KARHUNEN CLASS PROCESSES FORMING A BASIS

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Abstract

Necessary and sufficient conditions for a Karhunen class sequence to form an unconditional basis and a minimal basis are given.

Let $\{X_n\}_{n=-\infty}^{+\infty}$ be a centered weakly stationary sequence with a covariance function $R(\cdot)$. There is the interesting question whether each element ξ in $\mathcal{L}_2\left(\{X_n\}_{n=-\infty}^{+\infty}\right)$ can be expressed in the form

$$\xi = \sum_{n=-\infty}^{+\infty} \alpha_n X_n \tag{1}$$

where the summation is understood in the quadratic mean sense. This expression is a special form of a stochastic integral when a random measure $Z(\cdot,\cdot)$ is defined on $\boldsymbol{Z} \times \Omega$ ($\boldsymbol{Z}$ – integers). The question of the possibility to express each element ξ in $\mathcal{L}_2(\{X_n\})$ in the form (1) is very important for predicting next values of $\{X_n\}_{n=-\infty}^{+\infty}$. Knowing that (1) holds we can find the best predictors in the set of $\left\{\sum_{n\leq t} \alpha_n X_n\right\}$ only. This fact is very serious from both theoretical and practical reasons. Rozanov (1960) proved that $\{X_n\}_{n=-\infty}^{+\infty}$ is an unconditional basis for the range space $\mathcal{L}_2(\{X_n\})$, i.e. $\{\alpha_n\}_{n=-\infty}^{\infty}$ in the expression (4) are defined unambiguously, if and only if $\{X_n\}_{n=-\infty}^{+\infty}$ possesses a spectral density function $\varphi(\cdot)$ on $\langle -\pi, \pi\rangle$ satisfying

$$0 < c \leq \varphi(\lambda) \leq C < \infty$$

a.s. [Leb.]. We next generalize this result to the Karhunen class. Let $R(\cdot,\cdot)$ be a covariance function on $\boldsymbol{Z} \times \boldsymbol{Z}$ expressible in the form

$$R(n,m) = \int_S f_n(\lambda)\, \overline{f_m(\lambda)}\, \mu(d\lambda) \tag{2}$$

where μ is a measure such that there exists the Radon–Nikodym derivative $h(\cdot) = \frac{d\mu}{d\nu}$ and where $\nu(\cdot)$ is fixed a σ-finite measure, $\{f_n\}$ is a fixed orthonormal system in $\mathcal{L}_2(\nu)$, i.e.

$$\int_S f_n(\lambda)\, \overline{f_m(\lambda)}\, \nu(d\lambda) = \delta_{nm}.$$

In the case that $f_n(\lambda) = e^{in\lambda}$, ν the Lebesgue measure on $[-\pi,\pi]$, in this way we obtain the weakly stationary sequences with existing spectral density functions. If $\{f_n\}$ is a complete system in $\mathcal{L}_2(\nu)$ then

$$\mathcal{L}_2(\{X_n\}) \simeq \mathcal{L}_2(\mu, f_n) = \mathcal{L}_2(\mu),$$

where $\simeq$ means isometry. Now, we shall define random variables $Y_n = \int \frac{f_n(\lambda)}{h(\lambda)}\, Z(d\lambda)$ where $Z(\cdot,\cdot)$ is an orthogonal random measure generating the sequence $\{X_n\}_{n=-\infty}^{+\infty}$, i.e.

$$X_n = \int_S f_n(\lambda)\, Z(d\lambda) \quad \text{[a.e.]}.$$

Y_n will be defined when $E\{|Y_n|^2\} =$
$= \int_S \frac{|f_n|^2}{h^2}\, d\mu = \int_S \frac{|f_n|^2}{h}\, d\nu < \infty$ and, further

$$E\left\{X_n \overline{Y_m}\right\} = \int_S f_n(\lambda)\, \frac{\overline{f_m(\lambda)}}{h(\lambda)}\, \mu(d\lambda) =$$
$$= \int_S f_n \overline{f_m}\, d\nu = \delta_{nm}.$$

