VARIATIONAL SOLUTIONS OF THE PRICING PIDES FOR EUROPEAN OPTIONS IN LÉVY MODELS

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Abstract. One of the fundamental problems in financial mathematics is to develop efficient algorithms for pricing options in advanced models such as those driven by Lévy processes. Essentially there are three approaches in use. These are Monte Carlo, Fourier transform and PIDE based methods. We focus our attention here on the latter. There is a large arsenal of numerical methods for efficiently solving parabolic equations that arise in this context. Especially Galerkin and Galerkin inspired methods have an impressive potential. In order to apply these methods, what is required is a formulation of the equation in the weak sense.

The contribution of this paper is therefore to analyze weak solutions of the Kolmogorov backward equations which are related to prices of European options in (time-inhomogeneous) Lévy models and to establish a precise link between the prices and the weak solutions of these equations. The resulting relation is a Feynman–Kac representation of the solution as a conditional expectation. Our special concern is to provide a framework that is able to cover both, the common types of European options and a wide range of advanced models in which these derivatives are priced.

An application to financial models requires in particular to admit pure jump processes such as generalized hyperbolic processes as well as unbounded domains of the equation. In order to deal at the same time with the typical payoffs which can arise, the weak formulation of the equation is based on exponentially weighted Sobolev–Slobodeckii spaces. We provide a number of examples of models that are covered by this general framework. Examples of options for which such an analysis is required are calls, puts, digital and power options as well as basket options.

1. Introduction

There is little doubt that Lévy models constitute a big step forward in more realistic modeling in finance. Nevertheless there is a price to be paid for this progress. The gain in accuracy comes along with a substantial increase in mathematical and computational sophistication. Of course this phenomenon is observable allover in a technical world. Better products require more advanced techniques. As far as the use of Lévy models is concerned one of the fundamental problems is the explicit computation of prices of derivatives. Those prices appear as expectations under suitable martingale measures. In particular for calibration purposes efficient algorithms to get the prices are crucial. During an iterative calibration procedure

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typically a large number of model prices has to be compared to observed market prices. Essentially there are three approaches to compute the expectations: Monte Carlo simulation, Fourier based valuation methods and the representation of prices as solutions of partial integro-differential equations (PIDEs).

Monte Carlo simulation is a very powerful tool which usually works when other methods are not available. Its disadvantage is that it is computer intensive and consequently time-consuming and expensive. There is another more intrinsic problem when Monte Carlo simulation is used in the case of processes with jumps. The most attractive processes with jumps in financial modeling – such as generalized hyperbolic or CGMY processes – are processes with infinite activity. Their paths have an infinite number of jumps in every finite time interval. Such a behavior cannot be simulated. Therefore one usually neglects the very small jumps by cutting out the infinite mass which the Lévy measure of these processes accumulates around the origin. This way the Lévy measure becomes a finite measure. As a consequence the infinite activity process is replaced by a compound Poisson process. An alternative method which is in use consists in replacing the small jumps by an appropriately scaled Brownian motion. In any case the simulated paths are then by construction approximations of the model paths only.

The two remaining approaches, Fourier and PIDE-based methods, a priori look very different. To some degree they are employed by communities with a different mathematical background. Whereas Fourier methods are mainly used by probabilists, the analytic or numerical solution of PIDEs requires a background in analysis and numerics. References for the numerical solution of PIDEs – on the level of models driven by Lévy processes – are Matache, von Petersdorff and Schwab (2004), Matache, Schwab and Wihler (2005b), Matache, Nitsche and Schwab (2005a) and Winter (2009). For a compact survey see Hilber, Reich, Schwab and Winter (2009). Fourier based methods in finance started out with Carr and Madan (1999) and Raible (2000) and were in subsequent years pushed forward to compute prices for a large variety of options in equity, fixed income, foreign exchange and credit markets. For a recent survey see Eberlein (2013).

What is used in these two approaches is Fourier transformation and discretization of the resulting integrals on one and finite difference or finite element methods on the other side. As different as these approaches look from the point of view of numerics, the mathematical tools used to describe the solutions in an analytic way are rather similar if not the same. There is in fact a formal bridge, which shows that the conditional expectation which represents the price of an option is at the same time the solution of a partial integro-differential equation. This is the Feynman–Kac formula. It represents a deep relation between a stochastic and a deterministic quantity. The Feynman–Kac correspondence as a relation between conditional expectations and partial differential equations has a long history. It was initially studied in the case of diffusions and then pushed further to more general processes. It is one of the purposes of this paper to study this correspondence in a framework which covers advanced financial modeling approaches and at the same time large classes of derivatives which are priced in these models. From the point of view of processes which drive the models we choose the class of time-inhomogeneous Lévy processes, also called processes with independent increments and absolutely continuous characteristics (PIIAC) in Jacod and Shiryaev (1987). This class which generalizes Lévy processes turned out to be a natural choice in particular in the context of interest rate modeling (see e.g. Eberlein, Jacod and Raible (2005) and Eberlein and Kluge (2006)). One reason why one has to go beyond the class of Lévy processes in this context is the use of forward martingale measures.

