A Lévy-driven Asset Price Model with Bankruptcy and Liquidity Risk

Patrick Bäurer and Ernst Eberlein

Abstract We present a new asset price model, which is an enhancement of the exponential Lévy model. The possibility of bankruptcy is modelled by a single jump to zero, whereby higher probabilities for this event lead to lower asset prices. We emphasize in particular the dependence between the asset price and the probability of default. Explicit valuation formulas for European options are established by using the Fourier-based valuation method. The formulas can numerically be computed fast and thus allow to calibrate the model to market data. On markets which are not perfectly liquid, the law of one price does no longer hold and the cost of unhedgeable risks has to be taken into account. This aspect is incorporated in the recently developed two price theory (see Cherny and Madan (2010)), which is discussed and applied to the proposed defaultable asset price model.

1 Introduction

Standard models for asset prices do not take the possibility of bankruptcy of the underlying company into account. In real markets, however, there are plenty of cases where a listed company went bankrupt with the consequence of a total loss of the invested capital. Figure 1 shows an example. It is the purpose of this paper to expand an approach such that bankruptcy can occur. As underlying asset price model $S = (S_t)_{t\geq 0}$ we choose an exponential model which is driven by a Lévy process $L = (L_t)_{t\geq 0}$. A second Lévy process $Z = (Z_t)_{t\geq 0}$ is used as driver for the hazard rate which determines the default time. The asset price jumps to zero when this event happens.

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It is a well-known fact that there is a strong negative dependence between the value of the asset and the probability of default of the corresponding company. Figure 3 shows a striking example where we plotted CDS quotes of the German energy company E.ON against its stock price. In order to take this dependence into account in the modeling approach which will be developed, the process Z is not only used for the definition of the time point of default, but enters as an additional driver into the equation for the asset price. Negative dependence is generated via a minus sign in front of Z. The remaining terms in the definition of S are determined by the fact that the discounted asset price should be a martingale.

Earlier approaches where bankruptcy of the underlying company is taken into account are Davis and Lischka (2002), Andersen and Buffum (2004), Linetsky (2006) and Carr and Madan (2010). In these papers the driving process is a standard Brownian motion and the hazard rate of bankruptcy is chosen as a decreasing function of the stock price. A particular parsimonious specification for such a function is given by a negative power of the stock price. In order to improve the performance Carr and Madan (2010) use a stochastic volatility model and jointly employ price data on credit default swaps (CDSs) and equity options to simultaneously infer the risk neutral stock dynamics in the presence of the possibility of default.

Since we will use European option prices to calibrate the model, a Fourier-based valuation formula is derived. Several types of options are discussed explicitly. In order to get prices expressed as expectations in a form which is convenient from the point of view of numerics, the survival measure is introduced. The effect of the measure change is that expectations are those of a standard payoff function. Calibration is done with L being a normal inverse Gaussian (NIG) and the independent process Z being a Gamma process. As an alternative to the Fourier-based valuation method we derive also the corresponding partial integro-differential equations (PIDEs). In the last section we show that the defaultable asset price approach which is exposed here, provides also an appropriate basis for the recently developed two price theory. The latter allows to get bid and ask prices and thus to model in addition the liquidity component of the market.

2 The Defaultable Asset Price Model

A standard model for the price process $(S_t)_{t\geq 0}$ of a traded asset which goes back to Samuelson (1965) is given by

$$S_t = S_0 e^{X_t} \tag{1}$$

where $X = (X_t)_{t\geq 0}$ is a Brownian motion. This approach represented an essential improvement on the initial Bachelier (1900) model where S had been a Brownian motion itself. The main differences are that asset prices according to (1) are positive and behave in a multiplicative or geometric way. The geometric Brownian motion became well-known as the basis for the celebrated option pricing formula due to Black and Scholes (1973) and Merton (1973). A from the point of view of distributional assumptions more realistic modeling was achieved by replacing Brownian motion by jump-type Lévy processes like hyperbolic Lévy motions, see Eberlein and Keller (1995), Eberlein and Prause (2002) and Eberlein (2001). Similar results were obtained by using the class of Variance Gamma Lévy processes as seen in Madan and Seneta (1990), Madan and Milne (1991) and Carr et al. (2002). A virtually perfect adjustment of theoretical to real option prices across all strikes and maturities was achieved by using Sato processes (Carr et al., 2007).

In this paper, the asset price model (1) is enhanced by including the possibility of default. A meaningful dependence structure between the asset price and the probability of default is introduced. Since we shall use this model for valuation, the specification is done a priori in a risk-neutral setting, i.e. we assume the underlying measure P to be risk-neutral. The economic objects to be modeled are

- the hazard rate λ as a nonnegative stochastic process with càdlàg paths, which describes the behaviour of the default time τ,
- the asset price S as a nonnegative stochastic process with càdlàg paths.

We want the asset price S to be negatively dependent on the hazard rate λ. Therefore, we use two sources of randomness

(1) a Lévy process $Z = (Z_t)_{t\geq 0}$ as driver of the hazard rate λ,
(2) an independent Lévy process $L = (L_t)_{t\geq 0}$, which represents the market noise of the asset price.

In general a Lévy process is an $\mathbb{R}^d$-valued, adapted stochastic process $X = (X_t)_{t\geq 0}$ on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, P)$ which starts at zero and has independent and stationary increments. Any Lévy process is characterised by its Lévy triplet (b, c, ν_X), where $b \in \mathbb{R}^d$, c is a symmetric nonnegative $d \times d$ matrix and ν_X is a measure on $\mathbb{R}^d$, called the Lévy measure of X. The characteristic function of X_1 is given in its Lévy-Khintchine representation as follows

$$E[e^{i\langle u, X_1\rangle}] = \exp\left[i\langle u, b\rangle - \frac{1}{2}\langle u, cu\rangle + \int [e^{i\langle u, x\rangle} - 1 - i\langle u, h(x)\rangle]\nu_X(dx)\right].$$

If a random vector X has an exponential moment of order $v \in \mathbb{R}^d$, i.e. if $E[e^{\langle v, X\rangle}]$ is finite, we write $v \in \mathbb{EM}_X$ and in this case $E[e^{\langle z, X\rangle}]$ can be defined for all $z \in \mathbb{C}^d$ with $Re(z) \in \mathbb{EM}_X$. For Lévy processes X we have under the proper moment assumption that $E[e^{\langle z, X_t\rangle}] = e^{t\theta_X(z)}$, where

$$\theta_X(z) := \log E[e^{\langle z, X_1\rangle}] = \langle z, b\rangle + \frac{1}{2}\langle z, cz\rangle + \int [e^{\langle z, x\rangle} - 1 - \langle z, h(x)\rangle]\nu_X(dx)$$

is called the cumulant function of X. Since $\mathbb{EM}_{X_t}$ is independent of t for Lévy processes we use $\mathbb{EM}_X$ in this case to express that the moment condition holds for every t. The existence of exponential moments implies the finiteness of moments of arbitrary order, in particular the finiteness of the expectation. The latter entails that the truncation function h can be chosen to be the identity, i.e. $h(x) = x$. With the following lemma we are able to calculate explicitly the expectations of exponentials of stochastic integrals with respect to a Lévy process.

Lemma 1. *Let X be a Lévy process such that $[-M_X(1+\varepsilon), M_X(1+\varepsilon)]^d \subset \mathbb{EM}_X$ for constants $M_X, \varepsilon > 0$. If $f : \mathbb{R}_+ \to \mathbb{C}^d$ is a complex-valued, continuous function such that $|Re(f^i)| \leq M_X$ $(i = 1, \ldots, d)$, then*

$$E\left[\exp\left(\int_0^t f(s)dX_s\right)\right] = \exp\left(\int_0^t \theta_X(f(s))ds\right).$$

Proof. This is a straightforward extension of Lemma 3.1. in Eberlein and Raible (1999). A proof can be found in Kluge (2005). □

In the following we shall only use one-dimensional Lévy processes.

Example 1. A very flexible and useful subclass of Lévy processes is given by the normal inverse Gaussian (NIG) processes, which are generated by the NIG distribution with the simple characteristic function

$$\varphi_{NIG}(u) = e^{iu\mu} \frac{\exp(\delta\sqrt{\alpha^2 - \beta^2})}{\exp(\delta\sqrt{\alpha^2 - (\beta + iu)^2})}$$

and the four parameters $\mu, \beta \in \mathbb{R}$, $\delta > 0$ and $\alpha > |\beta| \geq 0$.

Example 2. The Gamma process, generated by the Gamma distribution, is an increasing Lévy process. The Gamma distribution has the parameters $p, b > 0$ and the characteristic function

$$\varphi_\Gamma(u) = \left(\frac{b}{b - iu}\right)^p.$$

The default time $\tau : \Omega \to [0, \infty]$ is constructed via

$$\tau = \inf\{t \geq 0 \mid e^{-\Gamma_t} \leq \xi\}.$$

where $\Gamma_t := \int_0^t \lambda_s ds$ is the integral over the hazard rate $\lambda = (\lambda_t)_{t\geq 0}$, a nonnegative $\mathbb{F}$-adapted process with càdlàg paths and ξ is a uniformly distributed random variable on $[0,1]$, independent of $\mathbb{F}$. This is the so-called intensity-based approach of default modelling. Details can be found in Bielecki and Rutkowski (2004). We need three properties of this construction:

1. One can easily show that
$$P(t < \tau \mid \mathscr{F}_t) = e^{-\Gamma_t}. \tag{2}$$
Thus, the survival probability can be calculated to be $P(t < \tau) = E\left[e^{-\Gamma_t}\right]$.
2. If $(M_t)_{t\geq 0}$ is a nonnegative $\mathbb{F}$-martingale, then
$$\left(M_t \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t}\right)_{t\geq 0}$$

follows a $\mathbb{G}$-martingale. $\mathbb{G} = (\mathscr{G}_t)_{t\geq 0}$ is defined by $\mathscr{G}_t := \mathscr{F}_t \vee \mathscr{H}_t$, where $\mathscr{H}_t := \sigma(\{\tau \leq u \mid u \leq t\})$ is the filtration which carries the information about the default time.

