tag{4.8}$$

for $x = v^m(t)$. For $m = 2$ we obtain from (4.7)

$$g_\infty^2(t,x) \leqslant 2g_\infty^1(t,x),$$

which implies (4.8). For $m \geqslant 3$ (4.8) is equivalent to

$$g_\infty^m(t,x) - g_\infty^{m-1}(t,x) \leqslant g_\infty^{m-1}(t,x) - g_\infty^{m-2}(t,x). \tag{4.9}$$

We prove (4.9) by induction. By induction hypothesis we assume that $v^{m-1} \leqslant v^{m-2} \leqslant \ldots \leqslant v^1$ and all of the v^i are monotonically nonincreasing. Also we have $\lim_{t\uparrow 1} v^m(t) = c$. Assume that $v^m(s) > v^{m-1}(s)$ for some $s \in [0,1)$. Since $\lim_{t\uparrow 1} v^m(t) = \lim_{t\uparrow 1} v^{m-1}(t) = c$ and since v^{m-1} is monotonically nonincreasing, there exists some $t \in [s,1)$ with $v^m(t) > v^{m-1}(t)$ and $v^m(t) \geqslant v^m(r)$ for all $r \geqslant t$. We establish (4.9) for $x = v^m(t)$.

From the definition of the thresholds follows $T_k^m(t,x) \leqslant T_k^{m-1}(t,x) \leqslant T_k^{m-2}(t,x)$ for all $1 \leqslant k \leqslant m-2$. We define the stopping time

$$\begin{aligned}\hat{T}(t,x) := T_1^m(t,x)\chi_{\{T_1^m(t,x)<T_1^{m-2}(t,x)\}} \\ &+ \sum_{i=2}^{m-2} T_i^m(t,x)\chi_{\{T_1^m(t,x)=T_1^{m-2}(t,x)\}\cap\cdots\cap\{T_{i-1}^m(t,x)=T_{i-1}^{m-2}(t,x)\}\cap\{T_i^m(t,x)<T_i^{m-2}(t,x)\}} \\ &+ T_{m-1}^m(t,x)\chi_{\{T_1^m(t,x)=T_1^{m-2}(t,x)\}\cap\cdots\cap\{T_{m-2}^m(t,x)=T_{m-2}^{m-2}(t,x)\}}.\end{aligned}$$

$\hat{T}(t,x)$ only stops at time points $T_1^m(t,x),\ldots,T_m^m(t,x)$ but not at time points $T_1^{m-2}(t,x),\ldots,T_{m-2}^{m-2}(t,x)$ when these are < 1. We now obtain from (4.7) the inequality

$$\begin{aligned}&g_\infty^m(t,x) - g_\infty^{m-1}(t,x) \\ &\quad\leqslant P(Y_{T_1^m(t,x)} \vee\cdots\vee Y_{T_m^m(t,x)} = x \vee \sup_{t<\tau_k\leqslant 1} Y_k) \\ &\qquad - P(Y_{T_1^{m-2}(t,x)} \vee\cdots\vee Y_{T_{m-2}^{m-2}(t,x)} = x \vee \sup_{t<\tau_k\leqslant 1} Y_k) - P(Y_{\hat{T}(t,x)} = x \vee \sup_{t<\tau_k\leqslant 1} Y_k) \\ &\quad= P(Y_{T_1^m(t,x)} \vee\cdots\vee Y_{T_m^m(t,x)} = x \vee \sup_{t<\tau_k\leqslant 1} Y_k, Y_{\hat{T}(t,x)} < x \vee \sup_{t<\tau_k\leqslant 1} Y_k) - g_\infty^{m-2}(t,x) \\ &\quad\leqslant P(Y_{T_1(t,x)} \vee\cdots\vee Y_{T_{m-1}(t,x)} = x \vee \sup_{t<\tau_k\leqslant 1} Y_k) - g_\infty^{m-2}(t,x) \\ &\quad\leqslant g_\infty^{m-1}(t,x) - g_\infty^{m-2}(t,x),\end{aligned}$$

with the stopping times

$$\begin{aligned}T_1(t,x) &:= T_1^m(t,x)\chi_{\{T_1^m(t,x)\neq\hat{T}(t,x)\}} + T_2^m(t,x)\chi_{\{T_1^m(t,x)=\hat{T}(t,x)\}} \\ T_i(t,x) &:= T_i^m(t,x)\chi_{\{T_1^m(t,x)\neq\hat{T}(t,x)\}\cap\cdots\cap\{T_i^m(t,x)\neq\hat{T}(t,x)\}} \\ &\qquad + T_{i+1}^m(t,x)\chi_{\{T_1^m(t,x)=\hat{T}(t,x)\}\cup\cdots\cup\{T_i^m(t,x)=\hat{T}(t,x)\}}\end{aligned}$$

for $i = 2,\ldots,m-1$.

