HYBRID LÉVY MODELS: DESIGN AND COMPUTATIONAL ASPECTS

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ABSTRACT. A hybrid model is a model, where two markets are studied jointly such that stochastic dependence can be taken into account. Such a dependence is well known for equity and interest rate markets on which we focus here. Other pairs can be considered in a similar way. Two different versions of a hybrid approach are developed. Independent time-inhomogeneous Lévy processes are used as the drivers of the dynamics of interest rates and equity. In both versions the dynamics of the interest rate side is described by an equation for the instantaneous forward rate. Dependence between the markets is generated by introducing the driver of the interest rate market as an additional term into the dynamics of equity in the first version. The second version starts with the equity dynamics and uses a corresponding construction for the interest rate side. Dependence can be quantified in both cases by a single parameter. Numerically efficient valuation formulas for interest rate and equity derivatives are developed. Using market quotes for liquidly traded assets we show that the hybrid approach can be successfully calibrated.

1. Introduction

It is a well known fact that equity and interest rate markets mutually influence each other, as extensively shown in many empirical studies such as Wainscott [25] and d'Addona and Kind [5], just to mention a few. Consequently both markets should be modelled jointly as soon as interest rates are assumed to be stochastic. Such a joint model is said to be hybrid when it allows to take stochastic dependence between the two markets into account. In this paper we focus on equity and fixed income markets; other combinations such as those which include foreign exchange, can be treated along the same lines.

The financial as well as the insurance industry have developed a great variety of hybrid derivatives, structured contracts as well as diversification products, whose payoffs depend on multiple underliers. They enable the investor to aim at a return which is greater than the one of the least risky asset and at the same time they allow to reduce the risk compared to an investment in a single risky asset. A performance basket which will be discussed in section 4 is a simple example. One can easily adapt this instrument to the risk aversion or the risk appetite of an individual investor.

One possibility to implement a hybrid model is to correlate the driving processes for both markets via an appropriate covariance matrix. This approach is natural in the framework of diffusions, in which the dependence

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structure is completely determined by covariances. Among the papers which follow this line of thought we mention Grzelak, Oosterlee and Van Weeren [16] and Grzelak, Oosterlee and Van Weeren [17] who investigate both, the combination of the Heston [18] and the Schöbel-Zhu [24] stochastic volatility models with the Hull-White [20] model for the short rate. Ahlip and Rutkowski [1] combine the Heston stochastic volatility model with CIR short rates for the pricing of forward start call options. In another paper Ahlip and Rutkowski [2] develop a pricing formula for foreign exchange options under the same combination of a stochastic volatility model with CIR interest rates. In this model the instantaneous volatility is correlated with the dynamics of the exchange rate, whereas the domestic and foreign short rates are assumed to be independent of the dynamics of the exchange rate. Concerning the difficulties which arise when correlating Lévy processes directly we refer to Eberlein and Madan [12] and the references given there.

Hybrid modelling of equity and interest rates is particularly important when managing long-dated contracts. As long maturities are an intrinsic feature of many insurance contracts, hybrid models for the joint movements of equity and fixed income markets are of crucial importance, but at the same time represent a significant challenge for the valuation and the risk management of contracts in the insurance industry. In addition to the long maturity aspect of such contracts, reference funds of variable annuity contracts – such as for example participating policies in the UK market – are typically composed of equity and bonds with percentages depending on the risk aversion of either the policyholder or the portfolio manager. Consequently this category of contracts requires de facto a tractable, market consistent hybrid model. A description of a typical participating policy contract can be found in Ballotta [3] and Ballotta [4] in which the impact of non-Gaussian dynamics on its fair valuation is analysed. In those studies though the interest rate is chosen as a constant.

In the present paper we develop two different versions of a hybrid model which is driven by independent, time-inhomogeneous Lévy processes, for which the specification of covariance matrices aimed at generating dependence is not necessary.

The first model uses for the bond market the Lévy forward rate approach of Eberlein and Raible [14], Eberlein, Jacod and Raible [10] and Eberlein and Kluge [11]. The equity dynamics is given by another exponential model. Dependence is generated by introducing the process which drives the forward rate as an additional term in the equity dynamics. A similar idea has already been used in Eberlein, Madan, Pistorius and Yor [13], where a gamma driven Ornstein-Uhlenbeck short rate model is combined with a variance gamma driven model for equity. We call this first approach the hybrid Lévy equity model.

The second approach starts with the specification of the equity dynamics as an exponential model with a time-inhomogeneous Lévy driver. For the interest rate dynamics we specify again the instantaneous forward rate in which we add now the driver of the equity part as an additional term. Differently from the first approach, in this specification interest rates and equity are mutually dependent. In the first setup, in fact, the interest rate model is not affected by the equity dynamics. We call the second approach the hybrid Lévy forward rate model.

The paper is structured as follows. In section 2 we describe the general setup for the use of time-inhomogeneous Lévy processes. In section 3 the first version of the hybrid model is introduced. The model allows to control the level of dependence via a specific parameter. In section 4 valuation of derivatives is discussed. We consider interest rate and equity derivatives as well as a performance basket. In subsection 4.5 we show how this model can be successfully calibrated to market quotes. Section 5 is devoted to the second version of the hybrid approach. Valuation of derivatives and calibration are investigated for this model as well. A key point is again the proper modelling of the dependence between the two markets. Specifically one can extract market consistent information on the level of dependence which is, for example, also relevant for the calculation of adequate capital requirements. Typically capital requirements are computed on the basis of risk measures strictly connected to dependence in the tails of the joint distribution.

2. The general setup

The interest rate and equity dynamics will be driven by two independent time–inhomogeneous Lévy processes $L^1 = (L^1_t)_{t\in[0,T^*]}$ and $L^2 = (L^2_t)_{t\in[0,T^*]}$, also called processes with independent increments and absolutely continuous characteristics (PIIAC) by Jacod and Shiryaev [21]. These processes are defined on a stochastic basis $\mathcal{B} := (\Omega, \mathcal{F}, \mathbb{F}, P)$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T^*]}$ which satisfies the usual conditions (see [21], Definition I 1.2, 1.3). For $i = 1, 2$ the distribution of L^i_t is determined by its characteristic function

$$E[e^{\mathrm{i}uL^i_t}] = \exp\Big(\int_0^t \Big(\mathrm{i}ub^i_s - \frac{1}{2}c^i_s u^2 + \int_{\mathbb{R}} \big(e^{\mathrm{i}ux} - 1 - \mathrm{i}uh(x)\big)\, F^i_s(dx)\Big) ds\Big).$$

The local characteristics $(b^i_s, c^i_s, F^i_s)_{s\in[0,T^*]}$ (relative to a truncation function h) are assumed to satisfy $\int_0^{T^*} (|b^i_s| + c^i_s + \int_{\mathbb{R}}(|x|^2 \wedge 1)F^i_s(dx))ds < \infty$. Later in the context of modelling markets we shall consider exponentials of the form $(\exp(\int_0^t f(s)dL^i_s))_{t\in[0,T^*]}$. For the markets to be free of arbitrage these exponential processes will be forced to become martingales. For this purpose we assume

Assumption ($\mathbb{EM}$)
There exist constants $M_i, \varepsilon_i > 0$ *such that for* $u \in [-(1+\varepsilon_i)M_i, (1+\varepsilon_i)M_i]$

$$\int_0^{T^*} \int_{\{|x|>1\}} \exp(ux)F^i_s(dx)ds < \infty \qquad (i = 1, 2).$$

Under this assumption the moment generating functions $M_{L^i_t}(u) := E[e^{uL^i_t}]$ exist for all $u \in [-(1+\varepsilon_i)M_i, (1+\varepsilon_i)M_i]$. Existence of the exponential moment entails in particular existence of the first moment, the expectation. As a consequence one can use the identity function $h(x) = x$ as truncation function and thus gets – of course with new drift characteristics b_s^i – the canonical representations in the simple form

$$L_t^i = \int_0^t b_s^i ds + \int_0^t \sqrt{c_s^i} dW_s^i + \int_0^t \int_{\mathbb{R}} x(\mu^i - \nu^i)(ds, dx),$$

where $W^i = (W_t^i)_{t \in [0,T^*]}$ $(i = 1, 2)$ are independent standard Brownian motions, μ^i is the random measure of jumps for L^i and ν^i its compensator. This is the representation which we shall use henceforth since from now on we shall always assume that $(\mathbb{EM})$ is in force.