3. For the t-survival measure

$$P^t(A) := P(A \mid t < \tau),$$

which is the measure P conditioned on no default until t, one gets $P^t \ll P$ and

$$\frac{dP^t|_{\mathscr{F}_t}}{dP|_{\mathscr{F}_t}} = \frac{e^{-\Gamma_t}}{E[e^{-\Gamma_t}]}. \tag{3}$$

Now we are ready to specify the asset price model in the form

$$S_t = S_0 \exp\left[rt + L_t - \zeta Z_t + \omega t + \Gamma_t\right] \mathbb{1}_{\{t<\tau\}} \tag{4}$$

with a constant r, representing the continuously compounded interest rate. Default is modeled by a single jump to zero at time point τ. This reflects the idea of no recovery for shareholders. This assumption seems to be reasonable if we look at the history of bankruptcies. As an example, the time series of stock prices showing the bankruptcy of the former German company Walter Bau is represented in Figure 1. Effectively, the default event, marked by the ellipse, is a jump to zero. In the sequel, this model will be denoted the **Defaultable Asset Price Model** (**DAM**).

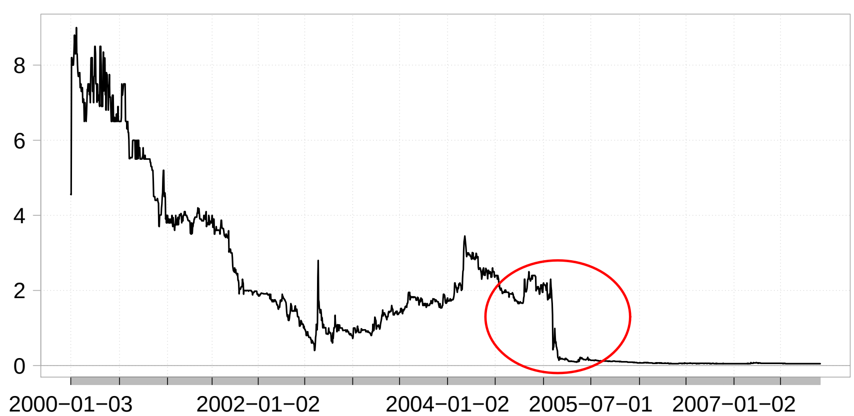

Fig. 1 The bankruptcy of Walter Bau

The term $-\zeta Z_t$ models the dependency between credit risk and asset price with an additional parameter $\zeta \geq 0$. A surge of the default probability leads to a decline of the asset price. A generalisation to a more complex functional dependence structure

$-f(Z_t)$ is possible and in line with the pricing methods below. The simple form $-\zeta Z_t$ was chosen for convenience.

Since we want $(S_t)_{t\geq 0}$ to be a martingale after discounting, the reason for the term $\omega t + \Gamma_t$ is a mathematical one. Using the well-known fact that $e^{X_t}/E[e^{X_t}]$ is a martingale for a process X with independent increments, we can choose the constant ω such that $\exp[L_t - \zeta Z_t + \omega t]$ is an $\mathbb{F}$-martingale:

$$\omega = -\log E[e^{L_1}] - \log E[e^{-\zeta Z_1}] = -\theta_L(1) - \theta_Z(-\zeta).$$

Thus, as indicated before, the discounted price process

$$e^{-rt} S_t = S_0 \exp\left[L_t - \zeta Z_t + \omega t\right] \;\cdot\; e^{\Gamma_t} \mathbb{1}_{\{t<\tau\}}$$

is a $\mathbb{G}$-martingale. This ensures that the considered financial market model is arbitrage-free, cf. Delbaen and Schachermayer (2006).

For the existence of $\omega \in \mathbb{R}$, we need the conditions

(i) $1 \in \mathbb{EM}_L$.
(ii) $-\zeta \in \mathbb{EM}_Z$.

A similar type of model for pricing convertible bonds was introduced by Davis and Lischka (2002). Their model, driven by a Brownian Motion $(W_t)_{t\geq 0}$ with volatility σ, is

$$S_t = S_0 \exp\left[rt + \sigma W_t - \frac{1}{2}\sigma^2 t + \int_0^t \lambda_s ds\right] \mathbb{1}_{\{t<\tau\}},$$

where $(\lambda_s)_{s\geq 0}$ is the hazard rate corresponding to the default time τ. This model approach was enhanced by Andersen and Buffum (2004), Linetsky (2006) and Carr and Madan (2010). Their idea of getting a reasonable dependence structure between credit risk and asset price was a different one. They choose the hazard rate as a function of the asset price, for example

$$\lambda_s = \lambda(S_s) = \alpha S_s^{-p},$$

which leads to a stochastic integral equation. Our approach, which is also an enhancement of this model, avoids this. Thus, we get a more direct analytical access.

As a model for the hazard rate $(\lambda_t)_{t\geq 0}$, we choose a positive Ornstein-Uhlenbeck (OU) process driven by an increasing Lévy process $(Z_t)_{t\geq 0}$ which is assumed to be independent of L

$$d\lambda_t = \kappa(\mu - \lambda_t)dt + dZ_t. \qquad (\kappa, \mu \geq 0). \tag{5}$$

This kind of processes moves up by the jumps of Z and then declines exponentially as if there is a restoring force measured by the parameter κ, see Figure 2. One main advantage is the analytical tractability, see for example Barndorff-Nielsen and Shephard (2001) or Cont and Tankov (2004), where OU processes are used as stochastic volatility models for financial assets. Schoutens and Cariboni (2009) investigated OU processes already as hazard rate models.

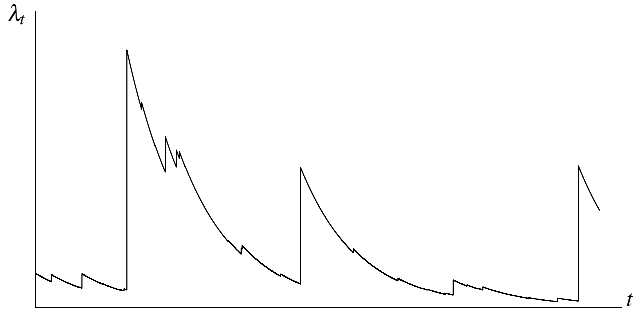

Fig. 2 OU process driven by a Γ process.

The upward jumps can be interpreted as bad news about the firm, like a profit alert, an essential loss of capital or a failed project. Other reasons could be major events or even catastrophes with consequences for a whole industrial sector or the global economy. Examples are the burst of the Dot-com bubble in 2000, the terror attacks of 9/11, the collapse of Lehman Brothers in 2008 or the Fukushima disaster in 2011. Hazard rates are not directly observable, but CDS quotes also reflect the default probability. Hence, the time evolution of hazard rates and short time CDS quotes should look quite similar. We take the one-year CDS quotes of the German energy company E.ON SE as an example, see Figure 3. There are two big jumps, one after the collapse of Lehman Brothers (left line) and one when the German government resolved the nuclear phase-out a few months after the Fukushima disaster (middle line). We can conclude that the model approach (5) looks quite reasonable in view of this example. The relation between the upward jumps of the CDS quotes and the downward movement of the stock price is clearly visible.

The explicit expression for (5) is

$$\lambda_t = \lambda_0 e^{-\kappa t} + \mu(1 - e^{-\kappa t}) + \int_0^t e^{\kappa(s-t)} dZ_s. \tag{6}$$

Using Fubini's Theorem for stochastic integrals, cf. Theorem 64 in Chapter IV of Protter (2005), we get for the hazard process

$$\Gamma_t = \Gamma_t^d + \int_0^t \gamma_s^t dZ_s \tag{7}$$

where we used the abbreviations

$$\Gamma_t^d := \frac{\lambda_0}{\kappa}(1 - e^{-\kappa t}) + \mu\left(t + \frac{e^{-\kappa t}}{\kappa} - \frac{1}{\kappa}\right)$$
$$\gamma_s^t := \frac{1 - e^{-\kappa(t-s)}}{\kappa}.$$

Fig. 3 One-year CDS quotes (top) and stock price (bottom) of the German energy company E.ON SE. The left line marks the collapse of Lehman Brothers, the middle line the German nuclear phase-out after the Fukushima disaster.

For the numerical calculation of the survival probability $P(t < \tau) = E[e^{-\Gamma_t}]$, we can now use Lemma 1

$$\begin{aligned} E[e^{-\Gamma_t}] &= e^{-\Gamma_t^d} E\left[\exp\left(-\int_0^t \gamma_u^t dZ_u\right)\right] \\ &= e^{-\Gamma_t^d} \exp\left(\int_0^t \theta_Z(-\gamma_u^t) du\right), \end{aligned} \tag{8}$$

where θ_Z is the cumulant function of Z. To obtain (8), we need the assumptions

(iii) There are constants $M_Z, \varepsilon > 0$ such that $\pm M_Z(1+\varepsilon) \in \mathbb{EM}_Z$.
(iv) κ satisfies $\frac{1}{\kappa} \leq M_Z$.