Thus (4.9) holds true for $x = v^m(t)$ and, therefore, $v^m(t) \leqslant v^{m-1}(t)$, a contradiction. This implies the statement $v^m \leqslant v^{m-1}$.

In the final step we prove that v^m is monotonically nonincreasing. Since $g_\infty^m(t,x) - h_\infty^m(t,x)$ is monotonically nonincreasing in x for any t, it suffices to prove that for $s < t$

$$g_\infty^m(s, v^m(t)) - h_\infty^m(s, v^m(t)) \geqslant 0. \tag{4.10}$$

We define for $(s,x) \in [0,1) \times (c,\infty]$

$$\begin{aligned}A_{(s,x)} := \{Y_{T_1^m(s,x)} \vee\cdots\vee Y_{T_m^m(s,x)} = x \vee \sup_{s<\tau_k\leqslant 1} Y_k, Y_{T_1^{m-1}(s,x)} \\ \vee\cdots\vee Y_{T_{m-1}^{m-1}(s,x)} < x \vee \sup_{s<\tau_k\leqslant 1} Y_k\}.\end{aligned}$$

Then

$$g_\infty^m(s,x) - g_\infty^{m-1}(s,x) = P(A_{(s,x)})$$

and for $s < t$

$$\begin{aligned}
&P(A_{(t,x)}) - P(A_{(s,x)}) \\
&\quad = P(A_{(s,x)}, N((s,t] \times (x,\infty]) = 0) + P(A_{(t,x)})P(N((s,t] \times (x,\infty]) \geqslant 1) \\
&\qquad - P(A_{(s,x)}) \\
&\quad = -P(A_{(s,x)}, N((s,t] \times (x,\infty]) \geqslant 1) + P(A_{(t,x)})(1 - e^{-\mu((s,t]\times(x,\infty])}) \\
&\quad \leqslant \frac{P(A_{(t,x)})}{h_\infty^1(t,x)}(h_\infty^1(t,x) - h_\infty^1(s,x)).
\end{aligned}$$

With $x = v^m(t)$ we obtain the inequality in (4.10) and thus monotonicity of v^m. □

To calculate the optimal thresholds we need to calculate the densities of record stopping times with general threshold v.

Lemma 4.3 *Let $v : [0,1] \to [c,\infty)$ be monotonically nonincreasing, right continuous with $v > c$ on $[0,1)$. Define the record stopping time associated to v for $(t,x) \in [0,1) \times (c,\infty)$ by*

$$T(t,x) := \inf\{\tau_k > t \mid Y_k = x \vee \sup_{\tau_j \in (t,\tau_k]} Y_j, Y_k > v(\tau_k)\},$$

(a) If $x \geqslant v(t)$, then $(T(t,x), Y_{T(t,x))\chi\{T(t,x)<1\}}$ has Lebesgue density

$$F_{(t,x)}(s,y) := \begin{cases} e^{-\mu((t,s]\times(x,\infty])} g(s,y), & \text{if } s \in (t,1), y > x, \\ 0, & \text{if } s \in [0,t] \text{ or } y \leqslant x \end{cases} \tag{4.11}$$

on $[0,1) \times \mathbb{R}$.

(b) For $x < v(t)$, $(T(t,x), Y_{T(t,x))}\chi_{\{T(t,x)<1\}}$ has Lebesgue density

$$\begin{aligned}
&F_{(t,x)}(s,y) \\
&\quad := \begin{cases} \left(1 + \displaystyle\int_t^{s\wedge v^{-1}(x)} \int_x^{y\wedge v(r)} e^{\mu((t,s]\times(x,z])} \mu(dr,dz)\right) & \\ \qquad \cdot e^{-\mu((t,s]\times(x,\infty])} g(s,y) \chi_{M_v}(s,y), & \text{if } s \in (t,1), y > x, \\ 0, & \text{if } s \in [0,t] \text{ or } y \leqslant x, \end{cases}
\end{aligned} \tag{4.12}$$

on $[0,1) \times \mathbb{R}$.