The cumulant function θ_s^i of L^i is then defined for any $z \in \mathbb{C}$ such that $\Re z \in [-(1+\varepsilon_i)M_i, (1+\varepsilon_i)M_i]$ and has the form

$$\theta_s^i(z) = b_s^i z + \frac{1}{2} c_s^i z^2 + \int_{\mathbb{R}} (e^{zx} - 1 - zx) F_s^i(dx).$$

For these arguments z we have $E[|\exp(zL_t^i)|] < \infty$ and the moment generating function of L_t^i has the form $E[\exp(zL_t^i)] = \exp\left(\int_0^t \theta_s^i(z) ds\right)$. By replacing z by the argument $\mathrm{i}u$ one gets the characteristic function in its Lévy-Khintchine form. For the theory which will be developed later we need the following result which is proved in [14] and generalized in [11]. Let $f : \mathbb{R}^+ \to \mathbb{C}$ be a continuous function with $|\Re f| \leq M^i$, then

$$E\Big[\exp\Big(\int_t^T f(s) dL_s^i\Big)\Big] = \exp\Big(\int_t^T \theta_s^i(f(s)) ds\Big), \tag{1}$$

where the integrals are defined component–wise for the real and the imaginary part.

Now let us briefly recap the Fourier approach in derivative pricing which leads to numerically very efficient valuation formulas. Speed of the numerical algorithms is crucial once one wants to calibrate a model to price or volatility quotes from the exchanges. For details see [9]. The price at time 0 of an option with maturity $t \in [0, T^*]$ is typically given by an expectation $E_Q[f(H_t - s)]$, where Q is a risk-neutral probability, f represents the payoff function, $H = (H_t)_{t \in [0,T^*]}$ is a suitably chosen process and s a log-transform of the starting value of the underlying. Since payoff functions are often unbounded we dampen them first with an exponential function and define $g(x) := e^{-Rx} f(x)$ for some $R \in \mathbb{R}$. Three integrability properties (C1)–(C3) are needed to get the Fourier formula. The expectation above can then be written as a Fourier integral

$$E_Q[f(H_t - s)] = \frac{1}{\pi} e^{-Rs} \int_0^\infty \Re\left(e^{\mathrm{i}us} \widehat{f}(u + \mathrm{i}R) M^{H_t}(R - \mathrm{i}u)\right) du. \tag{2}$$

3. **The hybrid Lévy equity model**

Two independent time-inhomogeneous Lévy processes L^1, L^2 as introduced in the previous section will be used to drive the interest rate and equity dynamics. The key quantities will be designed as exponentials of stochastic integrals with respect to these processes. Those exponentials will be properly compensated in order to become martingales. For this purpose ($\mathbb{EM}$) will always be assumed.

3.1. **The bond market.** For each maturity $T \in [0, T^*]$ and $t \in [0, T]$ we assume that the dynamics of the instantaneous forward rates is given by (see Eberlein et al. [10])

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds - \int_0^t \sigma_1(s,T)dL_s^1. \tag{3}$$

Here $f(0,T)$ is a deterministic, bounded, and in T measurable function which represents the currently observed rates. The drift coefficient α and the volatility σ_1 are assumed to satisfy the usual measurability and boundedness conditions (see [10], (2.5)). In order to keep the numerical algorithms as simple as possible, we will later consider only deterministic functions σ_1. The drift coefficient α will in the risk-neutral setting become a very specific one, namely a function of σ_1 and the cumulant function of the driving process L^1. Recall that the price of a default-free zero coupon bond is related to the instantaneous forward rate through $B(t,T) = \exp\Big(-\int_t^T f(t,s)ds\Big)$ and therefore – by using the version of Fubini's theorem for stochastic integrals – the dynamics for $f(t,T)$ translates into an equation for the bond prices

$$B(t,T) = B(0,T)\exp\Big(\int_0^t (r(s) - A(s,T))ds + \int_0^t \Sigma(s,T)dL_s^1\Big), \tag{4}$$

where $A(s,T) := \int_{s\wedge T}^T \alpha(s,u)du$ and $\Sigma(s,T) := \int_{s\wedge T}^T \sigma_1(s,u)du$. $r(t) := f(t,t)$ denotes as usual the short rate which is implicitly given by the forward rate dynamics. We will always assume that $\Sigma(s,T) \le M_1$, where M_1 is the constant from assumption ($\mathbb{EM}$). This guarantees that the exponential of the stochastic integral has finite expectation. A standard choice for σ_1 is the Vasiček structure

$$\sigma_1(s,T) := \begin{cases} \widehat{\sigma}\exp(-a(T-s)), & s \le T \\ 0, & s > T \end{cases} \tag{5}$$

with two parameters $a \neq 0$ and $\widehat{\sigma} > 0$, which entails

$$\Sigma(s,T) = \begin{cases} \frac{\widehat{\sigma}}{a}(1 - \exp(-a(T-s))), & s \le T \\ 0, & s > T. \end{cases} \tag{6}$$

Often $\widehat{\sigma}$ is chosen to be $|a|$. This helps to keep the dimension of the parameter space as low as possible in calibrating the model to data.

For the money market account $B(t) := \exp\Big(\int_0^t r(s)ds\Big)$ one gets the explicit equation $B(t) = B(0,t)^{-1}\exp(\int_0^t A(s,t)ds - \int_0^t \Sigma(s,t)dL_s^1)$.

In order to generate a bond market which is free of arbitrage, one has to make sure that discounted bond prices $B(t)^{-1}B(t,T)$ are martingales. This means that $\int_0^t A(s,T)ds$ has to be chosen as the exponential compensator of $\int_0^t \Sigma(s,T)dL_s^1$. One can easily achieve this goal since for deterministic volatility functions σ_1 the stochastic integral with respect to L^1 has still independent increments. Therefore its exponential divided by the expectation of this exponential is a martingale. The latter expectation can be obtained from (1). The bond market is therefore free of arbitrage if we choose

$$A(s,T) = \theta_s^1(\Sigma(s,T)) \qquad (s \in [0,T]). \tag{7}$$

This is the drift condition which will be assumed henceforth.

3.2. **The equity market.** The dynamics of the stock price will be defined in the following way

$$\ln S(t) = \ln S(0) + \int_0^t r(s)ds + \int_0^t \sigma_2(s)dL_s^2 + \int_0^t \beta(s)dL_s^1 - \omega_h(t). \tag{8}$$

In this hybrid approach the classical interest rate term $\int_0^t r(s)ds$ affects the stock price via the short rate in an endogenous way, whereas the additional term $\int_0^t \beta(s)dL_s^1$ allows for an exogenous influence of the fixed income market on the equity market. σ_2 is positive and denotes the volatility of the stock price. Both, σ_2 and β, could be chosen as random processes, but again having numerical aspects in mind we will only consider deterministic functions σ_2 and β in the following. They are assumed to satisfy $\sigma_2(s) \leq M_2$, $|\beta(s)| \leq M_1$, where M_1, M_2 are the constants from assumption ($\mathbb{EM}$). The drift term $\omega_h(t)$ is chosen such that the discounted stock price $(B(t)^{-1}S(t))_{t\in[0,T^*]}$ becomes a martingale. Following the same argument as above and exploiting the independence of the driving processes it is clear that $\omega_h(t) = \int_0^t [\theta_s^2(\sigma_2(s)) + \theta_s^1(\beta(s))]ds$ is the right choice to guarantee martingality.

Let us analyse the interaction between the two market segments further. From (8) we derive that

$$S(t) = d(t)\exp(H_t) = d(t)\exp(H_t^2 + H_t^1), \tag{9}$$

where $d(t) := S(0)B(0,t)^{-1}\exp(\int_0^t [\theta_s^1(\Sigma(s,t)) - \theta_s^1(\beta(s)) - \theta_s^2(\sigma_2(s))]ds)$ collects all deterministic terms and $H_t := H_t^1 + H_t^2$ is given by $H_t^1 := \int_0^t (\beta(s) - \Sigma(s,t))dL_s^1$, $H_t^2 := \int_0^t \sigma_2(s)dL_s^2$. According to (3) and (4) one can write

$$B(t,T) = D(t,T)\exp(X_t(T)), \tag{10}$$

where $D(t,T) := B(0,T)B(0,t)^{-1}\exp(\int_0^t (\theta_s^1(\Sigma(s,t)) - \theta_s^1(\Sigma(s,T)))ds)$ and $X_t(T) := \int_0^t (\Sigma(s,T) - \Sigma(s,t))dL_s^1$. Combining (9) and (10) one gets

$$S(t) = d(t,T)\exp(\mathcal{H}_t(T))B(t,T) \tag{11}$$

for $d(t,T) = d(t)D(t,T)^{-1}$ and $\mathcal{H}_t(T) = H_t^1 + H_t^2 - X_t(T)$. $\mathcal{H}_t(T)$ can be considered as the transition process of the hybrid model.

By choosing specific driving processes L^i and appropriate values for β one can see that significant levels of correlation between $S(t)$ and $B(t,T)$ can be achieved (see [23]).