This kind of model cannot be adjusted to an exogenously given survival function $t \mapsto P(t < \tau) = E[e^{-\Gamma_t}]$. The survival function can be recovered from CDS quotes using the methods described in Madan et al. (2004).

The same problem is known from short rate models for the term structure of interest rates (for an overview see the book of Brigo and Mercurio (2001)). The famous Vasicek (1977) model is not able to incorporate the current yield curve. Hull and White (1990) overcame this drawback by making one parameter in the Vasicek model time-dependent. The same idea could be used to extend (5) in the following way

$$d\lambda_t = \kappa(\mu(t) - \lambda_t)dt + dZ_t.$$

3 Option pricing

In this section, we price some European options under the Defaultable Asset Price Model. We define the $\mathbb{F}$-adapted semimartingale

$$X_t := \log S_0 + rt + L_t - \zeta Z_t + \omega t + \Gamma_t$$

such that $S_t = e^{X_t} \mathbb{1}_{\{t<\tau\}}$ and use the Fourier-based valuation method as given in Eberlein et al. (2010). This leads to the equation

$$E_Q[f(X_T)] = \frac{1}{2\pi} \int \varphi^Q_{X_T}(u - iR)\widehat{f}(iR - u)du, \tag{9}$$

where $\widehat{f}$ denotes the Fourier transform of f, which is defined by $\widehat{f}(u) = \int e^{iux} f(x)dx$ and where $\varphi^Q_{X_T}$ denotes the extended characteristic function of X_T under the probability measure Q. $R \in \mathbb{R}$ is a constant that must satisfy

(C1) $g \in L^1_{bc}(\mathbb{R}) = \{h \in L^1(\mathbb{R}) \mid h \text{ bounded and continuous}\}$,
(C2) $R \in \mathbb{EM}_{X_T}$,
(C3) $\widehat{g} \in L^1(\mathbb{R})$,

where $g(x) := e^{-Rx} f(x)$. The key point of (9) is the separation of the function f from the distribution Q_{X_T} of X_T.

In order to use the Fourier-based method within the Defaultable Asset Price Model one has to separate the indicator $\mathbb{1}_{\{t<\tau\}}$ from the payoff function. This means that we only consider payoff functions f which can be written as

$$f(S_T) = f(\mathbb{1}_{\{T<\tau\}} e^{X_T}) = \mathbb{1}_{\{T\geq\tau\}} f_1(X_T) + \mathbb{1}_{\{T<\tau\}} f_2(X_T) \tag{10}$$

for functions f_1 and f_2, that satisfy the assumptions for the valuation formula (9).

Lemma 2. *Let f be a payoff function of an option with maturity $T > 0$ which satisfies* (10)*. Then the following formula holds*

$$E\left[f(S_T)\right] = E\left[f_1(X_T)\right] - E\left[e^{-\Gamma_T}\right] E_T\left[f_1(X_T)\right] + E\left[e^{-\Gamma_T}\right] E_T\left[f_2(X_T)\right] \tag{11}$$

where $E_T := E_{P^T}$ is the expectation under the survival measure P^T.

Proof. For this calculation, we use the change-of-numeraire technique with the survival measure P^T

$$\begin{aligned}
E\left[f(S_T)\right] &\overset{(10)}{=} E\left[\mathbb{1}_{\{T\geq\tau\}} f_1(X_T)\right] + E\left[\mathbb{1}_{\{T<\tau\}} f_2(X_T)\right] \\
&= E\left[(1-\mathbb{1}_{\{T<\tau\}}) f_1(X_T)\right] + E\left[\mathbb{1}_{\{T<\tau\}} f_2(X_T)\right] \\
&= E\left[f_1(X_T) E\left[(1-\mathbb{1}_{\{T<\tau\}}) \mid \mathscr{F}_T\right]\right] + E\left[f_2(X_T) E\left[\mathbb{1}_{\{T<\tau\}} \mid \mathscr{F}_T\right]\right] \\
&\overset{(2)}{=} E\left[f_1(X_T)\right] - E\left[e^{-\Gamma_T} f_1(X_T)\right] + E\left[e^{-\Gamma_T} f_2(X_T)\right] \\
&\overset{(3)}{=} E\left[f_1(X_T)\right] - E\left[e^{-\Gamma_T}\right] E_T\left[f_1(X_T)\right] + E\left[e^{-\Gamma_T}\right] E_T\left[f_2(X_T)\right].
\end{aligned}$$

□

The elements on the right side of (11) can be calculated numerically. $E[e^{-\Gamma_T}]$ can be calculated by using Lemma 1. For the calculation of the expectations $E_T[f(X_T)]$ under the survival measure P^T for different functions f, we use (9). We shall calculate the extended characteristic function $\varphi_{X_T}^{P^T}$ of X_T under the survival measure P^T. We begin with a generic lemma of stochastic analysis.

Lemma 3. *Let X and Y be two independent semimartingales and H be a deterministic process with left-continuous paths. Then the processes X and $(\int_0^t H_s dY_s)_{t\geq 0}$ are independent as well.*

Proof. Fix $t \geq 0$ and define

$$H_t^n := \mathbb{1}_{\{0\}} H_0 + \sum_{k=1}^{2^n} \mathbb{1}_{](k-1)\frac{t}{2^n}, k\frac{t}{2^n}]} H_{k\frac{t}{2^n}}.$$

For each $n \geq 1$ and each $t' \geq 0$, $X_{t'}$ is independent from

$$\int_0^t H_s^n dY_s = \sum_{k=1}^{2^n} H_{k\frac{t}{2^n}} (Y_{k\frac{t}{2^n}} - Y_{(k-1)\frac{t}{2^n}}).$$

$\int_0^t H_s^n dY_s$ is a Riemann approximation for the stochastic integral $\int_0^t H_s dY_s$, i.e.

$$\int_0^t H_s^n dY_s \to \int_0^t H_s dY_s$$

in probability, see Proposition I.4.44 in Jacod and Shiryaev (2003). Independence is transferred to the stochastic limit, cf. Proposition 1.13 in Sato (1999), and thus the assertion follows. □

Lemma 4.
Let $R > 1$ ($R < 0$ resp.) such that

(v) $R \in \mathbb{EM}_L$*, i.e.* $E[e^{RL_T}]$ *exists for all* $T \geq 0$*,*
(vi) $\max\{\zeta R,\ \frac{R-1}{\kappa} - \zeta R\} \leq M_Z$ *(*$\max\{-\zeta R,\ \zeta R - \frac{R-1}{\kappa}\} \leq M_Z$ *resp.).*

Then $M^T_{X_T}(R) = E_T[e^{RX_T}]$ *exists, i.e. assumption (C2) of* (9) *is satisfied.*

Proof. Using Lemma 3, we obtain

$$\begin{aligned}
M^T_{X_T}(R) &= E_T[\exp(RX_T)] \\
&= \text{const.} \cdot E_T[\exp(RL_T)\exp(-\zeta RZ_T + R\Gamma_T)] \\
&\overset{(3)}{=} \text{const.} \cdot E[\exp(RL_T)\exp(-\zeta RZ_T + (R-1)\Gamma_T)] \\
&= \text{const.} \cdot M_{L_T}(R) \cdot E\left[\exp\left(\int_0^T (R-1)\gamma^T_s - \zeta R\, dZ_s\right)\right].
\end{aligned}$$

(vi) implies $|(R-1)\gamma^T_s - \zeta R| \leq M_Z$, and thus the existence of the last factor. □

To use (9), we need to calculate the extended characteristic function $\varphi^{P^T}_{X_T}$ of X_T under P^T. We abbreviate

$$\begin{aligned}
d_t &:= \ln S_0 + rt + \omega t \\
D_t(x) &:= \frac{\exp[x(d_t + \Gamma^d_t) - \Gamma^d_t]}{E[e^{-\Gamma_t}]},
\end{aligned}$$

and obtain for all $x \in \mathbb{C}$ with $\mathrm{Re}(x) = R$

$$\begin{aligned}
E_T[e^{xX_T}] &= e^{xd_T} E_T[e^{x(L_T - \zeta Z_T + \Gamma_T)}] \\
&= e^{xd_T} E\left[\frac{e^{-\Gamma_T}}{E[e^{-\Gamma_T}]} e^{x(L_T - \zeta Z_T + \Gamma_T)}\right] \\
&= D_T(x) E\left[e^{xL_T} e^{\int_0^T x\gamma^T_s - x\zeta - \gamma^T_s\, dZ_s}\right] \\
&\overset{(\star)}{=} D_T(x) E\left[e^{xL_T}\right] E\left[e^{\int_0^T x\gamma^T_s - x\zeta - \gamma^T_s\, dZ_s}\right] \\
&= D_T(x) \exp[T \cdot \theta_L(x)] \exp\left[\int_0^T \theta_Z(x\gamma^T_s - x\zeta - \gamma^T_s) ds\right],
\end{aligned}$$

where we have used Lemma 3 in equation $(\star)$. In the last step of this calculation, we used Lemma 1. The requirement

$$\left|\mathrm{Re}(x\gamma^t_s - x\zeta - \gamma^t_s)\right| \leq M_Z$$

is satisfied by the assumptions of Lemma 4. Hence, we have for all $u \in \mathbb{R}$ and suitable $R \in \mathbb{R}$

$$\varphi_{X_T}^{P^T}(u-iR) = E_T[e^{(R+iu)X_T}]$$
$$= D_T(R+iu)\exp[T\cdot\theta_L(R+iu)]\exp\left[\int_0^T \theta_Z((R+iu)\gamma_s^T-(R+iu)\zeta-\gamma_s^T)ds\right]. \tag{12}$$