Proof:

(a) If $x \geqslant v(t)$, then for $s \in (t,1)$ and $z \in (x,\infty)$

$$\begin{aligned}
&P(T(t,x) \leqslant s, Y_{T(t,x)} \leqslant z)\\
&\quad = P(\exists n_0 : \forall n \geqslant n_0 : \exists \tfrac{i}{2^n} \in (t,s] : \exists \tfrac{j}{2^n} \in (x,z] :\\
&\qquad\quad N((t, \tfrac{i-1}{2^n}] \times (x,\infty]) = 0, N((\tfrac{i-1}{2^n}, \tfrac{i}{2^n}] \times (\tfrac{j}{2^n},\infty]) = 0,\\
&\qquad\quad N((\tfrac{i-1}{2^n}, \tfrac{i}{2^n}] \times (x, \tfrac{j-1}{2^n}]) = 0, \text{and } N((\tfrac{i-1}{2^n}, \tfrac{i}{2^n}] \times (\tfrac{j-1}{2^n}, \tfrac{j}{2^n}]) = 1)\\
&\quad = \int_t^s \int_x^z e^{-\mu((t,r]\times(x,\infty])}\mu(dr,dy).
\end{aligned}$$

(b) If $x < v(t)$ then for $s \in (t,1)$ and $z \in (x,\infty)$ holds with $M_v = \{t,x) : x > v(t)\}$

$$P(T(t,x) \leqslant s, Y_{T(t,x)} \leqslant z) = A_1 + A_2$$

where

$$\begin{aligned}
A_1 &= P(\exists n_0 : \forall n \geqslant n_0 : \exists \tfrac{i}{2^n} \in (t,s] : \exists \tfrac{j}{2^n} \in (x,z] :\\
&\qquad N((t, \tfrac{i-1}{2^n}] \times (x,\infty]) = 0, N((\tfrac{i-1}{2^n}, \tfrac{i}{2^n}] \times (\tfrac{j}{2^n},\infty]) = 0,\\
&\qquad N((\tfrac{i-1}{2^n}, \tfrac{i}{2^n}] \times (x, \tfrac{j-1}{2^n}]) = 0, \text{and } N((\tfrac{i-1}{2^n}, \tfrac{i}{2^n}] \times (\tfrac{j-1}{2^n}, \tfrac{j}{2^n}] \cap M_v) = 1)\\
&= \int_t^s \int_x^z e^{-\mu((t,r]\times(x,\infty])}\chi_{M_v}(r,y)\mu(dr,dy)
\end{aligned}$$

and

$$\begin{aligned}
A_2 &= P(\exists n_0 : \forall n \geqslant n_0 : \exists t < \tfrac{i'}{2^n} < \tfrac{i}{2^n} \leqslant s : \exists x < \tfrac{j'}{2^n} < \tfrac{j}{2^n} \leqslant z :\\
&\qquad N((\tfrac{i'-1}{2^n}, \tfrac{i'}{2^n}] \times (\tfrac{j'-1}{2^n}, \tfrac{j'}{2^n}]) = 1 \text{ for } \tfrac{j'}{2^n} \leqslant v(\tfrac{i'}{2^n}) \text{ and } \tfrac{i'}{2^n} \leqslant v^{-1}(x),\\
&\qquad N((t, \tfrac{i-1}{2^n}] \times (\tfrac{j'}{2^n},\infty]) = 0, N((\tfrac{i-1}{2^n}, \tfrac{i}{2^n}] \times (\tfrac{j}{2^n},\infty]) = 0,\\
&\qquad \text{and } N((\tfrac{i-1}{2^n}, \tfrac{i}{2^n}] \times (\tfrac{j-1}{2^n}, \tfrac{j}{2^n}] \cap M_v) = 1)\\
&= \int_t^s \int_x^z \int_t^{r\wedge v^{-1}(x)} \int_x^{y\wedge v(r')} e^{-\mu((t,r]\times(y',\infty])}\mu(dr',dy')\chi_{M_v}(r,y)\mu(dr,dy). \quad \square
\end{aligned}$$

Conditioning by $(T_1^m(t,x), Y_{T_1^m(t,x)})\chi_{\{T_1^m(t,x)<1\}}$ we obtain for $x \geqslant v^m(t)$ as consequence a formula for $g_\infty^m(t,x)$ using that v^m is monotonically nonincreasing

$$g_\infty^m(t,x) = \int_t^1 \int_x^\infty h_\infty^m(s,y)e^{-\mu((t,s]\times(x,\infty])}g(s,y)dyds.$$

This allows to calculate the optimal threshold v^m. Finally we can calculate $g_\infty^m(t,x)$ for $x < v^m(t)$ using that

$$g_\infty^m(t,x) = \int_t^1 \int_x^\infty h_\infty^m(s,y)e^{-\mu((t,s]\times(x,\infty])}g(s,y)dyds.$$

Here $F^m_{(t,x)}$ is the density of $(T_1^m(t,x), Y_{T_1^m}(t,x))\chi_{\{T_1^m(t,x)<1\}}$ from Lemma 4.3 involving the optimal thresholds v^m.