4. **Valuation of derivatives**

Valuation formulas for derivatives simplify considerably when for a given maturity T expectations are taken with respect to the forward martingale measure P_T (see [15]) which lives on $(\Omega, \mathcal{F}_T)$. Its Radon–Nikodym density is defined by $\frac{dP_T}{dP} = (B(0,T)B(T))^{-1}$. By conditioning it on $\mathcal{F}_t$ for each $0 \leq t \leq T$ we get in the context of the approach above the corresponding density process in the explicit form

$$Z_t(T) = \exp\Big(\int_0^t \Sigma(s,T)dL_s^1 - \int_0^t \theta_s^1(\Sigma(s,T))ds\Big). \tag{12}$$

Following the exposition in [7] we shall develop numerically efficient Fourier based valuation formulas for the hybrid model. We start with interest rate derivatives. Write the price of a zero coupon bond as given in (10) in the form

$$B(t,T) = \exp(X_t(T) - s_1(t,T)), \tag{13}$$

where $s_1(t,T) := -\ln D(t,T)$. Assume that the payoff of an option with maturity $t \in [0,T]$ on a zero coupon bond with maturity $T \in [0,T^*]$ is given as a function $f(X_t(T) - s_1(t,T))$. Then the time-0 price of the option is $V_0(t,T) = B(0,t)E_{P_t}[f(X_t(T) - s_1(t,T))]$. If the integrability assumptions (C1)–(C3) hold true (see [7]) the last expression can be written as an integral

$$V_0(t,T) = \frac{1}{\pi}B(0,t)\int_0^\infty \Re\left(D(t,T)^{R-iu}\widetilde{M}^{X_t(T)}(R-iu)\widehat{f}(u+iR)\right)du.$$

Here $\widetilde{M}^{X_t(T)}$ denotes the moment generating function of $X_t(T)$ with respect to P_t, $\widehat{f}$ is the Fourier transform of f and $\Re$ is the real part of the complex integrand.

In order to calibrate this model to market data we will need explicit pricing formulas for the most liquid market instruments which are caps and floors for the fixed income market and options on individual shares or indices for the equity market.

4.1. **Caps and floors.**

Let $0 \leq T_0 < T_1 < \ldots < T_{n-1} < T_n = T$ be the tenor structure along which the payments of the cap or floor are made. We assume that the tenor length $\delta = \delta_k := T_k - T_{k-1}$ is equidistant. For a cap at each time point T_k for $k = 1, \ldots, n$ the payoff is $N\delta(L(T_{k-1}, T_{k-1}) - K)^+$, where N is the notional amount of the contract, K the strike rate and $L(T_{k-1}, T_{k-1})$ denotes the Euribor or Libor which prevails at time T_{k-1} for the tenor period $[T_{k-1}, T_k)$. In the case of floors the corresponding payoffs are given by $N\delta(K - L(T_{k-1}, T_{k-1}))^+$. We consider here only the setup, where rates are tenor independent. For an advanced model approach which allows to take tenor dependence into account see [8]. In the classical setting the Euribor or Libor is related to the zero coupon price via

$$L(T_{k-1}, T_{k-1}) = \frac{1}{\delta}\Big(\frac{1}{B(T_{k-1}, T_k)} - 1\Big). \tag{14}$$

Let us note that the payoff of a cap at time point T_k is up to a scaling factor nothing but the payoff of a call option on the reference rate $L(t, T_{k-1})$ – the Libor rate for the period $[T_{k-1}, T_k)$ observed at time $t \leq T_{k-1}$ – with strike K and maturity T_{k-1}. Each of these options is called a caplet. In the case of a floor each single payment corresponds to a put option on the reference rate and is called a floorlet. As a consequence of (14) the payoff of the T_{k-1}-caplet discounted to the time point T_{k-1} can be written as that of a put option on the price of the bond with maturity T_k, notional $\widetilde{N} := N(1 + \delta K)$ and strike $\widetilde{K} := (1 + \delta K)^{-1}$. In the same way each floorlet can be interpreted as a call. It is this representation which will be used for valuation. We denote by $\mathrm{Cap}(K, T; N)$ the price of the cap which is given as the sum of the caplet prices

$$\mathrm{Cap}(K, T; N) := \widetilde{N} \sum_{k=1}^{n} \mathrm{Cpl}(K, T_{k-1}, T_k).$$

The time-0 price of each caplet is

$$\mathrm{Cpl}(K, T_{k-1}, T_k) = B(0, T_{k-1}) E_{P_{T_{k-1}}}[(\widetilde{K} - B(T_{k-1}, T_k))^+].$$

The corresponding formulas for the floor are

$$\mathrm{Floor}(K, T; N) := \widetilde{N} \sum_{k=1}^{n} \mathrm{Flt}(K, T_{k-1}, T_k)$$

with

$$\mathrm{Flt}(K, T_{k-1}, T_k) = B(0, T_{k-1}) E_{P_{T_{k-1}}}[(B(T_{k-1}, T_k) - \widetilde{K})^+].$$

According to (13) the bond price can be written as

$$B(T_{k-1}, T_k) = \exp(X_{T_{k-1}}(T_k) - s_1(T_{k-1}, T_k))$$

and therefore using the short forms $f_C(x; K) = (e^x - K)^+$ and $f_P(x; K) = (K - e^x)^+$, we get

$$\mathrm{Cpl}(K, T_{k-1}, T_k) = B(0, T_{k-1}) E_{P_{T_{k-1}}}[f_P(X_{T_{k-1}}(T_k) - s_1(T_{k-1}, T_k); \widetilde{K})]$$

and

$$\mathrm{Flt}(K, T_{k-1}, T_k) = B(0, T_{k-1}) E_{P_{T_{k-1}}}[f_C(X_{T_{k-1}}(T_k) - s_1(T_{k-1}, T_k); \widetilde{K})].$$

These are the representations which are easily accessible for Fourier methods. If $R \in (-\infty, 0)$ is such that $\widetilde{M}^{X_{T_{k-1}}(T_k)}(R) < \infty$ then

$$\begin{aligned}\mathrm{Cpl}(K, T_{k-1}, T_k) &= \frac{1}{\pi} B(0, T_{k-1}) \\ &\cdot \int_0^\infty \Re\Big(D(T_{k-1}, T_k)^{R-iu} \frac{\widetilde{K}^{1-(R-iu)}}{(R-iu)(R-1-iu)} \widetilde{M}^{X_{T_{k-1}}(T_k)}(R-iu) \Big) du,\end{aligned}$$

and if $R \in (1,\infty)$ is such that $\widetilde{M}^{X_{T_{k-1}}(T_k)}(R) < \infty$ then

$$\mathrm{Flt}(K, T_{k-1}, T_k) = \frac{1}{\pi} B(0, T_{k-1})$$
$$\cdot \int_0^\infty \Re\Big(D(T_{k-1}, T_k)^{R-iu} \frac{\widetilde{K}^{1-(R-iu)}}{(R-iu)(R-1-iu)} \widetilde{M}^{X_{T_{k-1}}(T_k)}(R-iu)\Big) du.$$

The key input of these formulas is $\widetilde{M}^{X_t(T)}$, the moment generating function of $X_t(T)$ under the forward measure P_t. By using the explicit form of the density process in (12), we get

Proposition 4.1. *Suppose the constants M_1, ε in assumption* ($\mathbb{EM}$) *are such that $\Sigma(s,t) \le M'$ for $s,t \in [0,T]$ and a constant $M' < M_1$. Then*

$$\widetilde{M}^{X_t(T)}(R) < \infty \ \ \text{for} \ \ R \in \left[-(M_1 - M')M'^{-1}, 0\right] \cup \left(1, 1 + (M_1 - M')M'^{-1}\right],$$

and for each $z \in \mathbb{C}$ with $\Re z = R$ we get

$$\widetilde{M}^{X_t(T)}(z) = \exp\Big(\int_0^t [\theta^1_s(z\Sigma(s,T) + (1-z)\Sigma(s,t)) - \theta^1_s(\Sigma(s,t))] ds\Big).$$

4.2. **Options on equity.** Let us write the stock price dynamics (9) in the form $S(t) = \exp(H_t - s_2(t))$, where $s_2(t) := -\ln d(t)$. Consider an option on $S(t)$ with maturity $T \in [0, T^*]$ and payoff given in the form $f(H_T - s_2(T))$. The time-0 price of this option is then $V_0(T) = B(0,T) E_{P_T}[f(H_T - s_2(T))]$. If $\widetilde{M}^{H_T}$ denotes the moment generating function of H_T with respect to the forward measure P_T we get - provided the assumptions (C1)–(C3) hold true - the Fourier formula

$$V_0(T) = \frac{1}{\pi} B(0,T) \int_0^\infty \Re\left(d(t)^{R-iu} \widetilde{M}^{H_T}(R-iu) \widehat{f}(u+iR)\right) du. \quad (15)$$