Example 3. In the case of a call option, we have $f(x) = (e^x - K)^+$, i.e.

$$\widehat{f}(z) = \frac{K^{1+iz}}{iz(1+iz)}, \qquad \mathrm{Im}(z) \in (1,\infty).$$

Conditions (C1) and (C3) are fulfilled for $R > 1$. The payoff function is of type (10) with $f_1 \equiv 0$ and $f_2(x) = (e^x - K)^+$. For the put option, where $f(x) = (K - e^x)^+$, we have

$$\widehat{f}(z) = \frac{K^{1+iz}}{iz(1+iz)}, \qquad \mathrm{Im}(z) \in (-\infty,0).$$

Conditions (C1) and (C3) are fulfilled for $R < 0$. We have $f_1 \equiv K$ and $f_2(x) = (K - e^x)^+$. By using (11), we obtain the call prices

$$C_0(T,K) = e^{-rT}E[e^{-\Gamma_T}]E_T[(e^{X_T} - K)^+] \tag{13}$$

and the put prices

$$P_0(T,K) = e^{-rT}\left[E[e^{-\Gamma_T}]E_T[(K - e^{X_T})^+] + K(1 - E[e^{-\Gamma_T}])\right]. \tag{14}$$

Example 4. The payoff function of a digital call option with barrier $B > 0$ and maturity $T > 0$ is $f(x) = \mathbb{1}_{\{x>B\}}$, i.e. it is of type (10) with $f_1 \equiv 0$ and $f_2 = \mathbb{1}_{\{e^x>B\}}$. We use (11) and obtain

$$E\left[e^{-rT}\mathbb{1}_{\{S_T>B\}}\right] = e^{-rT}E[e^{-\Gamma_T}]E_T\left[\mathbb{1}_{\{e^{X_T}>B\}}\right].$$

The Fourier transform of f_2 is

$$\widehat{f_2}(z) = -\frac{B^{iz}}{iz} \quad \text{for} \quad \mathrm{Im}(z) > 0.$$

The assumptions for applying (9) are satisfied for $R > 0$, cf. Eberlein et al. (2010). For the digital put option, we have

$$E\left[e^{-rT}\mathbb{1}_{\{S_T<B\}}\right] \overset{(11)}{=} e^{-rT}\left(1 - E[e^{-\Gamma_T}] + E[e^{-\Gamma_T}]E_T\left[\mathbb{1}_{\{e^{X_T}<B\}}\right]\right).$$

The Fourier transform of $f_2(x) = \mathbb{1}_{\{e^x<B\}}$ is

$$\widehat{f_2}(z) = \frac{B^{iz}}{iz} \quad \text{for} \quad \mathrm{Im}(z) < 0.$$

In this case, we need $R < 0$. To give a numerical example, we take $S_0 = 30$, $T = 260$ and the parameters

$$\begin{array}{lll} \alpha = 50.0 & \beta = -0.1 & \delta = 0.012 \\ p = 0.0035 & b = 66 & \kappa = 0.11 \\ & \zeta = 9.0 & \end{array} \qquad (*)$$

which correspond to a one-year default probability of about 10.7 %. The results can be seen in Figure 4. The main difference to a non-defaultable model is that the prices tend to $1 - P(T \geq \tau)$ for $B \searrow 0$ and not to 1.

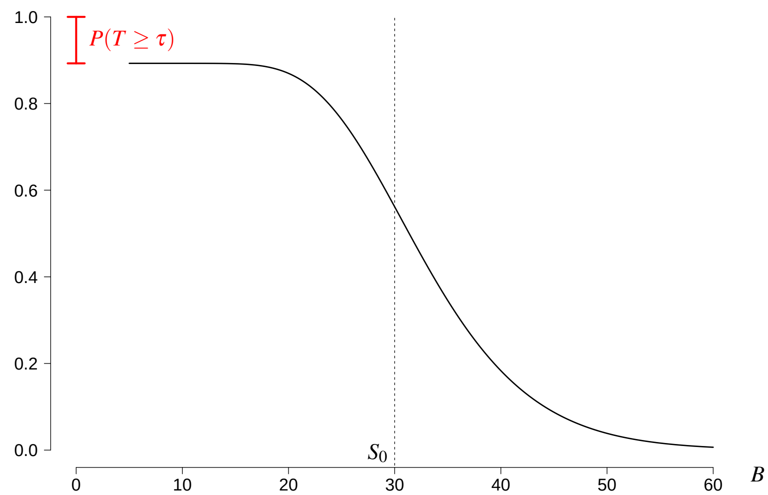

Fig. 4 Prices of digital call options with barrier B.

Example 5. The payoff of a self-quanto call option with strike $K > 0$ is $e^x(e^x - K)^+$, i.e. we have

$$e^{-rT} E\left[\mathbb{1}_{\{T<\tau\}} e^{X_T}(e^{X_T} - K)^+\right] = e^{-rT} E[e^{-\Gamma_T}] E_T\left[e^{X_T}(e^{X_T} - K)^+\right].$$

The Fourier transform of $f_2(x) = e^x(e^x - K)^+$ is

$$\widehat{f_2}(z) = \frac{K^{2+iz}}{(1+iz)(2+iz)} \quad \text{for} \quad \mathrm{Im}(z) > 2.$$

For a self-quanto put option with payoff $e^x(K - e^x)^+$ we have

$$e^{-rT} E\left[\mathbb{1}_{\{T<\tau\}} e^{X_T}(K - e^{X_T})^+\right] = e^{-rT} E[e^{-\Gamma_T}] E_T\left[e^{X_T}(K - e^{X_T})^+\right].$$

The Fourier transform of $f_2(x) = e^x(K - e^x)^+$ is the same as above, but for $\mathrm{Im}(z) < 1$.

For calculating expectations $E[f(S_T)]$, we can also use Monte Carlo simulations, i.e. we can simulate the random variable S_T for example N times and approximate $E[f(S_T)]$ by $\frac{1}{N}\sum_{i=1}^{N} f(s_T^i)$, where $(s_T^i)_{i=1,...,N}$ denotes a simulated sample of S_T. For the pathwise simulation of the Defaultable Asset Price Model

$$S_t = S_0 \exp\left[rt - qt + L_t - \zeta Z_t + \omega t + \Gamma_t\right] \mathbb{1}_{\{t<\tau\}},$$

we have to be able to simulate the Lévy processes L_t and Z_t pathwise. This means, that it is necessary to simulate whole paths $(S_t)_{0\leq t\leq T}$ if we want to create a simulation for S_T. If we have to do that already, with only little additional effort one can price path-dependent options or options with different maturities $T_k \leq T$ $(k = 1,...,n)$ simultaneously.

Example 6. An Asian option is a derivative, whose payoff depends on the average price

$$\overline{S}_T := \frac{1}{T}\int_0^T S_t dt$$

of the underlying price process $(S_t)_{0\leq t\leq T}$. We simulate the price path on an equidistant time grid $0 = t_0 < t_1 < ... < t_n = T$. The simulated value $\overline{s}_T^i$ of the average price is then given as the mean

$$\overline{s}_T^i = \frac{1}{n}\sum_{k=0}^{n} s_{t_k}^i$$

of the simulated prices $(s_{t_k}^i)_{k=0,...,n}$ for each simulation $i \in \{1,...,N\}$. Figure 5 shows an example.

4 Calibration

Calibration is conducted by minimising the sum of the squared differences between observed market prices and model prices

$$\mathrm{SD}(\alpha) := \sum_j \left(\pi_j^{\mathrm{Model}}(\alpha) - \pi_j^{\mathrm{Market}}\right)^2$$

over the model parameters $\alpha = (\alpha_1,...,\alpha_n)$ in a parameter space $A_1 \times ... \times A_n \subset \mathbb{R}^n$. This space is given by constraints on the mathematical model. In our case, we have to consider the parameter spaces of the processes L and Z and have to check the conditions (i) - (vi).

We choose a $\mathrm{NIG}(\alpha,\beta,\delta,\mu)$ process for L and a $\Gamma(p,b)$ process for Z as an example. This leads to a model with the seven parameters

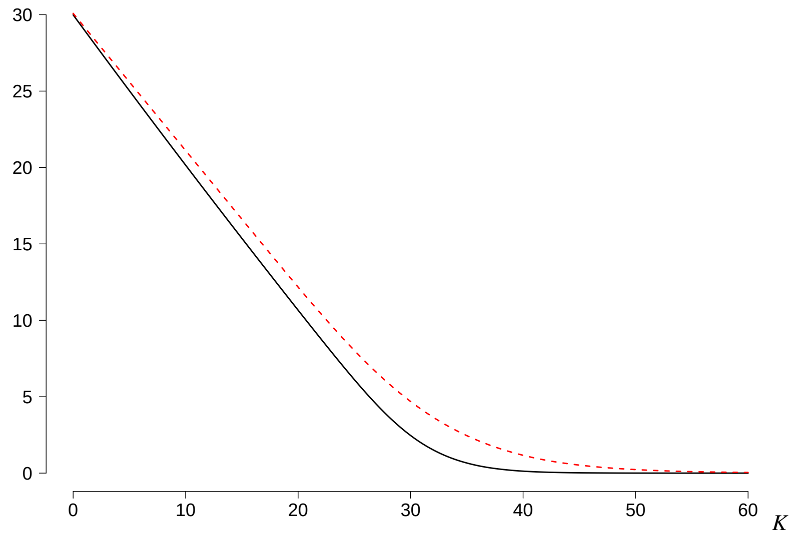

Fig. 5 Prices of average price calls with payoff $(\bar{S}_T - K)^+$ (solid line). For comparison, prices of ordinary calls (dashed line).