For certain densities $g(t,x)$ the optimal stopping curves can be determined directly. To that purpose define constants d_m and functions H_m in the following way: Let for $m \in \mathbb{N}$ d_m be the uniquely determined constant with

$$e^{-d_m} \int_0^{d_m} \frac{e^x}{x} \int_0^x H^m(y) dy dx = H^m(d_m). \tag{4.13}$$

Here the functions H^m for $z \geqslant 0$ are inductively defined by $H^1(z) := e^{-z}$ and

$$H^m(z) := \begin{cases} e^{-z} + e^{-z} \int_0^z \frac{e^x}{x} \int_0^x H^{m-1}(y) dy dx, & z \leqslant d_{m-1}, \\ e^{-z} + \int_0^1 \int_0^{z \wedge \left(\frac{d_{m-1}}{x}\right)} H^{m-1}(xy) R(x,y,z,d_{m-1}) dy dx, & z > d_{m-1} \end{cases}$$

for $m \geqslant 2$, where

$$R(x,y,z,d) := \begin{cases} \frac{1}{1-x} e^{-(1-x)y} + \frac{x \vee (dz^{-1}) - x}{1-x} e^{-(1-x)z} \\ \quad - \frac{x \vee (dz^{-1})}{1-x} e^{-\frac{1-x}{x \vee (dz^{-1})} d} - d \int_{(1-x)y}^{\frac{1-x}{x \vee (dz^{-1})} d} \frac{e^{-u}}{u} du, & \text{if } y > d, \\ d \int_{(1-x)d}^{\frac{1-x}{x \vee (dz^{-1})} d} \frac{e^{-u}}{u^2} du + \frac{x \vee (dz^{-1}) - x}{1-x} e^{-(1-x)z}, & \text{if } y \leqslant d. \end{cases}$$

In the following theorem we given an application to the case where the intensity measure μ of the Poisson process N has a density of the form

$$g(t,y) = -a(t) F'(y)$$

with a continuous integrable function $a : [0,1] \to [0,\infty]$ not identical zero in any neighbourhood of 1 and $F : [c,\infty] \to \mathbb{R}$ monotonically nonincreasing, continuous with $\lim_{x \downarrow c} F(x) = \infty$ and $F(\infty) = 0$.

In this situation we get the following result for the optimal m-choice problem for N.

Theorem 4.4 *Under the conditions stated above the optimal threshold $v^m(t)$ for $t \in [0,1)$ is a solution of*

$$F(v^m(t)) = \frac{d_m}{A(t)} \quad \textit{with } A(t) := \int_t^1 a(s) ds. \tag{4.14}$$

Further $h^m_\infty(t,x) = H^m(A(t)F(x))$ and the optimal m-choice probability is given by

$$s^m = H^{m+1}(\infty).$$

Proof: At first let $v(t)$, $t \in [0,1)$ be a solution of

$$F(v(t)) = \frac{d}{A(t)}$$

where $d > 0$ and $T(t,x)$ is the record stopping time associated to v. Let $v : [0,1] \to [c,\infty)$ be chosen right continuous. To calculate $F_{(t,x)}$ the density of $(T(t,x), Y_{T(t,x)})$ in case $x < v(t)$ we have to calculate

$$\int_t^{s\wedge v^{-1}(x)} \int_x^{y\wedge v(r)} e^{\mu((t,s]\times(x,z])}\mu(dr,dz).$$