Again the key input is $\widetilde{M}^{H_T}$ which can be obtained according to

Proposition 4.2. *Suppose the constants M_i, ε_i $(i = 1,2)$ in assumption* ($\mathbb{EM}$) *are such that $\sigma_2(s) \le H_p$, $|\beta(s)| \le H_p/2$ and $\Sigma(s,T) \le H_p/2$ for a constant $H_p > 0$ $(p \ge 1)$ with $pH_p < M_1 \wedge M_2$. Then*

$$\widetilde{M}^{H_T}(R) < \infty \ \ \text{for} \ \ R \in \Big[-\frac{M_1 \wedge M_2 - pH_p}{H_p}, 0\Big] \cup \Big(p, p + \frac{M_1 \wedge M_2 - pH_p}{H_p}\Big],$$

and for all $z \in \mathbb{C}$ with $\Re z = R$ we get $\widetilde{M}^{H_T}(z) = M^{H^2_T}(z) \widetilde{M}^{H^1_T}(z)$, where

$$M^{H^2_T}(z) = \exp\Big(\int_0^T \theta^2_s(z\sigma_2(s)) ds\Big) \quad \textit{and}$$

$$\widetilde{M}^{H^1_T}(z) = \exp\Big(\int_0^T [\theta^1_s(z\beta(s) + (1-z)\Sigma(s,T)) - \theta^1_s(\Sigma(s,T))] ds\Big).$$

Proof. Since $H_T = H_T^1 + H_T^2$ one gets exploiting independence of H_T^1 and H_T^2

$$\begin{aligned}\widetilde{M}^{H_T}(z) &= B(0,T)^{-1} E\left[\exp(zH_T^2) B(T)^{-1} \exp(zH_T^1)\right] \\ &= E\left[\exp(zH_T^2)\right] E_{P_T}\left[\exp(zH_T^1)\right].\end{aligned}$$

Using formula (1) we conclude further

$$M^{H_T^2}(z) = E\Big[\exp\Big(\int_0^T z\sigma_2(s) dL_s^2\Big)\Big] = \exp\Big(\int_0^T \theta_s^2(z\sigma_2(s))ds\Big)$$

since for $\Re z \in \left[-(M_1 \wedge M_2 - pH_p)H_p^{-1}, 0\right]$ and $s \in [0, T^*]$ $|\Re(z\sigma_2(s))| \leq |\Re z| H_p < M_2$ and similarly for the other domain of $\Re z$. The second factor is

$$\begin{aligned}\widetilde{M}^{H_T^1}(z) = E\Big[\exp\Big(\int_0^T (z\beta(s) + (1-z)\Sigma(s,T)) dL_s^1\Big)\Big] \\ \cdot \exp\Big(-\int_0^T \theta_s^1(\Sigma(s,T))ds\Big).\end{aligned}$$

From this the result follows since for both ranges of R given in the statement above one can verify (see [23]) that $|\Re(z\beta(s) + (1-z)\Sigma(s,T))| \leq M_1$. ☐

4.3. **Power calls and puts.** The stage is now set to value specific options. We will discuss in detail power calls and power puts. Other types such as for example self-quanto options can be treated along the same lines. For power options the payoff functions are for a given strike $K \geq 0$ amd $p \in \mathbb{N}$

$$f_{C^p}(x; K) := \left[(e^x - K)^+\right]^p \quad \text{und} \quad f_{P^p}(x; K) := \left[(K - e^x)^+\right]^p.$$

This includes for $p = 1$ standard calls and puts with payoffs f_C and f_P as defined earlier. For some analytical purposes it is convenient to consider first the case $K = 1$ and only then to pass to the general case. This is possible since

$$f_{C^p}(x; K) = K^p\Big[\Big(e^{x - \ln K} - 1\Big)^+\Big]^p = K^p f_{C^p}(x - \ln K; 1). \tag{16}$$

Recall that Euler's beta function is defined for all $z_1, z_2 \in \mathbb{C}$ such that $\Re z_1, \Re z_2 > 0$ by

$$B(z_1, z_2) := \int_0^1 t^{z_1 - 1}(1-t)^{z_2 - 1} dt$$

and is related to the gamma function by $B(z_1, z_2) = \Gamma(z_1)\Gamma(z_2)\Gamma(z_1 + z_2)^{-1}$. An integration exercise leads to the following explicit formula

$$\widehat{f}_{C^p}(z; 1) = B(-p - iz, p + 1) = p! \frac{\Gamma(-p - iz)}{\Gamma(1 - iz)}$$

for $z \in \mathbb{C}$ such that $\Im z \in (p, \infty)$. By using (16) one can derive from this result the Fourier transform of f_{C^p} for arbitrary strikes K as follows

$$\widehat{f}_{C^p}(z; K) = K^{p+iz} B(-p - iz, p + 1).$$

The result simplifies for standard call options to

$$\widehat{f}_C(z; K)) = \frac{K^{1+iz}}{iz(1 + iz)}.$$

Inserting the above expression for $\widehat{f}_{C^p}$ into the general formula (15) one gets for $R \in (p, \infty)$ with $\widetilde{M}^{H_T}(R) < \infty$ an integral representation for the time-0 price $C(K, T; p)$ of a power call. That the range for R has to be chosen to be (p, ∞) becomes clear when one verifies (C1) and (C3) for the dampened function $g_{C^p}(x) := e^{-Rx} f_{C^p}(x)$.

Along the same lines one can derive the corresponding expressions for power puts. For z with $\Im z \in (-\infty, 0)$ one gets $\widehat{f}_{P^p}(z; K) = K^{p+iz} B(iz, p+1)$. The same analytical expression as for calls results for standard puts but the domain for z differs. $\Im z \in (1, \infty)$ is replaced by $\Im z \in (-\infty, 0)$.

Proposition 4.2 shows how one can choose $R = R(p)$ as a function of p such that $\widetilde{M}^{H_T}(R) < \infty$. Given processes L^1, L^2, the constants M_1, M_2 from assumption ($\mathbb{EM}$) are known. Now choose H_p as large as possible such that $pH_p < M_1 \wedge M_2$. This defines the range of R for which the statement in Proposition 4.2 holds true. Note that this puts restrictions on the coefficients $\sigma_2(s)$, $\beta(s)$ and $\Sigma(s, T)$.

4.4. **Valuation of hybrid derivatives.** For a hybrid financial product we consider a performance basket. This is a diversification product which takes the performance of both markets, equity and fixed income, into account. We consider a portfolio which consists of a specific stock (or a stock index) with price process $S(t)$ and a bond with price process $B(t, T)$. The payoff at maturity $t \le T$ of the derivative with this portfolio as underlying is given by $\Big(w_1 S(t) S(0)^{-1} + w_2 B(t, T) B(0, T)^{-1}\Big)^+$, where $w_1, w_2 \in \mathbb{R}$ are weights. A positive weight stands for a long position whereas a negative weight means a short position. If both positions are long, the payoff is strictly positive. It is 0 if both positions are short. We shall therefore focus on mixed positions, where one is long and the other is short. The time-0 value of the performance basket is

$$C(t, T; w_1, w_2) := E\Big[B(t)^{-1}\Big(w_1 \frac{S(t)}{S(0)} + w_2 \frac{B(t, T)}{B(0, T)}\Big)^+\Big].$$

This expectation is 0 for negative weights and it is $w_1 + w_2$ for positive weights because of the martingale property of discounted price processes. Therefore we will consider now the cases $w_1 > 0, w_2 < 0$ and $w_1 < 0, w_2 > 0$.

Using $F_S(t, T) = B(t, T)^{-1} S(t)$ for the forward price of S with respect to the bond $B(\cdot, T)$ as numeraire, the pricing formula above can be written as

$$C(t, T; w_1, w_2) = E_{P_t^T}\Big[\Big(w_1 \frac{F_S(t, T)}{F_S(0, T)} + w_2\Big)^+\Big], \tag{17}$$

where P_t^T denotes the T-forward measure on $\mathcal{F}_t$. From (11) we get $F_S(t,T) = d(t,T)\exp(\mathcal{H}_t(T))$ with

$$d(t,T) = F_S(0,T)\exp\Big(\int_0^t [\theta_s^1(\Sigma(s,T)) - \theta_s^1(\beta(s)) - \theta_s^2(\sigma_2(s))]ds\Big).$$

This can be written in the form $F_S(t,T) = \exp(\mathcal{H}_t(T) - s(t,T))$ which is suitable for a Fourier based formula by setting $s(t,T) = -\ln d(t,T)$. To exploit (17), we will therefore need the moment generating function of the transition process $\mathcal{H}_t(T)$ with respect to the forward measure P_t^T. Its derivation can be done in the same way as in Proposition 4.2.