$\alpha > 0,\ \beta \in (-\alpha, \alpha),\ \delta > 0$	for the NIG process
$p,\ b > 0$	for the Γ process
$\kappa \geq 0$	for the OU restoring force
$\zeta \geq 0$	as dependence parameter.

We note here that the drift parameter μ of the NIG process is redundant. The reason is the martingale setting. If L_1 is NIG-distributed, then $L_1 - \log E[e^{L_1}]$ is also NIG-distributed, but independent of μ.

The model assumptions (i)-(vi) can be reduced to restrictions on the process parameters. For the NIG process L, we have $\mathbb{EM}_L = (-\alpha - \beta, \alpha - \beta)$ and for the Γ process Z, we get $\mathbb{EM}_Z = (-\infty, b)$. Consequently we can convert the conditions to

(i) $1 < \alpha - \beta$
(ii) $-\zeta < b$
(iii) is always satisfied
(iv) $\frac{1}{\kappa} < b$
(v) $1 < R < \alpha - \beta$
(vi) $\max\{\zeta R,\ \frac{R-1}{\kappa} - \zeta R\} < b \quad (\max\{-\zeta R,\ \zeta R - \frac{R-1}{\kappa}\} < b$ resp.),

which can all be checked easily.

We calibrate all parameters, i.e. the parameters for L, the credit parameters and the dependence parameter ζ, to the option price surface. Hence, we obtain the required risk-neutral parameters of the model which are needed to price other financial products based on this asset. Accordingly, we can extract credit risk information about the firm from option quotes. This enables us to calculate default probabilities. Alternatively, one could calibrate the credit parameters to the CDS term structure, fix them and calibrate the remaining ones using option prices.

We consider the stocks of the European banks BNP Paribas, Commerzbank, Credit Agricole, Credit Suisse, Deutsche Bank, UBS and UniCredit and look at the corresponding call prices on March 20, 2014. We restrict ourselves to calls with expiration date T_1 in December 2014 and T_2 in December 2015. As a riskless interest rate, we take the EONIA rate. The current stock prices are dividend-adjusted via

$$S_0 \quad \rightsquigarrow \quad S_0 - e^{-rT_D} \cdot D,$$

where we take the estimated or promised dividend payment of each bank for D and the day following the annual general assembly for T_D. The results of the calibrations can be found in the Table 1 and Table 2.

	BNP Paribas		Commerzbank		Credit Agricole		Credit Suisse	
	T_1	T_2	T_1	T_2	T_1	T_2	T_1	T_2
α	53.0	52.6	50.3	49.9	45.2	46.1	45.8	44.0
β	-0.09	-0.05	-0.23	-0.17	-0.10	0.03	-0.08	-0.1
δ	0.0087	0.0091	0.0229	0.0213	0.0088	0.0095	0.0056	0.0060
p	0.00218	0.00182	0.00134	0.00122	0.004	0.00366	0.00312	0.00244
b	51	81	91	101	90	119	78	112
κ	0.162	0.402	0.47	0.402	0.16	0.234	0.18	0.25
ζ	5.0	5.0	5.5	5.5	4.6	5.1	4.0	3.0

Table 1 Calibration results 1

	Deutsche Bank		UBS		UniCredit	
	T_1	T_2	T_1	T_2	T_1	T_2
α	61.3	60.4	69.0	69.1	45.0	45.0
β	-0.95	-1.1	-0.5	-0.8	-3.2	-3.2
δ	0.0109	0.0106	0.0120	0.0110	0.013	0.013
p	0.00314	0.00276	0.0028	0.0025	0.0022	0.0020
b	87	126	142	144	154	146
κ	0.182	0.26	0.28	0.27	0.16	0.18
ζ	3.5	3.8	3.0	3.5	6.0	5.8

Table 2 Calibration results 2

In Figure 6, we observe a virtually perfect fit of the DAM to the real market data of BNP Paribas.

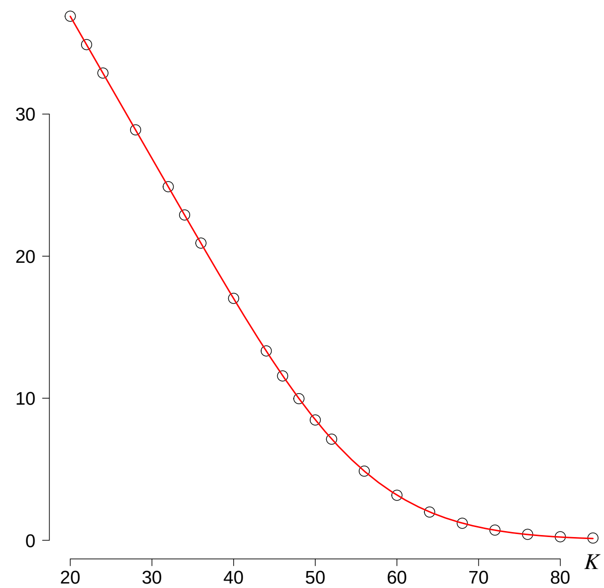

Fig. 6 Quoted call prices of BNP Paribas (circles) and the model prices (line) after the calibration.

5 A Differential Equation for the Option Pricing Function

In the former sections, the calculation of the desired expectation $E[f(S_T)]$ is accomplished by combining the change of measure with the Fourier-based valuation method. Now we shall investigate another common method, namely pricing by solving a partial integro-differential equation (PIDE). The procedure is the following. Write the martingale $E[f(S_T) \mid \mathscr{F}_t]$ as a C^2-function g of t and some underlying process $V_t = (V_t^1, \ldots, V_t^d)$

$$E[f(S_T) \mid \mathscr{F}_t] = g(V_t, t). \tag{15}$$

We assume that the processes V^i are special semimartingales, i.e. they possess a (unique) decomposition $V^i = V_0 + M^i + A^i$ with a local martingale M^i and a predictable process A^i with paths of bounded variation. By applying Itô's formula we obtain

$$\begin{aligned} g(V_t,t) = g(V_0,0) &+ \sum_{i\leq d}\int_0^t \partial_i g(V_{s-},s)dV_s^i + \int_0^t \partial_{d+1} g(V_{s-},s)ds \\ &+ \frac{1}{2}\sum_{i,j\leq d}\int_0^t \partial_{ij} g(V_{s-},s)d\langle (V^i)^c,(V^j)^c\rangle_s \\ &+ \sum_{s\leq t}\left[g(V_s,s) - g(V_{s-},s) - \sum_{i\leq d}\partial_i g(V_{s-},s)\Delta V_s^i\right]. \end{aligned} \tag{16}$$

$g(V_t,t)$ is a special semimartingale, but also a martingale by (15). Consequently, any decomposition

$$g(V_t,t) = g(V_0,0) + M_t + A_t$$

with a local martingale M and a predictable process A with paths of bounded variation has to satisfy $A \equiv 0$. Expanding and sorting the the right-hand side of (16) in this sense leads to the desired PIDE

$$\begin{aligned} 0 = &\sum_{i\leq d}\int_0^t \partial_i g(V_{s-},s)dA_s^i + \int_0^t \partial_{d+1} g(V_{s-},s)ds \\ &+ \frac{1}{2}\sum_{i,j\leq d}\int_0^t \partial_{ij} g(V_{s-},s)d\langle (V^i)^c,(V^j)^c\rangle_s \\ &+ \int_{[0,t]\times\mathbb{R}^d}\left[g(V_{s-}+x,s) - g(V_{s-},s) - \sum_{i\leq d}\partial_i g(V_{s-},s)x\right](\mu^V)^p(ds,dx), \end{aligned} \tag{17}$$

where $(\mu^V)^p$ is the predictable compensator of the jump measure μ^V of V, cf. Theorem II.1.8 in Jacod and Shiryaev (2003). The boundary condition is set at the maturity date T of the contingent claim

$$g(x^1,\ldots,x^d,T) = f(l(x^1,\ldots,x^d)),$$

where l is the function, such that $S_T = l(V_T^1,\ldots,V_T^d)$. Solving the PIDE (numerically) on $\mathbb{R}^d\times[0,T]$ gives us the desired value

$$E[f(S_T)] = g(V_0,0).$$

The boundary condition determines the solution $g(x,t)$ at the end of the considered time interval $[0,T]$, but the value we are looking for is the one at the beginning.