For $y \in (x, v(t)]$ holds

$$\begin{aligned}
&\int_t^{s\wedge v^{-1}(y)} \int_x^{y} e^{\mu((t,s]\times(x,z])}\mu(dr,dz) \\
&\quad = -\int_t^{s\wedge v^{-1}(y)} a(r) \int_x^y e^{(A(t)-A(s))(F(x)-F(z))} F'(z)dzdr \\
&\quad = \int_t^{s\wedge v^{-1}(y)} a(r)dr \int_x^y \tfrac{1}{A(t)-A(s)} \tfrac{\partial}{\partial z} e^{-(A(t)-A(s))F(z)} dz e^{(A(t)-A(s))F(x)} \\
&\quad = \tfrac{A(t)-A(s)\vee(\frac{d}{F(y)})}{A(t)-A(s)} e^{(A(t)-A(s))F(x)} \left(e^{-(A(t)-A(s))F(y)} - e^{-(A(t)-A(s))F(x)}\right) \\
&\quad = \tfrac{A(t)-A(s)\vee(\frac{d}{F(y)})}{A(t)-A(s)} \left(e^{(A(t)-A(s))(F(x)-F(y))} - 1\right).
\end{aligned}$$

Using some substitutions we obtain further

$$\begin{aligned}
&\int_{s\wedge v^{-1}(y)}^{s\wedge v^{-1}(x)} \int_x^{v(r)} e^{\mu((t,s]\times(x,z])}\mu(dr,dz) = \int_{s\wedge v^{-1}(y)}^{s\wedge v^{-1}(x)} a(r) \int_x^{v(r)} \tfrac{1}{A(t)-A(s)} \tfrac{\partial}{\partial z} \\
&\quad = \int_{s\wedge v^{-1}(y)}^{s\wedge v^{-1}(x)} a(r) \left(e^{-(A(t)-A(s))\frac{d}{A(r)}} - e^{-(A(t)-A(s))F(x)}\right) dr \tfrac{e^{(A(t)-A(s))F(x)}}{A(t)-A(s)} \\
&\quad = \int_{\frac{1}{A(s)\vee(\frac{d}{F(y)})}}^{\frac{1}{A(s)\vee(\frac{d}{F(x)})}} \tfrac{1}{z^2} e^{-(A(t)-A(s))dz} dz \tfrac{e^{(A(t)-A(s))F(x)}}{A(t)-A(s)} - \tfrac{A(s)\vee(\frac{d}{F(y)})-A(s)\vee(\frac{d}{F(x)})}{A(t)-A(s)}.
\end{aligned}$$

With the substitution $z' = (A(t)-A(s))z$ this is identical to

$$\begin{aligned}
&\int_{\frac{A(t)-A(s)}{A(s)\vee(\frac{d}{F(y)})}d}^{\frac{A(t)-A(s)}{A(s)\vee(\frac{d}{F(x)})}d} \tfrac{e^{-z}}{z^2} dz d e^{(A(t)-A(s))F(x)} - \tfrac{A(s)\vee(\frac{d}{F(y)})-A(s)\vee(\frac{d}{F(x)})}{A(t)-A(s)} \\
&\quad = \left\{ \left[-\tfrac{1}{z}e^{-z}\right]_{\cdots}^{\cdots} - \int_{\cdots}^{\cdots} \tfrac{e^{-z}}{z}dz \right\} d e^{(A(t)-A(s))F(x)} - \tfrac{A(s)\vee(\frac{d}{F(y)})-A(s)\vee(\frac{d}{F(x)})}{A(t)-A(s)} \\
&\quad = \Bigg\{ -\tfrac{A(s)\vee(\frac{d}{F(x)})}{A(t)-A(s)} e^{-\frac{A(t)-A(s)}{A(s)\vee(\frac{d}{F(x)})}d} + \tfrac{A(s)\vee(\frac{d}{F(y)})}{A(t)-A(s)} e^{-\frac{A(t)-A(s)}{A(s)\vee(\frac{d}{F(y)})}d}
\end{aligned}$$

$$- d \int_{\frac{A(t)-A(s)}{A(s)\vee(\frac{d}{F(y)})}d}^{\frac{A(t)-A(s)}{A(s)\vee(\frac{d}{F(x)})}d} \frac{e^{-z}}{z} dz \Big\} e^{(A(t)-A(s))F(x)} - \frac{A(s)\vee(\frac{d}{F(y)})-A(s)\vee(\frac{d}{F(x)})}{A(t)-A(s)}.$$