Proposition 4.3. *Suppose the constants M_i, ε_i $(i = 1,2)$ in assumption $(\mathbb{EM})$ are such that $\sigma_2(s) \leq H$, $|\beta(s)| \leq H/2$ and $\Sigma(s,T) \leq H/2$ for a constant $H > 0$ with $H < M_1 \wedge M_2$. Then*

$$M_{P_t^T}^{\mathcal{H}_t(T)}(R) < \infty \quad \textit{for} \quad R \in \Big[-\frac{M_1 \wedge M_2 - H}{H}, 0\Big] \cup \Big(1, 1 + \frac{M_1 \wedge M_2 - H}{H}\Big]$$

and $M_{P_t^T}^{\mathcal{H}_t(T)}(z) = M^{H_t^2}(z) M_{P_t^T}^{H_t^1 - X_t(T)}(z)$ for all z with $\Re z = R$, where

$$M^{H_t^2}(z) = \exp\Big(\int_0^t \theta_s^2(z\sigma_2(s))ds\Big) \quad \textit{and}$$

$$M_{P_t^T}^{H_t^1 - X_t(T)}(z) = \exp\Big(\int_0^t [\theta_s^1(z\beta(s) + (1-z)\Sigma(s,T)) - \theta_s^1(\Sigma(s,T))]ds\Big).$$

For the case $w_1 > 0, w_2 < 0$ we have to consider the valuation of a call with maturity t on the forward price F_S, since (17) reads as

$$C(t,T;w_1,w_2) = \frac{w_1}{F_S(0,T)} E_{P_t^T}[f_C(\mathcal{H}_t(T) - s(t,T);K)]$$

for a strike defined as $K := -w_2 w_1^{-1} F_S(0,T)$. The corresponding Fourier formula which holds for $R \in \big(1, 1 + (M_1 \wedge M_2 - H)H^{-1}\big]$ is

$$C(t,T;w_1,w_2) = \frac{1}{\pi}\frac{w_1}{F_S(0,T)} \cdot \int_0^\infty \Re\Big(d(t,T)^{R-iu}\frac{K^{1-(R-iu)}}{(R-iu)(R-1-iu)} M_{P_t^T}^{\mathcal{H}_t(T)}(R-iu)\Big)du.$$

In the case $w_1 < 0, w_2 > 0$ we consider instead a put option with maturity t on the forward price F_S, since (17) can be written in the form

$$C(t,T;w_1,w_2) = -\frac{w_1}{F_S(0,T)} E_{P_t^T}[(K - F_S(t,T))^+]$$

for the same strike K as above. The corresponding Fourier formula which holds for $R \in \big[-(M_1 \wedge M_2 - H)H^{-1}, 0\big)$ is exactly the same as in the case above, but has a minus in front.

In order to illustrate the influence of the dependence parameter β on the valuation, we chose for Figure 1 two NIG Lévy processes L^1, L^2 (see e.g. [6] for details on NIG processes). The numbers on the x-axis represent the

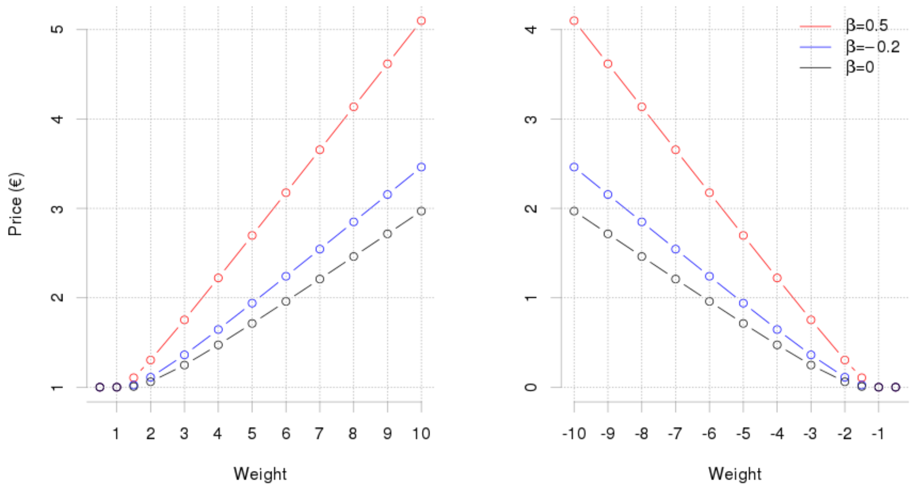

Figure 1 Prices as a function $w_1 \mapsto C(5, 10; w_1, w_2)$ with varying values β. Calls on the left side, puts on the right side.

weight w_1. The second weight is given by $w_2 = 1 - w_1$ for the call and by $w_2 = -1-w_1$ for the put. The lines correspond top down to $\beta = 0.5, -0.2, 0$.

4.5. **Calibration of the hybrid Lévy equity model.** Calibration of the hybrid Lévy equity model is done in two steps which is natural since we started with modelling the interest rate market and only in a second step coupled it to the equity market. The data which is used for the first step are market quotes for caps. One could base the calibration upon quotes for floors as well or upon data from both categories. We shall use Euro cap quotes given by Bloomberg for August 15, 2006. The data is as usual given in terms of flat volatilities as a surface along maturities $T \in \mathcal{T}^1$ and cap rates $K \in \mathcal{K}^1$. On the basis of a consistency requirement flat volatilities can be transformed into implied spot volatilities $v_s = v_s(T, K)$ for $(T, K) \in \mathcal{T}^1 \times \mathcal{K}^1$ (see Hull [19, 26.3]). One can then compute caplet and cap market prices (in Euros) by using the standard lognormal market model.

If $\mathrm{Cpl}^{mdl}(K, T, T + \delta; p_1)$ and $\mathrm{Cpl}^{mkt}(K, T, T + \delta; v_s)$ denote the model respectively market price of a caplet with cap rate $K \in \mathcal{K}^1$, maturity $T \in \mathcal{T}^1$ and payment at time $T+\delta$, we minimize over all admissible parameter vectors p_1 the sum

$$Z^1(p_1) := \sum_{(T,K)\in\mathcal{T}^1\times\mathcal{K}^1} [\mathrm{Cpl}^{mdl}(K, T, T + \delta; p_1) - \mathrm{Cpl}^{mkt}(K, T, T + \delta; v_s)]^2.$$

p_1 consists of the parameters of the driving process L^1 and of those of the volatility structure σ_1.

The resulting parameter vector $\widehat{p}_1$ is used as an input for the second step in which we calibrate the model to data from the equity market. For this purpose we shall use quotes from the same date August 15, 2006 for call options on Deutsche Bank [DBK], Commerzbank [CBK], Allianz [ALV] and Daimler [DAI]. Again the quotes are given as a surface of implied volatilities

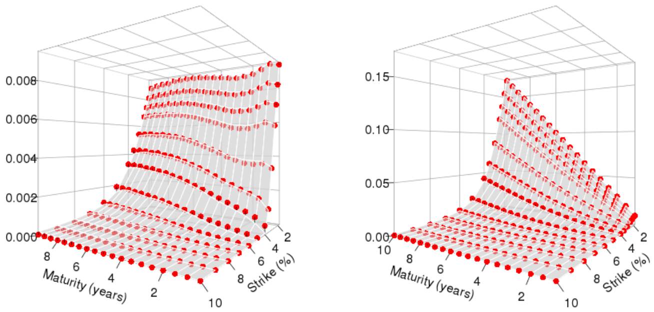

Figure 2 Calibrated NIG model prices (dots) and market prices (surfaces in grey). Left: Caplets. Right: Caps.

$v = v(T, K)$ along maturities given by a set of dates $\mathcal{T}^2$ and strike rates given by a set $\mathcal{K}^2$. Market call prices $\mathrm{C}^{mkt}(K, T; v)$ expressed in Euros are derived from the volatilities by using the Black-Scholes formula (see Hull [19, 13.12]). Note that one has to correct for dividend payments. Finally in order to get a parameter vector $\widehat{p}_2$ we compare model prices $\mathrm{C}^{mdl}(K, T; p_2, \widehat{p}_1)$ to the market prices by minimizing the function

$$Z^2(p_2, \widehat{p}_1) := \sum_{(T,K)\in\mathcal{T}^2\times\mathcal{K}^2} [\mathrm{C}^{mdl}(K, T; p_2, \widehat{p}_1) - \mathrm{C}^{mkt}(K, T; v)]^2.$$

over all admissible parameter vectors p_2. p_2 consists of the parameters of the driving process L^2, plus those of the volatility structure σ_2 and the dependence parameter β. As a consequence of the restrictions which the Lévy parameters and the volatility functions have to satisfy, the minimization procedure is an optimization under a number of constraints.

We used normal inverse Gaussian (NIG) processes for L^1 and L^2 and got in the first step of the calibration the parameter values $\alpha_1 = 4, \beta_1 = -3.8, \delta_1 = 1.34$ for the process L^1. Since the location parameter μ does not influence the valuation, it can be set to be 0. The value for the volatility parameter is $a = 0.0020898$, where we chose the simplified version of (5), where $\widehat{\sigma} = |a|$. The final value for Z^1 was $3.355532 \cdot 10^{-6}$. Figure 2 shows the derived market prices of caplets and caps (in Euros) as a grey surface. The dots mark the calibrated model prices. Caplet prices are shown on the left hand side, cap prices on the right hand side of the figure.

The result of the second step of the calibration is shown in Table 1, where $\alpha_2, \beta_2, \delta_2$ are the NIG Lévy parameters, σ_2 and β represent the volatility and the dependence parameter respectively and Z^2 is the final value achieved in the optimization procedure.