In order to apply this approach to the DAM

$$S_t = \exp\left[\log S_0 + rt + L_t - \zeta Z_t + \omega t + \Gamma_t\right]\mathbb{1}_{\{t<\tau\}} = e^{X_t}\mathbb{1}_{\{t<\tau\}},$$

we firstly have to take care of the indicator function $\mathbb{1}_{\{t<\tau\}}$. Therefore, we shall only consider payoff functions f of type (10), i.e. we assume that

$$f(S_T) = f(\mathbb{1}_{\{T<\tau\}}e^{X_T}) = \mathbb{1}_{\{T\geq\tau\}}f_1(X_T) + \mathbb{1}_{\{T<\tau\}}f_2(X_T)$$

for functions f_1 and f_2. As seen before, most of the common payoff functions have this form. In this case, we can eliminate the indicator function $\mathbb{1}_{\{t<\tau\}}$ in the time-0 pricing formula

$$\begin{aligned}\pi_0 &= e^{-rT}E[f(S_T)] \overset{(10)}{=} e^{-rT}E[\mathbb{1}_{\{T\geq\tau\}}f_1(X_T)+\mathbb{1}_{\{T<\tau\}}f_2(X_T)]\\&= e^{-rT}\{E[E[\mathbb{1}_{\{T\geq\tau\}}f_1(X_T)\mid\mathscr{F}_T]]+E[E[\mathbb{1}_{\{T<\tau\}}f_2(X_T)\mid\mathscr{F}_T]]\}\\&= e^{-rT}\{E[f_1(X_T)E[\mathbb{1}_{\{T\geq\tau\}}\mid\mathscr{F}_T]]+E[f_2(X_T)E[\mathbb{1}_{\{T<\tau\}}\mid\mathscr{F}_T]]\}\\&= e^{-rT}\{E[f_1(X_T)(1-e^{-\Gamma_T})]+E[f_2(X_T)e^{-\Gamma_T}]\}\\&= e^{-rT}E[f_1(X_T)(1-e^{-\Gamma_T})+f_2(X_T)e^{-\Gamma_T}] =: e^{-rT}E[\tilde{f}(X_T,\Gamma_T)].\end{aligned}$$

In the next step, we write the martingale $E[\tilde{f}(X_T,\Gamma_T)\mid\mathscr{F}_t]$ as a function of the processes

$$V_t^1 := L_t,\quad V_t^2 := Z_t,\quad V_t^3 := Y_t := \int_0^t e^{\kappa s}dZ_s \quad\text{and}\quad t.$$

We remark here that $e^{-r(T-t)}E[\tilde{f}(X_T,\Gamma_T)\mid\mathscr{F}_t]$ does not represent the option price at time t. It is only an auxiliary function that is needed for the calculation of π_0. The correct option price at time t would be given by $e^{-r(T-t)}E[\tilde{f}(X_T,\Gamma_T)\mid\mathscr{G}_t]$.

Lemma 5. *Let $(X_t)_{t\geq0}$ be a semimartingale with independent increments and let $f:[0,\infty)\to\mathbb{R}$ be a locally bounded, deterministic and left-continous function. Then the semimartingale $(Y_t)_{t\geq0}$ defined by*

$$Y_t := \int_0^t f(s)dX_s$$

has independent increments as well.

Proof. Due to Theorem II.4.15 in Jacod and Shiryaev (2003), there is a version of the characteristics of X, which is deterministic. The characteristics of Y can be calculated by only using the characteristics of X and the function f, see Proposition IX.5.3 in Jacod and Shiryaev (2003). Consequently, there is a version of the characteristics of Y, which is deterministic. So Theorem II.4.15 gives us the intended result. □

Lemma 6. *The conditional expectation $E[\tilde{f}(X_T,\Gamma_T)\mid\mathscr{F}_t]$ is a function of L_t, Z_t, Y_t and t*

$$E[\tilde{f}(X_T,\Gamma_T)\mid\mathscr{F}_t] = g(L_t,Z_t,Y_t,t). \tag{18}$$

Proof. First of all, we note that Γ_t is a function of Z_t, Y_t and t

$$\Gamma_t = \Gamma_t^d + \int_0^t \frac{1-e^{-\kappa(t-s)}}{\kappa}dZ_s = \Gamma_t^d + \frac{1}{\kappa}\left[Z_t - e^{-\kappa t}\int_0^t e^{\kappa s}dZ_s\right],$$

and that $\Gamma_T-\Gamma_t$ is a function of Z_T-Z_t, Y_T-Y_t, Y_t and t

$$\begin{aligned}\Gamma_T - \Gamma_t &= \Gamma_T^d - \Gamma_t^d + \frac{1}{\kappa}\big[Z_T - Z_t - e^{-\kappa T} Y_T + e^{-\kappa t} Y_t\big]\\ &= \Gamma_T^d - \Gamma_t^d + \frac{1}{\kappa}\big[Z_T - Z_t - (e^{-\kappa T} - e^{-\kappa t})Y_t - e^{-\kappa T}(Y_T - Y_t)\big].\end{aligned}$$

Consequently,

$$\begin{aligned}X_T =& \log S_0 + rT + \omega T + L_T - \zeta Z_T + \Gamma_T\\ =& \log S_0 + rT + \omega T + L_T - L_t + L_t - \zeta(Z_T - Z_t + Z_t)\\ &+ \Gamma_T - \Gamma_t + \Gamma_t\end{aligned}$$

is a function of

(a) the increments $L_T - L_t$, $Z_T - Z_t$, $Y_T - Y_t$,
(b) the random variables L_t, Z_t, Y_t and t.

L and Z are Lévy processes, and so Lemma 5 shows that all increment terms under (a) are independent of $\mathscr{F}_t$. The terms under (b) are $\mathscr{F}_t$-measurable. Hence, we get the intended result

$$\begin{aligned}E[\tilde{f}(X_T, \Gamma_T) \mid \mathscr{F}_t] &= E[\widehat{f}(L_T - L_t, Z_T - Z_t, Y_T - Y_t, L_t, Z_t, Y_t, t) \mid F_t]\\ &= E[\widehat{f}(L_T - L_t, Z_T - Z_t, Y_T - Y_t, x, y, z, t)]\big|_{x=L_t, y=Z_t, z=Y_t}.\end{aligned}$$

□

Theorem 1. *Assume that the function $g(x,y,z,t)$, defined in* (18)*, is of class $C^2(\mathbb{R}^4)$ and that L_1 and Z_1 have a finite first moment. Then g satisfies the following integro-differential equation*

$$\begin{aligned}0 =& E[L_1]\partial_1 g + E[Z_1]\partial_2 g + E[Z_1]e^{\kappa t}\partial_3 g + \partial_4 g + \frac{1}{2}c_L \partial_{11} g\\ &+ \int_{\mathbb{R}} [g(x+\xi, y, z, t) - g - \xi\partial_1 g]\nu_L(d\xi)\\ &+ \int_{\mathbb{R}} [g(x, y+\xi, z+e^{\kappa t}\xi, t) - g - \xi\partial_2 g - e^{\kappa t}\xi\partial_3 g]\nu_Z(d\xi)\end{aligned} \tag{19}$$

with boundary condition

$$g(x,y,z,T) = f_1(b_2(x,y,z,T))(1 - e^{-b_1(x,y,z,T)}) + f_2(b_2(x,y,z,T))e^{-b_1(x,y,z,T)},$$

where we have abbreviated $g = g(x,y,z,t)$ and

$$\begin{aligned}b_1(x,y,z,t) &:= \Gamma_t^d + \frac{1}{\kappa}(y - e^{-\kappa t} z),\\ b_2(x,y,z,t) &:= \log S_0 + rt + \omega t + x - \zeta y + b_1(x,y,z,t).\end{aligned}$$

ν_L and ν_Z are the Lévy measures of the processes L and Z. c_L denotes the variance of the Brownian part of L.

Proof. We denote $V_t = (V_t^1 = L_t, V_t^2 = Z_t, V_t^3 = Y_t)$ and apply Itô's formula (16), cf. Theorem I.4.57 in Jacod and Shiryaev (2003). The existence of the first moment gives us a simple semimartingale representation for the Lévy process L

$$L_t = L_t - tE[L_1] + tE[L_1] =: M_t^L + tE[L_1].$$

As a consequence, we obtain the semimartingale representation of the stochastic integral $\int H_s dL_s$

$$\int_0^t H_s dL_s = \int_0^t H_s dM_s^L + E[L_1]\int_0^t H_s ds,$$

where H is a locally bounded predictable process. The first summand is a local martingale, cf. I.4.34 (b) in Jacod and Shiryaev (2003). We are interested in the second one, which is a predictable process with paths of bounded variation. The same procedure can be applied to the increasing Lévy process Z. Therefore, we get the representations

$$\int_0^t H_s dZ_s = \int_0^t H_s dM_s^Z + E[Z_1]\int_0^t H_s ds\ ,$$
$$\int_0^t H_s dY_s = \int_0^t H_s e^{\kappa s} dZ_s = \int_0^t H_s e^{\kappa s} dM_s^Z + E[Z_1]\int_0^t H_s e^{\kappa s} ds\ .$$

Since Z is an increasing Lévy process, we have $Z^c \equiv 0$ and also $Y^c \equiv 0$. Thus, the second term of Itô's formula is simplified considerably

$$\frac{1}{2}\sum_{i,j\leq d}\int_0^t \partial_{ij} g(V_{s-},s)d\langle (V^i)^c,(V^j)^c\rangle_s = \frac{1}{2}c_L\int_0^t \partial_{11} g(V_{s-},s)ds.$$