Together we obtain for $s \in (t,1)$ and $y \in (x, v(t)]$

$$\begin{aligned} F_{(t,x)}(s,y) = & \Big\{ -\frac{A(s)\vee(\frac{d}{F(x)})}{A(t)-A(s)} e^{-\frac{A(t)-A(s)}{A(s)\vee(\frac{d}{F(x)})}d} + \frac{A(s)\vee(\frac{d}{F(x)})-A(s)}{A(t)-A(s)} e^{-(A(t)-A(s))F(x)} \\ & + \frac{A(t)}{A(t)-A(s)} e^{-(A(t)-A(s))F(y)} - d \int_{(A(t)-A(s))F(y)}^{\frac{A(t)-A(s)}{A(s)\vee(\frac{d}{F(x)})}d} \frac{e^{-z}}{z} dz \Big\} \\ & \cdot a(s)(-F'(y))\chi_{M_v}(s,y) \\ = & R\left(\tfrac{A(s)}{A(t)}, A(t)F(y), A(t)F(x), d\right) a(s)(-F'(y)), \chi_{M_v}(s,y). \end{aligned}$$

Similarly we obtain for $y \geqslant v(t)$

$$\begin{aligned} & \int_t^{s\wedge v^{-1}(x)} \int_x^{v(r)} e^{\mu((t,s]\times(x,z])} \mu(dr,dz) \\ & = \int_{\frac{A(t)-A(s)}{A(t)}d}^{\frac{A(t)-A(s)}{A(s)\vee(\frac{d}{F(x)})}d} \frac{e^{-z}}{z^2} dz d e^{(A(t)-A(s))F(x)} - \frac{A(t)-A(s)\vee(\frac{d}{F(x)})}{A(t)-A(s)}. \end{aligned}$$

In consequence also in this case for $s \in (t,1)$ and $y \in [v(t), \infty)$ (thus $A(t)F(y) \leqslant d$) holds

$$F_{(t,x)}(s,y) = R\left(\frac{A(s)}{A(t)}, A(t)F(y), A(t)F(x), d\right) a(s)(-F'(y))\chi_{M_v}(s,y).$$

This representation of $F_{(t,x)}$ allows to prove the statement by induction. The case $m = 1$ has already been given in Example 3.1.

For the induction step $m-1 \to m$ we get for $x \geqslant v^{m-1}(t)$ (thus $A(t)F(x) \leqslant d_{m-1}$) using the induction hypothesis

$$\begin{aligned} g_\infty^{m-1}(t,x) &= \int_t^1 \int_x^\infty h_\infty^{m-1}(s,y) e^{-\mu((t,s]\times(x,\infty])} \mu(ds,dy) \\ &= \int_t^1 \int_x^\infty H^{m-1}(A(s)F(y)) e^{-(A(t)-A(s))F(x)} a(s)(-F'(y)) dy ds \\ &= \int_t^1 \frac{a(s)}{A(s)} e^{A(s)F(x)} \int_0^{A(s)F(x)} H^{m-1}(z) dz ds e^{-A(t)F(x)} \\ &= \int_0^{A(t)F(x)} \frac{e^y}{y} \int_0^y H^{m-1}(z) dz dy e^{-A(t)F(x)}. \end{aligned}$$

In the last equalities we used the substitutions $z = A(s)F(y)$ resp. $y = A(s)F(x)$. Thus for $x \geqslant v^{m-1}(t)$ we have

$$h_\infty^m(t,x) = H^m(A(t)F(x)).$$

Similarly in the case $x < v^{m-1}(t)$:

$$\begin{aligned}
g_\infty^{m-1}(t,x) &= \int_t^1 \int_x^\infty h_\infty^{m-1}(s,y) F_{(t,x)}^{m-1}(s,y)\,dyds \\
&= \int_t^1 \int_{x \vee v^{m-1}(s)}^\infty H^{m-1}(A(s)F(y)) R\left(\tfrac{A(s)}{A(t)}, A(t)F(y), A(t)F(x), d_{m-1}\right) \\
&\qquad \cdot a(s)(-F'(y))\,dyds \\
&= \int_t^1 \int_0^{(A(t)F(x)) \wedge \left(\frac{A(t)}{A(s)} d_{m-1}\right)} H^{m-1}\left(\tfrac{A(s)}{A(t)} z\right) \\
&\qquad \cdot R\left(\tfrac{A(s)}{A(t)}, z, A(t)F(x), d_{m-1}\right) \tfrac{a(s)}{A(t)}\,dzds \\
&= \int_0^1 \int_0^{(A(t)F(x)) \wedge \left(\frac{d_{m-1}}{y}\right)} H^{m-1}(yz) R(y,z,A(t)F(x),d_{m-1})\,dzdy.
\end{aligned}$$