Parameter	DBK	CBK	ALV	DAI
α_2	8.38	5.6	0.97	5.73
β_2	-5.06	-2.8	-0.39	-2.13
δ_2	6.31	15.93	3.77	8.3
σ_2	0.194	0.144	0.1114	0.1818
β	0.0153	0.0284	-0.0153	0.0065
Z^2	0.3537774	3.595755	1.307042	0.07381847

Table 1 Calibrated NIG model parameters and final Z^2-values

5. The hybrid Lévy forward rate model

Let again L^1, L^2 be two independent Lévy processes which satisfy assumption ($\mathbb{EM}$). We define the dynamics of the stock price in the form

$$\ln S(t) = \ln S(0) + \int_0^t r^h(s)ds + \int_0^t \sigma_2(s)dL_s^2 - \omega_2(t)$$

and couple then the interest rate market to the equity market by setting the dynamics of the instantaneous forward rate as

$$f^h(t,T) = f^h(0,T) + \int_0^t \alpha(s,T)ds - \int_0^t \sigma_1(s,T)dL_s^1 + \int_0^t \beta(s,T)dL_s^2.$$

The second equation defines also the short rate $r^h t) := f^h(t,t)$ from which the money market account can be derived. The process $\omega_2(t)$ will be chosen appropriately later. $\beta(s,T)$ determines the exogenous influence of the equity market on the interest rate market. The volatilities $\sigma_2(s)$ and $\sigma_1(s,T)$ as well as the drift coefficient $\alpha(s,T)$ and $\beta(s,T)$ could be chosen as random processes which satisfy the usual measurability and boundedness conditions (see again [10], (2.5)). For the sake of numerical simplicity we will consider only deterministic functions $\sigma_1(\cdot,T) \geq 0, \sigma_2 \geq 0$ and $\beta(\cdot,T)$. The dynamics of $f^h(t,T)$ translates into an equation for zero coupon prices

$$\begin{aligned} B^h(t,T) &= B^h(0,T) \\ &\cdot \exp\Big(\int_0^t (r^h(s) - A(s,T))ds + \int_0^t \Sigma(s,T)dL_s^1 - \int_0^t C(s,T)dL_s^2\Big), \end{aligned} \tag{18}$$

where $C(s,T) := \int_{s\wedge T}^T \beta(s,u)du$. To make sure that $S(t)$ and $B^h(t,T)$ are well defined we assume in the following that $\Sigma(s,T) \leq M_1,\ |C(s,T)| \leq M_2$ and $\sigma_2(s) \leq M_2$. For calibration purposes β will later be chosen as

$$\beta(s,T) = \begin{cases} b\exp(-b(T-s)), & s \leq T \\ 0, & s > T \end{cases} \tag{19}$$

for a constant $b \neq 0$. This means that $C(s,T) = 1 - \exp(-b(T-s))$. Note that this choice will guarantee the Markov property for the short rate r^h since for $0 < T < U$, $\beta(\cdot,U)$ is a scalar multiple of $\beta(\cdot,T)$ (for details see [14], Lemma 4.2).

If we define the vector $v(s,T) := (\Sigma(s,T), -C(s,T))$ and write the driving process as $L = (L^1, L^2)$ then the discounted bond prices $U_t(T) := B^h(t)^{-1}B^h(t,T)$ can be represented in the compact form

$$U_t(T) = B^h(0,T)\exp\Big(\int_0^t v(s,T)dL_s - \int_0^t A(s,T)ds\Big).$$

Following the reasoning that led to (7) we see that under the drift condition

$$A(s,T) = \theta_s^1(\Sigma(s,T)) + \theta_s^2(-C(s,T)) \qquad (s \in [0,T]) \tag{20}$$

the process $(U_t(T))_{t\in[0,T]}$ is a martingale. In the same way the choice

$$\omega_2(t) = \int_0^t \theta_s^2(\sigma_2(s))ds \qquad (t \in [0,T]) \tag{21}$$

guarantees that the discounted stock price is a martingale. The choices (20) and (21) for the drift processes will be assumed henceforth.

Let us analyse the interaction between the equity and the fixed income market as in section 3.2. The money market account can be represented in the explicit form $B^h(t) = B^h(0,t)^{-1}\exp\Big(\int_0^t A(s,t)ds - \int_0^t \Sigma(s,t)dL_s^1 + \int_0^t C(s,t)dL_s^2\Big)$. Plugging this in (18) we get

$$\begin{aligned} B^h(t,T) &= \frac{B^h(0,T)}{B^h(0,t)} \\ &\quad \cdot \exp\Big(\int_0^t \Sigma(s,t,T)dL_s^1 - \int_0^t C(s,t,T)dL_s^2 - \int_0^t A(s,t,T)ds\Big), \end{aligned}$$

where $A(s,t,T) := A(s,T) - A(s,t)$, $\Sigma(s,t,T) := \Sigma(s,T) - \Sigma(s,t)$ and $C(s,t,T) := C(s,T) - C(s,t)$. Consequently the interaction between the two market segments is described by the process

$$X_t^h(T) := \int_0^t \Sigma(s,t,T)dL_s^1 - \int_0^t C(s,t,T)dL_s^2$$

with the component $X_t(T) := \int_0^t \Sigma(s,t,T)dL_s^1$ from the bond market and the component $\mathcal{D}_t^2(T) := \int_0^t C(s,t,T)dL_s^2$ which gives the impact from the equity market.

Similarly we can decompose the equity hybrid process

$$H_t := \int_0^t (\sigma_2(s) + C(s,t))dL_s^2 - \int_0^t \Sigma(s,t)dL_s^1$$

into two independent components $H_t^2 := \int_0^t (\sigma_2(s) + C(s,t))dL_s^2$ and $\mathcal{D}_t^1 := \int_0^t \Sigma(s,t)dL_s^1$. If we collect now the deterministic terms in $B^h(t,T)$ and $S(t)$

by defining

$$D^h(t,T) := \frac{B^h(0,T)}{B^h(0,t)} \exp\Big(\int_0^t - \left[\Theta^1(\Sigma(s,t,T) + \Theta^2(C(s,t,T)))\right] ds \Big),$$

with $\Theta^1(\Sigma(s,t,T)) := \theta^1_s(\Sigma(s,T)) - \theta^1_s(\Sigma(s,t))$ as well as $\Theta^2(C(s,t,T)) := \theta^2_s(-C(s,T)) - \theta^2_s(-C(s,t))$ and

$$d(t) := \frac{S(0)}{B^h(0,t)} \exp\Big(\int_0^t [\theta^1_s(\Sigma(s,t)) + \theta^2_s(-C(s,t)) - \theta^2_s(\sigma_2(s))] ds \Big),$$

we get $B^h(t,T) = D^h(t,T)\exp(X^h_t(T))$ and $S(t) = d(t)\exp(H_t)$. By subtracting $X^h_t(T)$ from H_t one obtains the process

$$\mathcal{H}_t(T) := \int_0^t (\sigma_2(s) + C(s,T)) dL^2_s - \int_0^t \Sigma(s,T) dL^1_s$$

with two independent components which allows to get the relation

$$S(t) = d(t,T)\exp(\mathcal{H}_t(T)) B^h(t,T)$$

with $d(t,T) := d(t)D^h(t,T)^{-1}$.

5.1. **Valuation of interest rate derivatives.** Let us first introduce the hybrid forward martingale measure P^h_T for a maturity $T \in [0,T^*]$ which lives on $(\Omega, \mathcal{F}_T)$. Its Radon–Nikodym density is defined by

$$\frac{dP^h_T}{dP} := \frac{1}{B^h(0,T)B^h(T)}.$$

with corresponding density process

$$Z_t(T) = \exp\Big(\int_0^t \Sigma(s,T) dL^1_s - \int_0^t C(s,T) dL^2_s - \int_0^t A(s,T) ds \Big).$$

This density can be decomposed into a product $\frac{dP^h_T}{dP} = \frac{dP^1_T}{dP}\frac{dP^2_T}{dP}$, where

$$\frac{dP^1_T}{dP} := \exp\Big(\int_0^T \Sigma(s,T) dL^1_s - \int_0^T \theta^1_s(\Sigma(s,T)) ds \Big),$$

$$\frac{dP^2_T}{dP} := \exp\Big(- \int_0^T C(s,T) dL^2_s - \int_0^T \theta^2_s(-C(s,T)) ds \Big).$$

The measure P^1_T is called T-forward interest rate measure whereas P^2_T is called T-forward dependence measure since it controls the influence of the equity market.