The jump term in Itô's formula can be written in terms of the jump measure $\mu^{(L,Z)}$ of the two-dimensional Lévy process (L,Z)

$$\begin{aligned}
&\sum_{s\leq t}\left[g(V_{s-}+\Delta V_s,s) - g(V_{s-},s) - \sum_{i\leq d}\partial_i g(V_{s-},s)\Delta V_s^i\right]\\
&=\sum_{s\leq t}\Big[g(L_{s-}+\Delta L_s, Z_{s-}+\Delta Z_s, Y_{s-}+e^{\kappa s}\Delta Z_s,s) - g(V_{s-},s)\\
&\qquad -\partial_1 g(V_{s-},s)\Delta L_s - \partial_2 g(V_{s-},s)\Delta Z_s - \partial_3 g(V_{s-},s)e^{\kappa s}\Delta Z_s\Big]\\
&=\int_{[0,t]\times\mathbb{R}^2}\Big[g(L_{s-}+x, Z_{s-}+y, Y_{s-}+e^{\kappa s}y,s) - g(V_{s-},s)\\
&\qquad -\partial_1 g(V_{s-},s)x - \partial_2 g(V_{s-},s)y - \partial_3 g(V_{s-},s)e^{\kappa s}y\Big]\mu^{(L,Z)}(ds,(dx,dy)).
\end{aligned}$$

The semimartingale representation of this type of integral is

$$W*\mu^V = \underbrace{W*\mu^V - W*(\mu^V)^p}_{\text{martingale}} + \underbrace{W*(\mu^V)^p}_{\text{pred. + bounded variation}}\ ,$$

cf. Theorem II.1.8. in Jacod and Shiryaev (2003). So, we have to investigate the predictable compensator of the jump measure $\mu^{(L,Z)}$, which is

$$\left(\mu^{(L,Z)}\right)^p(\omega;dt,(dx,dy)) = dt \otimes \nu_{(L,Z)}(dx,dy),$$

where $\nu_{(L,Z)}$ is the Lévy measure of (L,Z). Since the processes L and Z are independent, $\nu_{(L,Z)}$ is supported on the union of the coordinate axes and we can write

$$\nu_{(L,Z)}(A) = \nu_L(A_x) + \nu_Z(A_y),$$

where $A_x := \{(x,0) \mid x \in A\}$ is the projection on the x-axis and $A_y := \{(0,y) \mid y \in A\}$ the projection on the y-axis. This result can be found in Sato (1999), E 12.10.(i) or Cont and Tankov (2004), Proposition 5.3. Consequently, each two-dimensional integral w.r.t. $\nu_{(L,Z)}$ is the sum of two one-dimensional integrals

$$\int g(x,y)\nu_{(L,Z)}(dx,dy) = \int g(x,0)\nu_L(dx) + \int g(0,y)\nu_Z(dy). \tag{20}$$

As a result, the predictable and bounded variation part of the jump term is

$$\begin{aligned}
&\int_{[0,t]\times\mathbb{R}^2} \Big[g(L_{s-}+x, Z_{s-}+y, Y_{s-}+e^{\kappa s}y, s) - g(V_{s-},s) \\
&\qquad - \partial_1 g(V_{s-},s)x - \partial_2 g(V_{s-},s)y - \partial_3 g(V_{s-},s)e^{\kappa s}y\Big] ds \otimes \nu_{(L,Z)}(dx,dy) \\
&= \int_0^t \int_{\mathbb{R}} \Big[g(L_{s-}+x, Z_{s-}, Y_{s-}, s) - g(V_{s-},s) - \partial_1 g(V_{s-},s)x\Big] \nu_L(dx) \\
&\qquad + \int_{\mathbb{R}} \Big[g(L_{s-}, Z_{s-}+y, Y_{s-}+e^{\kappa s}y, s) - g(V_{s-},s) \\
&\qquad\qquad - \partial_2 g(V_{s-},s)y - \partial_3 g(V_{s-},s)e^{\kappa s}y\Big] \nu_Z(dy)\ ds.
\end{aligned}$$

If we now zero all the predictable parts of Itô's formula with bounded variation, we obtain

$$0 = \int_0^t H(L_{s-}, Z_{s-}, Y_{s-}, s)ds \qquad (\forall t \geq 0)$$

for

$$\begin{aligned}
H(x,y,z,t) :=& E[L_1]\partial_1 g + E[Z_1]\partial_2 g + E[Z_1]e^{\kappa t}\partial_3 g + \partial_4 g + \frac{1}{2}c_L \partial_{11} g \\
&+ \int_{\mathbb{R}} [g(x+\xi,y,z,t) - g - \xi\partial_1 g]\nu_L(d\xi) \\
&+ \int_{\mathbb{R}} [g(x,y+\xi,z+e^{\kappa t}\xi,t) - g - \xi\partial_2 g - e^{\kappa t}\xi\partial_3 g]\nu_Z(d\xi),
\end{aligned}$$

where we wrote for short $g = g(x,y,z,t)$. By continuity, $H(x,y,z,t)$ has to be zero for every $t \geq 0$, every $x \in S(L_t)$, every $y \in S(Z_t)$ and every $z \in S(Z_t)$, whereby $S(X)$ denotes the support of the random variable X. This is the desired equation (19). $\square$

In many cases, we have $S(L_t) = \mathbb{R}$ and $S(Z_t) = S(Y_t) = \mathbb{R}_+$, such that we have to solve equation (19) for $x \in \mathbb{R}$, $y, z \in \mathbb{R}_+$ and $t \in \mathbb{R}_+$.

To apply the stated theorem, we have to verify that the function g, defined in (18), is of class $C^2(\mathbb{R}^4)$. The validity of this condition depends on the specific processes L and Z and on the payoff function f of the claim which we consider. Cont and Voltchkova (2005) investigated a similar issue in the simpler case of exponential Lévy models. The problem is more complicated in our model setting and is not pursued in this paper.

6 Two Price Theory

In the classical risk-neutral valuation theory for financial derivatives it is implicitly assumed that the product is traded in a perfectly liquid market, which means that it can be bought and sold at once within the trading session and that this does not cause any substantial price movement. Typical examples for assets which are traded in rather liquid markets are shares of big listed companies, the corresponding plain vanilla options on these shares and government bonds of countries with a high rating. Neglecting processing, inventory and transaction costs of the market makers, in these markets the law of one price prevails, which means that the price for buying an asset is the same as the one for selling it.

In reality however there are two prices, one for buying from the market - the ask price - and one for selling to the market - the bid price. ”The difference between these two prices can be quite large and may have little connection to processing, inventory, transactions costs or information considerations. The differences instead reflect the very real and substantial costs of holding unhedgeable risks in incomplete markets.”[1]. In particular a large part of the products financial institutions are dealing with are very specialised. The markets for these over-the-counter (OTC) traded structured products are very narrow with the consequence of large spreads between bid and ask prices..

Cherny and Madan (2010) started to develop a two price theory, which models bid and ask prices in a way which takes the cost of unhedgeable risks into account. In classical financial mathematics, cf. Delbaen and Schachermayer (2006), the price $\pi_0(X)$ of a derivative with discounted payoff X is calculated via

$$\pi_0(X) = E_P[X],$$

where P is a risk-neutral pricing measure. This formula is now substituted by the non-linear pricing formulas

[1] Cherny and Madan (2010), Introduction, p. 1150

$$b(X) = \inf_{Q \in \mathscr{D}} E_Q[X]$$
$$a(X) = \sup_{Q \in \mathscr{D}} E_Q[X]$$

for the bid and the ask price of an asset with discounted payoff X. $\mathscr{D}$ is a convex set of probability measures which contains a risk-neutral measure P. The size of $\mathscr{D}$ is related to the degree of uncertainty (liquidity) in the market under consideration. With increasing uncertainty more measures (scenarios) should be added to the set. Conversely, $\mathscr{D}$ could be shrunk when the uncertainty in the market decreases.

Under slight additional assumptions, namely comonotonicity and law-invariance, these two values can be calculated using concave distortions Ψ, more exactly

$$b(X) = \int_{\mathbb{R}} y \Psi(F_X(dy)) \tag{21}$$

$$a(X) = -\int_{\mathbb{R}} y \Psi(F_{-X}(dy)), \tag{22}$$

where F_X is the distribution function of X under P. Very useful parametrized families of distortions $(\Psi_\gamma)_{\gamma \geq 0}$ are presented in the following example.

Example 7. The MINVAR-family of distortions is defined by

$$\Psi_\gamma^{\mathsf{MI}}(y) := 1 - (1-y)^{\gamma+1}, \quad \gamma \geq 0,\ y \in [0,1].$$

Another family is given by

$$\Psi_\gamma^{\mathsf{MA}}(y) := y^{\frac{1}{\gamma+1}}, \quad \gamma \geq 0,\ y \in [0,1]$$

and is called MAXVAR. One possible combination of MINVAR and MAXVAR is

$$\Psi_\gamma^{\mathsf{MAMI}}(y) := (1 - (1-y)^{\gamma+1})^{\frac{1}{1+\gamma}}, \quad \gamma \geq 0,\ y \in [0,1]$$

and is called MAXMINVAR. The other possible combination is

$$\Psi_\gamma^{\mathsf{MIMA}}(y) := 1 - (1 - y^{\frac{1}{\gamma+1}})^{\gamma+1}, \quad \gamma \geq 0,\ y \in [0,1]$$

and is called MINMAXVAR.

The existence of the integrals in (21) and (22) depends on the payoff X and the used distortion Ψ. The existence under the four introduced distortions is ensured, if X possesses exponential moments, as seen in the following proposition.