In the last equalities we used the substitutions $z = A(t)F(y)$ resp. $y = \frac{A(s)}{A(t)}$. Thus for $x < v^{m-1}(t)$ (and, therefore, $A(t)F(x) > d_{m-1}$) holds

$$h_\infty^m(t,x) = H^m(A(t)F(x)).$$

Thus for $x \geqslant v^m(t)$ we have

$$g_\infty^m(t,x) = \int_0^{A(t)F(x)} \frac{e^y}{y} \int_0^y H^m(z)\,dzdy\, e^{-A(t)F(x)}.$$

Equalizing this with $h_\infty^m(t,x) = H^m(A(t)F(x))$ it follow that the optimal threshold $v^m(t)$ is determined by

$$A(t)F(v^m(t)) = d_m \tag{4.15}$$

with d_m as determined in (4.12). This completes the induction. $\square$

We next will determine the solution of the optimal 2-choice problem for this type of densities. The constant d_1 is given by

$$d_1 = 0.8043522\ldots, \tag{4.16}$$

the unique solution of

$$\int_0^{d_1} \frac{e^y}{y}\,dy = 1. \tag{4.17}$$

With some detailed calculations we obtain

$$H^2(z) = \begin{cases} e^{-z} + e^{-z} \int_0^z \frac{e^y - 1}{y}\,dy, & \text{if } z \leqslant d_1, \\ e^{-z} + e^{-d_1} + (e^{d_1} - 1 - d_1) \int_{d_1}^z \frac{e^{-y}}{y}\,dy, & \text{if } z > d_1. \end{cases}$$

This implies the optimal one-choice probability

$$s_1 = H^2(\infty) = e^{-d_1} + (e^{d_1} - 1 - d_1) \int_{d_1}^\infty \frac{e^{-y}}{y}\,dy = 0.5801642\ldots \tag{4.18}$$

The calculation of H^3 is involved. We obtain

$$H^3(z) = \begin{cases} H^2(z) + e^{-z}\Big\{\int_0^z \frac{e^x}{x}\int_0^x \frac{1-e^{-y}}{y}dydx - \int_0^z \frac{1}{x}\int_0^x \frac{e^y-1}{y}dydx\Big\}, & \text{if } z \leqslant d_1, \\ H^2(z) + e^{-z}\Big\{K_1 \int_{d_1}^z \frac{e^y-1}{y}dy + K_2\Big\}, & \text{if } d_1 < z \leqslant d_2, \end{cases}$$

where

$$\begin{aligned} K_1 &:= \int_0^{d_1} \frac{1-e^{-y}}{y}dy = 0.6676616\ldots \\ K_2 &:= \int_0^{d_1} \frac{e^x}{x}\int_0^x \frac{1-e^{-y}}{y}dydx - \int_0^{d_1}\frac{1}{x}\int_0^x \frac{e^y-1}{y}dydx = 0.2144351\ldots \end{aligned}$$

The case $d_2 < z$ is left open. In consequence we obtain

$$d_2 = 1.5817197\ldots$$

is that constant $> d_1$, which solves the equation

$$K_1\int_{d_1}^{d_2} \frac{e^y}{y}dy - \log\left(\frac{d_2}{d_1}\right) + K_2 = 1.$$

Numerical calculation yields the optimal two-choice probability

$$s_2 = H^3(\infty) = 0.8443\ldots$$

The functions H^m, $m \geqslant 4$ seem to be too difficult to be calculated explicitly and the corresponding optimal m-choice probability can only be calculated numerically (and even that is a challenge).

Example 4.1 *Let (X_i) be iid with continuous standard normal distribution function $F = \Phi$. Then normalizing constants from extreme value theory are given by*

$$a_n = \frac{1}{\sqrt{2\log n}}, \qquad b_n = \sqrt{2\log n} - \frac{\log\log n + \log 4\pi}{2\sqrt{2\log n}}.$$

Then we obtain that

$$(X_{T_1^{n,m}} \vee \cdots \vee X_{T_m^{n,m}} = M_n) \to s_m = H^{m+1}(\infty).$$

$$\begin{aligned} \hat{T}_1^{n,m} &:= \left\{1 \leqslant i \leqslant n-m+1 \mid X_i = M_i, F(X_i) > \Phi\left(a_n \log\left(\frac{1-\frac{i}{n}}{d_m}\right) + b_n\right)\right\}, \\ \hat{T}_l^{n,m} &:= \left\{\hat{T}_{l-1}^{n,m} < i \leqslant n-m+l \mid X_i = M_i, F(X_i) > \Phi\left(a_n \log\left(\frac{1-\frac{i}{n}}{d_{m-l+1}}\right) + b_n\right)\right\} \end{aligned} \tag{4.19}$$

define for $2 \leqslant l \leqslant m$ an asymptotically optimal sequence of m-choice stopping times.