Interest rate options. Let us write the hybrid bond price in the form $B^h(t,T) = \exp(X_t^h(T) - s_1(t,T))$, where we have set $s_1(t,T) := -\ln D^h(t,T)$. Assume that the payoff f of an option with maturity $t \in [0,T]$ on a zero coupon bond with maturity $T \in [0,T^*]$ can be represented as $f(X_t^h(T) - s_1(t,T))$, then the time-0 price of the option is

$$V_0(t,T) = B^h(0,t) E_{P_t^h}[f(X_t^h(T) - s_1(t,T))],$$

where we made use of the hybrid t-forward measure. If we denote by $M_{P_t^h}^{X_t^h(T)}$ the moment generating function of $X_t^h(T)$ under P_t^h, then this can be written as a Fourier integral

$$V_0(t,T) = \frac{1}{\pi} B^h(0,t) \int_0^\infty \Re\Big(D^h(t,T)^{R-iu} M_{P_t^h}^{X_t^h(T)}(R-iu) \widehat{f}(u+iR)\Big)\, du.$$

provided (C1)–(C3) hold true. The crucial input is again the moment generating function.

Proposition 5.1. *Suppose the constants* M_i, ε_i $(i = 1,2)$ *in assumption* $(\mathbb{EM})$ *are such that for* $\Sigma(s,t) \leq H$ *and* $|C(s,t)| \leq H/2$ *for a constant* $H > 0$ *with* $H < M_1 \wedge M_2$. *Then*

$$M_{P_t^h}^{X_t^h(T)}(R) < \infty \quad \text{for} \quad R \in \Big[-\frac{M_1 \wedge M_2 - H}{H}, 0\Big] \cup \Big(1, 1 + \frac{M_1 \wedge M_2 - H}{H}\Big]$$

and for all z *with* $\Re z = R$ *we get* $M_{P_t^h}^{X_t^h(T)}(z) = M_{P_t^1}^{X_t(T)}(z) M_{P_t^2}^{-\mathcal{D}_t^2(T)}(z)$, *where*

$$M_{P_t^1}^{X_t(T)}(z) = \exp\Big(\int_0^t \big[\theta_s^1(z\Sigma(s,T) + (1-z)\Sigma(s,t)) - \theta_s^1(\Sigma(s,t))\big]\, ds\Big),$$

$$M_{P_t^2}^{-\mathcal{D}_t^2(T)}(z) = \exp\Big(\int_0^t \big[\theta_s^2((z-1)C(s,t) - zC(s,T)) - \theta_s^2(-C(s,t))\big]\, ds\Big).$$

Proof. Recall that $X_t^h(T) = X_t(T) - \mathcal{D}_t^2(T)$. The factorization of $M_{P_t^h}^{X_t^h(T)}(z)$ follows since the variables $\exp(X_t(T))\frac{dP_t^1}{dP}$ und $\exp(-\mathcal{D}_t^2(T))\frac{dP_t^2}{dP}$ are independent. For the latter component a measure change back to P provides the formula for $M_{P_t^2}^{-\mathcal{D}_t^2(T)}(z)$. The explicit expression for $M_{P_t^1}^{X_t(T)}(z)$ follows in an analogous manner. □

The valuation of caps and floors produces formulas which are identical to those which were derived in section 4. One has just to replace $B(T_{k-1}, T_k)$ by $B^h(T_{k-1}, T_k)$ and $P_{T_{k-1}}$ by $P_{T_{k-1}}^h$ as well as $D(T_{k-1}, T_k)$ by $D^h(T_{k-1}, T_k)$ and $\widetilde{M}^{X_{T_{k-1}}(T_k)}$ by $M_{P_{T_{k-1}}^h}^{X_{T_{k-1}}^h(T_k)}$. The range for R for which $M_{P_{T_{k-1}}^h}^{X_{T_{k-1}}^h(T_k)}(R) < \infty$ is the one given in Proposition 5.1.

Swaptions. We consider interest rate swaps, where along a tenor structure $\mathcal{T}$ given by $0 \leq T_0 < T_1 < \ldots < T_{n-1} < T_n$ with $T_k - T_{k-1} = \delta_k$, floating rates are exchanged against fixed rates. An investor holding a long position in a payer swap at time T_k will have to make a payment according to the fixed rate K and in exchange will receive the variable rate $L(T_{k-1}, T_k)$ given by

$$L(T_{k-1}, T_{k-1}) = \frac{1}{\delta_k}\left(\frac{1}{B^h(T_{k-1}, T_k)} - 1\right).$$

For $1 \leq k \leq n$ the cashflow at these time points is $\delta_k(L(T_{k-1}, T_{k-1}) - K)$, where we assume for simplicity that the notional amount N of the contract equals 1. For a receiver swap the cashflow $\delta_k(K - L(T_{k-1}, T_{k-1}))$ has the opposite sign. From this it follows that the value of a (forward start) payer swap at time $t \leq T_0$ is

$$FS_t(K) := E\Big[\sum_{k=1}^{n} B^h(t)B^h(T_k)^{-1}\delta_k(L(T_{k-1}, T_{k-1}) - K)\Big|\mathcal{F}_t\Big].$$

This pricing formula can be written in the form (see [22], Lemma 13.1.1.)

$$FS_t(K) = B^h(t, T_0) - \sum_{k=1}^{n} c_k(K)B^h(t, T_k) \tag{22}$$

with $c_k(K) := K\delta_k$ for $k = 1, \ldots, n-1$ and $c_n(K) := 1 + K\delta_n$.

The owner of a payer swaption with strike rate K and maturity at time $T = T_0$ has the right to enter at this time point the underlying payer swap with fixed rate K. The payer swaption has therefore at time $t \leq T$ the value

$$PS_t(T, K) := E[B^h(t)B^h(T)^{-1}(FS_T(K))^+|\mathcal{F}_t]$$

which according to (22) is equal to

$$PS_t(T, K) = E\Big[B^h(t)B^h(T)^{-1}\Big(1 - \sum_{k=1}^{n} c_k(K)B^h(T, T_k)\Big)^+\Big|\mathcal{F}_t\Big].$$

The corresponding valuation formula for a receiver swaption is

$$RS_t(T, K) = E\Big[B^h(t)B^h(T)^{-1}\Big(\sum_{k=1}^{n} c_k(K)B^h(T, T_k) - 1\Big)^+\Big|\mathcal{F}_t\Big].$$

Under the hybrid forward measure P_T^h the latter formula becomes for $t = 0$

$$RS_0(T, K) = B^h(0, T)E_{P_T^h}\Big[\Big(\sum_{k=1}^{n} c_k(K)D^h(T, T_k)\exp(X_T^h(T_k)) - 1\Big)^+\Big].$$

In order to transform this expectation into an integral which can be evaluated fast, we need to put restrictions on the volatility $\sigma_1(\cdot, T)$ and the dependence function $\beta(\cdot, T)$.

Assumption ($\mathbb{VOL}$).
For all $s, T \in [0, T^*]$ with $s \leq T$, $\sigma_1(s, T) \neq 0$ and $\beta(s, T) \neq 0$. Furthermore $\sigma_1(s, T) = \sigma^1(s)h(T)$ and $\beta(s, T) = \beta^1(s)h(T)$, where $\sigma^1, h : [0, T^*] \to \mathbb{R}^+$ as well as $\beta^1 : [0, T^*] \to \mathbb{R}$ are continuously differentiable functions.

All standard functions which are used for $\sigma_1(\cdot, T)$ and $\beta(\cdot, T)$ factorize in the sense of this assumption. The factorization implies that $\Sigma(s,t,T_k) = \Sigma(s,t,T_n)H(t,T_k,T_n)$ and $C(s,t,T_k) = C(s,t,T_n)H(t,T_k,T_n)$ with $H(t,T_k,T_n) := \int_t^{T_k} h(u)du \ (\int_t^{T_n} h(u)du)^{-1}$. The deterministic factor $H(t,T_k,T_n)$ absorbs the dependence of $X^h(T_k)$ on T_k since $X_t^h(T_k) = H(t,T_k,T_n)X_t^h(T_n)$.

We define now $f_C(x) := (f(x))^+$ with $f(x) := \sum_{k=1}^n d_k e^{H_k x} - 1$, where $d_k := c_k(K)D^h(T,T_k)$ and $H_k := H(T,T_k,T_n)$. Then we get the representation $RS_0(T,K) = B^h(0,T)E_{P_T^h}[f_C(X_T^h(T_n))]$ which is the appropriate form for the transformation in a Fourier integral. If $R \in (1,\infty)$ is such that $M_{P_T^h}^{X_T^h(T_n)}(R) < \infty$, then

$$RS_0(T,K) = \frac{1}{\pi} B^h(0,T) \int_0^\infty \Re\Big(\widehat{f}_C(u+iR) M_{P_T^h}^{X_T^h(T_n)}(R-iu)\Big) du.$$

Since f_C is a more sophisticated function, we shall derive its Fourier transform which is an input to the integral representation. Note that f is a strictly increasing, continuous function with positive and negative values and therefore has a unique zero.