Proposition 1. *Let X be a random variable with $E[e^{tX}] < \infty$ for $|t| \leq t_0$. Then the integrals* (21) *and* (22) *exist for the distortion families Ψ^{MA}, Ψ^{MI}, Ψ^{MAMI}, Ψ^{MIMA} and any $\gamma \geq 0$.*

Proof. The assumption implies that the distribution function F_X of X decays exponentially. We consider the left tail of Ψ^{MA}

$$\int_{-\infty}^{0} \Psi_\gamma^{\mathsf{MA}}(F_X(y))dy \le \int_{-\infty}^{0} \Psi_\gamma^{\mathsf{MA}}(Ce^{t_0 y})dy = C^{\frac{1}{1+\gamma}} \int_{-\infty}^{0} e^{\frac{t_0}{1+\gamma}y}dy < \infty$$

and the left tail of Ψ^{MI}

$$\begin{aligned}
\int_{-\infty}^{0} \Psi_\gamma^{\mathsf{MI}}(F_X(y))dy &\le \int_{-\infty}^{0} \Psi_\gamma^{\mathsf{MI}}(Ce^{t_0 y})dy \\
&= \int_{-\infty}^{0} 1-(1-Ce^{t_0 y})^{1+\gamma}dy \\
&\le C_1 + \int_{-\infty}^{-d^2} 1-(1+(1+\gamma)(-Ce^{t_0 y}))dy \\
&= C_1 + \int_{-\infty}^{-d^2} (1+\gamma)Ce^{t_0 y}dy < \infty,
\end{aligned}$$

where we have used Bernoulli's inequality

$$(1+x)^r \ge 1+rx \qquad (x > -1,\ r \ge 1).$$

The same arguments show the statement for the right tails of Ψ^{MI} and Ψ^{MA} and for both tails of the distortion families Ψ^{MAMI} and Ψ^{MIMA}. □

Example 8. Since the payoff $P = (K - S_T)^+$ of a put option always possesses exponential moments if $S_T \ge 0$, the bid and ask prices always exist and are given by

$$a_\gamma(P) = \int_0^K \Psi_\gamma(F_{S_T}(x))dx \tag{23}$$

$$b_\gamma(P) = \int_0^K (1 - \Psi_\gamma(1 - F_{S_T}(x)))dx. \tag{24}$$

The payoff $C = (S_T - K)^+$ of a call option does not possess exponential moments in general for nonnegative random variables S_T. Consider $S_T = S_0 \exp(Y)$ for a random variable Y with exponential moment at $u_0 > 1$. Let Ψ be the MINVAR-family of distortions. Then the integrals (21) and (22) exist for every $\gamma \ge 0$ and we get

$$a_\gamma(C) = \int_K^\infty \Psi_\gamma(1 - F_{S_T}(x))dx \tag{25}$$

$$b_\gamma(C) = \int_K^\infty (1 - \Psi_\gamma(F_{S_T}(x)))dx. \tag{26}$$

Let Ψ be the MAXVAR-, MAXMINVAR- or MINMAXVAR-family of distortions. Then the integrals exist for every $\gamma \in [0, u_0 - 1)$ and the formulas (25) and (26) are in force for $\gamma \in [0, u_0 - 1)$. The proofs are similar to that of Proposition 1. Details can be found in Bäurer (2015).

We now apply the two price theory to the Defaultable Asset Price Model and derive bid and ask prices for options. As a consequence, we get prices for which market, credit and liquidity risk is taken into account. The bid and ask price formulas

(21) and (22) depend on the distribution function F_X of the option payoff X. In many cases, it can be reduced to a dependence on F_{S_T}, the distribution function of the underlying S_T, cf. Example 8. In the DAM, the distribution function

$$F_{S_T}(x) = P(T \geq \tau) + P(e^{X_T} \leq x \text{ and } T < \tau)$$

of the asset price S_T is not known explicitly, because of the dependence between X_T and τ. Nevertheless one can calculate the desired values numerically. Using Lemma 1, the quantities $P(T < \tau)$ and $P(T \geq \tau) = 1 - P(T < \tau)$ are given by a simple integral

$$P(T < \tau) = E[e^{-\Gamma_T}] = e^{-\Gamma_T^d} \exp\left(\int_0^T \theta_Z(-\gamma_u^T)du\right).$$

We use the T-survival measure $P^T(A) := P(A \mid T < \tau)$ to determine

$$\begin{aligned} P(e^{X_T} \leq x \text{ and } T < \tau) &= P(e^{X_T} \leq x \mid T < \tau) \cdot P(T < \tau) \\ &= P^T(e^{X_T} \leq x) \cdot P(T < \tau). \end{aligned}$$

The probability $P^T(e^{X_T} \leq x)$ can be calculated numerically by Fourier inversion

$$\begin{aligned} P^T(e^{X_T} \leq x) &= P^T(X_T \leq \log(x)) \approx P^T(C \leq X_T \leq \log(x)) \\ &= \frac{1}{2\pi} \int \frac{e^{-itC} - e^{-it\log(x)}}{it} \varphi_{X_T}^{P^T}(t)dt, \end{aligned} \tag{27}$$

where the constant $C \in \mathbb{R}$ has to be chosen properly. $\varphi_{X_T}^{P^T}$ is the characteristic function of X_T under P^T, which can be calculated by integration via (12). Thus, the computational cost for calculating the distribution function at one point is that of two simple integrations and one double integration.

Alternatively, we can compute the distribution function F_{S_T} by Monte Carlo simulations. We can then also assess the bid and ask prices for path-dependent options.

For the existence of the integrals in (21) and (22), we often need the existence of exponential moments of

$$X_T := \log S_0 + rT + L_T - \zeta Z_T + \omega T + \Gamma_T.$$

Lemma 7. *Suppose that*

(I) L_T has an exponential moment of order $u_0 > 0$.
(II) Z_T has an exponential moment of order $u_0[(\frac{1}{\kappa} - \zeta) \vee \zeta]$.

Then X_T has an exponential moment of order u_0.

Proof. First we observe that $|\gamma_s^T - \zeta| \leq (\frac{1}{\kappa} - \zeta) \vee \zeta$ and therefore we can conclude

$$
\begin{aligned}
E[\exp(u_0 X_T)] &= \text{const.} \cdot E\left[\exp\left(u_0 L_T - u_0 \zeta Z_T + u_0 \Gamma_T\right)\right] \\
&= \text{const.} \cdot E[\exp(u_0 L_T)] E\left[\exp\left(\int_0^T \gamma_s^T u_0 - \zeta u_0 dZ_s\right)\right] \\
&\leq \text{const.} \cdot M_{L_T}(u_0) \cdot E\left[\exp\left(\int_0^T u_0 |\gamma_s^T - \zeta| dZ_s\right)\right] \\
&\leq \text{const.} \cdot M_{L_T}(u_0) \cdot E\left[\exp\left(u_0 \left[\left(\frac{1}{\kappa} - \zeta\right) \vee \zeta\right] Z_T\right)\right] < \infty.
\end{aligned}
$$

□

Example 9. For pricing calls and puts, we can use (23), (24), (25) and (26). Suppose X_T has an exponential moment at $u_0 > 1$. If Ψ is the MINVAR-family of distortions, then the integrals in (25) and (26) exist for every $\gamma \geq 0$. If Ψ is the MAXVAR-, MAXMINVAR- or MINMAXVAR-family of distortions Ψ^{MI}, then the integrals exist for every $\gamma \in [0, u_0 - 1)$. A numerical example with the parameter set

$$
\begin{array}{lll}
\alpha = 50.0 & \beta = -0.1 & \delta = 0.012 \\
p = 0.0035 & b = 66 & \kappa = 0.11 \\
& \zeta = 9.0. &
\end{array}
\qquad (**)
$$

is shown in Figure 7.

Fig. 7 Bid and ask prices of a put with $S_0 = 30$, DAM with parameters $(**)$, $T = 260$, $\gamma = 0.1$, MAXVAR.

Example 10. For a digital call option with barrier $B > 0$ and payoff $X = \mathbb{1}_{\{S_T > B\}}$, we can use the simple formulas

$$
\begin{aligned}
a_\gamma(X) &= \Psi_\gamma(1 - F_{S_T}(B)) \qquad \text{and} \\
b_\gamma(X) &= 1 - \Psi_\gamma(F_{S_T}(B)).
\end{aligned}
$$

Figure 8 shows a numerical example. For this option, there are no constraints concerning the integrability.

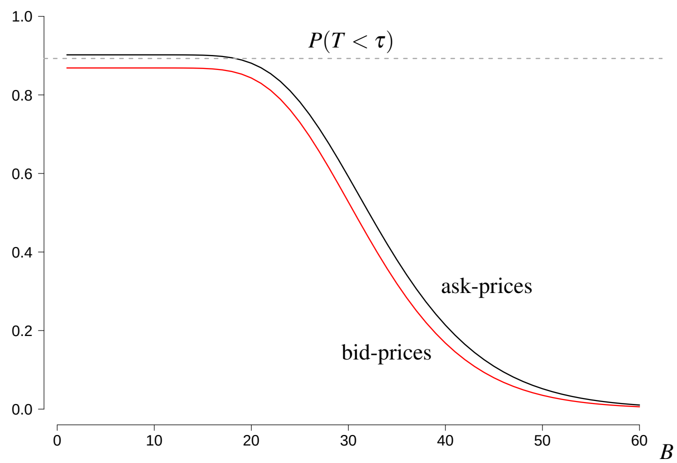

Fig. 8 Bid and ask prices of a digital call with $S_0 = 30$, DAM with parameters $(**)$, $T = 260$, $\gamma = 0.1$, MAXVAR.

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