As in Section 3 for one-choice problem the proof of optimality of the stopping times $T_1^m(t,x),\dots,T_m^m(t,x)$ in Theorem 4.2 can be modified to hold true for Poisson processes N with finite intensity measure μ. As result we obtain

Theorem 4.5 *Let the Poisson process N on $[0,1]\times(c,\infty]$ fulfill the finite intensity condition I_f.*

a) The optimal m-choice stopping times for N are given by $(T_1^m,\dots,T_m^m) := (T_1^m(0,c),\dots,T_m^m(0,c))$ with

$$T_1^m(t,x) = \inf\{\tau_k > t \mid Y_k = x \vee \sup_{\tau_j\in(t,\tau_k]} Y_j, Y_k > v^m(\tau_k)\},$$
$$T_l^m(t,x) \ = \inf\{\tau_k > T_{l-1}^m(t,x) \mid Y_k = x \vee \sup_{\tau_j\in(t,\tau_k]} Y_j, Y_k > v^{m-l+1}(\tau_k)\},$$

for $2 \leqslant l \leqslant m$. The corresponding stopping thresholds v^m are determined as solutions of

$$\begin{aligned} g_\infty^m(t,v^m(t)) &= h_\infty^m(t,v^m(t)) && \textit{for } t\in[0,t_0^m),\\ v^m(t) &= c && \textit{for } t\in[t_0^m,1], \end{aligned} \tag{4.20}$$

where $t_0^m := \sup\{t\in[0,1] \mid g_\infty^m(t,c) > h_\infty^m(t,c)\}$.

The functions g_∞^m, h_∞^m are given by

$$g_\infty^m(t,x) := P(Y_{T_1^m(t,x)} \vee\cdots\vee Y_{T_m^m(t,x)} = x \vee \sup_{t<\tau_k\leqslant 1} Y_k)$$

and

$$\begin{aligned} h_\infty^1(t,x) &:= e^{-\mu((t,1]\times(x,\infty])} && \textit{for } m=1,\\ h_\infty^m(t,x) &:= g_\infty^{m-1}(t,x) + h_\infty^1(t,x) && \textit{for } m\geqslant 2. \end{aligned}$$

$v^m : [0,1]\to[c,\infty]$ is monotonically nonincreasing and can be chosen right continuous. Furthermore,

$$v^m(t) \leqslant v^{m-1}(t) \textit{ for } t\in[0,1], m\geqslant 2.$$

b) The optimal m-choice probability is given by

$$s_m := P(Y_{T_1^m}\vee\cdots\vee Y_{T_m^m} = \sup_k Y_k, T_1^m < 1) = \lim_{x\downarrow c} g_\infty^m(t,x).$$

Theorem 4.5 allows to generalize the optimal (one-)choice results in Gnedin and Sakaguchi (1992) resp. Example 3.1 for iid sequences with distribution function F arriving at Poisson distributed time points to the m-choice case. Let the intensity measure μ of a Poisson process N have Lebesgue density

$$g(t,y) = a(t)F'(y) \tag{4.21}$$

with continuous and integrable function $a:[0,1]\to[0,\infty]$ and continuous distribution function F. Define $A(t) := \int_t^1 a(s)ds$ and let the constants d_m and the functions H^m be defined as in (4.13). The point t_0^m is defined as for $m=1$ in Example 3.1.

Theorem 4.6 *Let N be a Poisson process with finite intensity* (4.21)*. Then we have*

a) If $A(0) \leqslant d_m$, then $t_0^m = 0$ and $v^m \equiv c$.

b) If $A(0) > d_m$, then t_0^m is the minimal solution of $A(t_0^m) = d_m$. For $t \in [0, t_0^m) v^m(t)$ is a solution of

$$F(v^m(t)) = 1 - \frac{d_m}{A(t)}. \tag{4.22}$$

c) $h_\infty^m(t,x) = H^m(A(t)(1 - F(x)))$ for $x > -\infty$ and the optimal m-choice probability is given by

$$s_m = H^{m+1}(A(0)) - e^{-A(0)}. \tag{4.23}$$

Proof: The proof is similar to that of Theorem 4.4 replacing the decreasing function F there by the decreasing function $1 - F$ here. □

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