This fact proves that the random variables $\{Y_n, X_n\}_{n=-\infty}^{+\infty}$ form a biorthonormal system for $\mathcal{L}_2(\{X_n\})$ under the assumption $\int_S \frac{|f_n|^2}{h}\, d\nu < \infty$ for every $n \in \boldsymbol{Z}$. The question of representability of random variables $\xi \in \mathcal{L}_2\{\{X_n\}_{n=-\infty}^{+\infty}\}$ in the form (1) is the question of when the system of variables $\{X_n\}_{n=-\infty}^{+\infty}$ forms a basis in the Hilbert space $\mathcal{L}_2\{\{X_n\}\}$. In order to investigate this question it is natural to be limited to the case when the system $\{X_n\}_{n=-\infty}^{+\infty}$

is minimal. The minimality of a weakly stationary sequence $\{X_n\}_{n=-\infty}^{+\infty}$ was introduced by Kolmogorov (1941) who proved a necessary and sufficient condition for minimality, which is the existence of the following integral

$$\int_{-\pi}^{\pi} \frac{1}{\varphi(\lambda)} d\lambda,$$

where $\varphi(\cdot)$ is the spectral function of $\{X_n\}_{n=-\infty}^{+\infty}$. As is known the system $\{X_n\}_{n=-\infty}^{+\infty}$ is minimal if and only if there exists in a Hilbert space a conjugate system of variables $\{Y_n\}_{n=-\infty}^{+\infty}$ forming a biorthogonal system in $\mathcal{L}_2\{\{X_n\}\}$. If this conjugate system $\{Y_n\}$ is complete, then every variable $\xi \in \mathcal{L}_2\{X_n\}$ is uniquely determined by (1). The existence of a biorthogonal system for $\mathcal{L}_2(\{X_n\})$ under the assumption $\int_s \frac{|f_n|^2}{h} d\nu < \infty$ for every $n \in Z$ proves the minimality of $\{X_n\}_{n=-\infty}^{+\infty}$, i.e. for each $n \in Z$

$$X_n \notin \mathcal{L}_2\{X_m;\, m \in Z,\, m \neq n\}.$$

From a general result in Hilbert spaces a minimal system $\{X_n\}_{n=-\infty}^{+\infty}$ is an unconditional basis in $\mathcal{L}_2(\{X_n\})$ if and only if the system $\{X_n\}_{n=-\infty}^{+\infty}$ is Besselian and Hilbertian, see [3].

Definition 1 *The minimal system $\{X_n\}_{n=-\infty}^{+\infty}$ is Besselian in $\mathcal{L}_2(\{X_n\})$ if for each $h \in \mathcal{L}_2(\{X_n\})$*

$$\sum_{n=-\infty}^{\infty} \left|E\{h\overline{Y_n}\}\right|^2 < \infty$$

and is Hilbertian if for any sequence $\{c_n\}_{n=-\infty}^{+\infty}$ with

$$\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty$$

there exists $h \in \mathcal{L}_2(\{X_n\})$ such that $E\{h\overline{Y_n}\} = c_n$.

As is well known, cf. Rozanov (1960), the system $\{X_n\}_{n=-\infty}^{+\infty}$ is Besselian if and only if there exists a constant C such that

$$\sum_{n=-\infty}^{\infty} \left|E\{h\overline{Y_n}\}\right|^2 \leq C\, E\{|h|^2\}.$$

If we put $h = \int_S \varphi(\lambda)\, Z(d\lambda)$, $\varphi \in \mathcal{L}_2(\mu)$, then this is equivalent to

$$\sum_{-\infty}^{+\infty} \left|E\{h\overline{Y_n}\}\right|^2 = \sum_{-\infty}^{+\infty} \left|\int \varphi(\lambda)\,\overline{f_n(\lambda)}\,\nu(d\lambda)\right|^2$$
$$= \int_S |\varphi(\lambda)|^2\, \nu(d\lambda) \leq C \int_S |\varphi(\lambda)|^2\, h(\lambda)\,\nu(d\lambda).$$