Proposition 5.2. *Let Z be the unique zero of f, then for $z \in \mathbb{C}$ with $\Im z > 1$*

$$\widehat{f}_C(z) = \sum_{k=1}^n d_k \frac{-e^{(H_k+iz)Z}}{H_k + iz} + \frac{e^{izZ}}{iz}.$$

Proof.

$$\widehat{f}_C(z) = \int_Z^\infty e^{izx}\Big(\sum_{k=1}^n d_k e^{H_k x} - 1\Big) dx.$$

With the substitution $t = e^{Z-x}$ one gets

$$\widehat{f}_C(z) = e^{izZ} \int_0^1 t^{-iz-1}\Big(\sum_{k=1}^n d_k t^{-H_k} e^{H_k Z} - 1\Big) dt.$$

Therefore

$$\begin{aligned} e^{-izZ}\widehat{f}_C(z) &= \sum_{k=1}^n d_k e^{H_k Z} B(-iz - H_k, 1) - B(-iz, 1) \\ &= \sum_{k=1}^n d_k e^{H_k Z} \frac{\Gamma(-iz-H_k)}{\Gamma(1-iz-H_k)} - \frac{\Gamma(-iz)}{\Gamma(1-iz)} \\ &= \sum_{k=1}^n d_k \frac{-e^{H_k Z}}{iz + H_k} + \frac{1}{iz}. \end{aligned}$$

For the beta and gamma functions one has to make use of the fact that $0 \le H_k \le 1$ in order to guarantee that $\Re(-iz - H_k) > 0$. □

Parameter	ALV	CBK
α_1	3.12	7.913
β_1	1.87	4.348
δ_1	9.24	5.803
α_2	3.31	5.384
β_2	-1.43	-3.441
δ_2	6.01	7.836
a	0.00258	0.00084
σ_2	0.1559	0.1802
b	0.00143	-0.0035
Z^1	$5.813372 \cdot 10^{-6}$	$3.857768 \cdot 10^{-6}$
Z^2	2.683827	3.146725

Table 2 Calibrated parameters and final values of the objective functions

5.2. **Calibration of the hybrid Lévy forward rate model.** Whereas the calibration of the hybrid equity model was done in a two step procedure, this is not possible for the hybrid forward rate approach. The reason is that L^1, the process which drives the forward rates, appears already implicitly through the short rate r^h in the dynamics of the stock price. Consequently one has to estimate the parameters for the equity and the interest rate market simultaneously. We shall again use NIG processes for L^1 and L^2. This means that we have to determine the parameters $\alpha_i, \beta_i, \delta_i$ (i=1,2) for these processes. Again the location parameters μ_i are irrelevant in the risk-neutral setting and can be chosen to be 0. Furthermore we shall need according to (5) the parameter a which determines the volatility function σ_1. This function is again used in the simplified form, where $\widehat{\sigma} = |a|$. The function $\beta(s,T)$ is parametrized by b (see (19)) and finally we shall choose $\sigma_2(s) = \sigma$ as a constant function. The objective functions $Z^1(q)$ and $Z^2(p)$, where $q = (\alpha_1, \beta_1, \delta_1, \alpha_2, \beta_2, \delta_2, a, b)$ and $p = (q, \sigma)$ are defined as in section 4.5.

The calibration will be based on market quotes for caps and call options which were exploited already in section 4.5, namely those given for August 15, 2006. For the equity side we shall use here two different sets of quotes, namely the quotes for calls on Allianz (ALV) and on Commerzbank (CBK). The valuation formulas for options on equity are identical to the ones which we derived in section 4.2. One has just to replace $B(t,T)$ by $B^h(t,T)$ and $\widetilde{M}^{H_T}$ by$M^{H_T}_{P^h_T}$. As far as the latter moment generating function is concerned, we state the result which corresponds to Proposition 4.2 for completeness. Suppose $\sigma_2(s) \leq H/2$, $|C(s,t)| \leq H/2$ and $\Sigma(s,t) \leq H$ for a constant $H > 0$ such that $H < M_1 \wedge M_2$, then

$$M^{H_t}_{P^h_t}(R) < \infty \text{ for } R \in [-(M_1 \wedge M_2 - H)H^{-1}, 0] \cup (1, 1 + (M_1 \wedge M_2 - H)H^{-1}],$$

and for all z such that $\Re z = R$ we get $M^{H_t}_{P^h_t}(z) = M^{H^2_t}_{P^2_t}(z) M^{-\mathcal{D}^1_t}_{P^1_t}(z)$ with

$$M^{H^2_t}_{P^2_t}(z) = \exp\Big(\int_0^t \left[\theta^2_s(z\sigma_2(s) + (z-1)C(s,t)) - \theta^2_s(-C(s,t))\right] ds \Big)$$

and

$$M_{P_t^1}^{-\mathcal{D}_t^1}(z) = \exp\Big(\int_0^t \big[\theta_s^1((1-z)\Sigma(s,t)) - \theta_s^1(\Sigma(s,t))\big]\, ds\Big).$$

In Table 2 we list the calibrated parameters as well as the final values of the objective functions Z^1 and Z^2.

References

[1] Ahlip, R. and M. Rutkowski (2009)
Forward start options under stochastic volatility and stochastic interest rates.
Int. Journal of Theoretical and Applied Finance **12**(2), 209-225

[2] Ahlip, R. and M. Rutkowski (2013)
Pricing of foreign exchange options under the Heston stochastic volatility model and CIR interest rates.
Quantitative Finance **13**(6), 955-966

[3] Ballotta, L. (2005)
A Lévy process-based framework for the fair valuation of participating life insurance contracts.
Insurance: Mathematics and Economics **37**(2), 173-196

[4] Ballotta, L. (2009)
Pricing and capital requirements for with profit contracts: modelling considerations.
Quantitative Finance **9**(7), 803-817

[5] d'Addona, S. and A. H. Kind (2006)
International stock-bond correlations in a simple affine asset pricing model.
Journal of Banking and Finance **30**(10), 2747-2765

[6] Eberlein, E. (2009)
Jump-type Lévy processes.
In: Handbook of Financial Time Series
Andersen, T. G.; Davis, R. A.; Kreiss, J.-P.; Mikosch, T. (Eds.)
Springer Verlag, 439-455

[7] Eberlein, E. (2013)
Fourier based valuation methods in mathematical finance.
In: Quantitative Energy Finance
Benth, F.; Kholodnyi, V.; Laurence, P. (Eds.)
Springer Verlag, 85-114

[8] Eberlein, E. and Chr. Gerhart (2018)
A multiple-curve Lévy forward rate model in a two-price economy.
Quantitative Finance **18**(4), 537-561

[9] Eberlein, E., K. Glau and A. Papapantoleon (2010)
Analysis of Fourier transform valuation formulas and applications.
Applied Mathematical Finance **17**(3), 211-240

[10] Eberlein, E., J. Jacod and S. Raible (2005)
Lévy term structure models: no-arbitrage and completeness.
Finance and Stochastics **9**, 67-88

[11] Eberlein, E. and W. Kluge (2006)
Exact pricing formulae for caps and swaptions in a Lévy term structure model.
Journal of Computational Finance **9**(2), 99-125

[12] Eberlein, E. and D. Madan (2010)
On correlating Lévy processes.
The Journal of Risk **13**(1), 3-16

[13] Eberlein, E., D. Madan, M. Pistorius and M. Yor (2013)
A simple stochastic rate model for rate equity hybrid products.
Applied Mathematical Finance **20**(5-6), 461-488

[14] Eberlein, E. and Raible, S. (1999)
Term structure models driven by general Lévy processes.
Mathematical Finance **9**(1), 31-53
[15] Geman, H., N. El Karoui and J.-C. Rochet (1995)
Changes of numéraire, changes of probability measure and option pricing.
J. Appl. Prob. **32**, 443-458
[16] Grzelak, L. A., C. W. Oosterlee and S. Van Weeren (2011)
The affine Heston model with correlated Gaussian interest rates for pricing hybrid derivatives.
Quantitative Finance **11**(11), 1647-1663
[17] Grzelak, L. A., C. W. Oosterlee and S. Van Weeren (2012)
Extension of stochastic volatility equity models with the Hull-White interest rate process.
Quantitative Finance **12**(1), 89-105
[18] Heston, S. (1993)
A closed-form solution for options with stochastic volatility with applications to bond and currency options.
The Review of Financial Studies **6**(2), 327-343
[19] Hull, J. C. (2006)
Optionen, Futures, und andere Derivate., 6. Auflage
Pearson
[20] Hull, J. and A. White (1990)
Pricing interest-rate derivative securities.
The Review of Financial Studies **3**(4), 573-592
[21] Jacod, J. and A. N. Shiryaev (2003)
Limit theorems for stochastic processes., 2nd edition
Springer, Berlin
[22] Musiela, M. and M. Rutkowski (2005)
Martingale methods in financial modelling., 2nd edition
Springer, Berlin
[23] Rudmann, M. (2017)
Lévy-Hybridmodelle: Modellierung, Bewertung, Kalibrierung.
Dissertation, Universität Freiburg
[24] Schöbel, R. and J. Zhu (1999)
Stochastic volatility with an Ornstein-Uhlenbeck process: an extension.
The European Financial Review **3**, 23-46
[25] Wainscott, C. B. (1990)
The stock-bond correlation and its implications for asset allocation.
Financial Analysts Journal **46**(4), 55-60+79