This inequality holds for each $\varphi \in \mathcal{L}_2(\mu)$. Using this fact we obtain the equivalent condition

$$h(\cdot) \geq \frac{1}{C} \qquad \text{a.e. } [\nu].$$

On the other hand, the system $\{X_n\}_{n=-\infty}^{+\infty}$ is Hilbertian if and only if there is a constant $c > 0$ such that

$$\sum_{-\infty}^{+\infty} \left|E\{g\overline{Y_n}\}\right|^2 \geq c\, E\{|g|^2\}$$

for each $g \in \mathcal{L}_2(\{X_n\})$. With $g = \int_S \varphi\, dZ$, $\varphi \in \mathcal{L}_2(\mu)$ this gives similarly as above

$$h(\lambda) \leq \frac{1}{c} \qquad \text{a.e. } [\nu].$$

Based on these considerations we obtain the following theorem.

Theorem 1. *Let $\{X_n\}$ be a random sequence with $R(\cdot,\cdot)$ on $Z \times Z$ as a covariance function belonging to the Karhunen class, i.e.*

$$R(n,m) = \int_S f_n(\lambda)\,\overline{f_m(\lambda)}\,\mu(d\lambda), \qquad n, m \in Z$$

where ν is a fixed σ-finite measure, $\{f_n\}$ is a fixed complete orthonormal systems in $\mathcal{L}_2(\nu)$ and $d\mu = h\,d\nu$. Then

1. *$\{X_n\}_{n=-\infty}^{+\infty}$ is minimal iff*

$$\int_S \frac{|f_n|^2}{h} d\nu < \infty \quad \text{for every } n \in Z$$

2. *A minimal system $\{X_n\}_{n=-\infty}^{+\infty}$ is an unconditional basis of $\mathcal{L}_2(\{X_n\})$ iff there exist $0 < c \leq C < \infty$ such that*

$$\frac{1}{C} \leq h(\cdot) \leq \frac{1}{c} \qquad \text{a.e. } [\nu].$$

An interesting open problem seems to be to find a characterization of the class of processes with representation as in (2) analogously to the characterization of weakly stationary processes by the Herglotz lemma.

In the case of $S = (-\pi, \pi)$, ν = Lebesgue measure, $f_n(\lambda) = \frac{1}{\sqrt{2\pi}} e^{in\lambda}$, for $n \in Z$, μ and μ is a finite measure on S, then all the covariance functions of the form

$$R(n,m) = \int_{-\pi}^{\pi} f_n(\lambda)\,\overline{f_m(\lambda)}\,\mu(d\lambda)$$

are stationary ones and vice versa. In general, if $(S, \mathcal{S}, \nu)$ is a measure space with an orthonormal system of functions $\{f_n\}_{n=-\infty}^{+\infty}$, i.e.

$$\int_S f_n(\lambda)\,\overline{f_m(\lambda)}\,\nu(d\lambda) = \delta_{nm}$$

and μ is a measure on $(S, \mathcal{S})$ such that

$$\int_S |f_n(\lambda)|^2 \mu(d\lambda) < \infty$$

for each $n \in \mathbb{Z}$, then the question is how to describe the class of all covariance functions of the form

$$R(n,m) = \int_S f_n(\lambda) \overline{f_m(\lambda)} \mu(d\lambda)?$$

References

[1] Rozanov Yu. A. (1960), On stationary sequences forming a basis. Soviet. Math. Doklady 1 (1960), 155–158.

[2] Kolmogorov A. N. (1941), Stationary sequences in a Hilbert space. Bull. MGV, Matematika, Tom 2, No 6, 1–40 (in Russian).

[3] Gelfand I. M. (1951), Moskov. Gos. Univ. Uč. Zap. 148, Mat. 4, 224–237.