The Feynman–Kac relation has also been investigated in the framework of viscosity solutions. For a recent reference in this context see Cont and Voltchkova (2005) where PIDEs related to Lévy driven models are studied. Nevertheless the notion of viscosity solution is not the optimal choice if one looks for efficient numerical algorithms. With respect to well-developed variational discretization procedures the relation has to be available for variational solutions. Results for processes with jumps were proved already much earlier, see Bensoussan and Lions (1982). However the application in financial models requires in particular to admit pure jump processes such as generalized hyperbolic processes as well as unbounded domains. Processes without diffusion part are not covered by the last mentioned reference. In order to deal at the same time with the typical payoffs, the weak formulation of the equation is based on weighted Sobolev–Slobodeckii spaces. By consequence, the results from Bensoussan and Lions (1982) are not applicable.

PIDEs on unbounded domains arise very naturally in financial models since the domain is identical to the range of values of the underlying stochastic process. For example in the case of a European option whose payoff is written as a function of the logarithmic stock price in an exponential Lévy model, this range is the whole real line. For completeness we mention that studying bounded domains can be sufficient for specific types of exotic options. As an example let us consider an option which has a positive payoff only as long as during its entire life the underlying stock price stays inside the interval $(S_0 \,\mathrm{e}^a, S_0 \,\mathrm{e}^b)$. The price of such a two-sided barrier option corresponds to a PIDE on the bounded interval (a, b). In general a systematic study of PIDEs for option prices should include – if not start with – unbounded domains. The issue of unbounded domains is also important with regard to numerical procedures. Of course, one possibility when solving the PIDE numerically is to start with a truncation of the domain to a bounded one as described in e.g. Hilber, Reich, Schwab and Winter (2009). Then the numerical procedure does not reflect the unboundedness of the domain. Recent developments in numerical analysis and in computational finance show that specific Galerkin and Galerkin-inspired methods – that rely on the formulation of the variational equation on an unbounded domain – are an attractive alternative. We mention here the use of reduced basis methods for pricing and calibration, see Cont, Lantos and Pironneau (2011) and an application of adaptive wavelet Galerkin methods designed for unbounded domains, see Kestler and Urban (2012). The advantage of the latter comes from the fact that during the numerical procedure the truncated domain is chosen adaptively. The Feynman–Kac relation which is achieved in Section 6 refers to weak solutions of the evolution problem in the appropriate generality.

The second important aspect which has to be taken into account in applications in finance is the boundary or initial condition. The theory has to be developed in such a way that the payoffs of standard options – which actually appear as initial conditions – are covered. Let us mention that assuming polynomial boundedness of the payoff function, written as a function of the logarithm of the stock price, eliminates standard options such as a plain vanilla European calls, since the payoff is in this case exponential. More specifically it is of the form

$$h(x) = (S_0 \,\mathrm{e}^x - K)^+.$$

Cont and Voltchkova (2005) assume in addition to polynomial boundedness Lipschitz-continuity of the payoff written as a function of the stock price itself. This continuity assumption eliminates digital options as well as any other product with discontinuous payoff which is a frequent feature in structured products. As an example we mention the contingent premium option which is one of the simplest structured products. In this case the buyer of the option pays no premium upfront but agrees to pay a predetermined premium if the option has any value at expiration.

Although this article is devoted to pricing European options, the present framework is useful in the context of pricing path dependent instruments such as barrier, lookback, or American options. PDE methods allow efficient pricing for these options as well. The variational formulation of the equation related to barrier options relies on Sobolev spaces restricted to functions vanishing outside of a certain domain. For American options, one has to treat a variational inequality instead of an equality. In both cases, the variational formulation is closely linked to the European option pricing equation. Moreover, an efficient numerical algorithm implemented for computing European option prices can be adapted to compute American, barrier and lookback options, see Hilber, Reich, Schwab and Winter (2009) and Glau (2010).

Technically the key point in the following is to enforce sufficient integrability for the initial condition. This is achieved by dampening the payoff, i.e. by multiplying it with an exponential function. Dampening is also a major tool used in Fourier based methods (see e.g. Eberlein, Glau and Papapantoleon (2010)) and allows one to include not only standard options like calls and puts, but also many exotics. As simple as dampening by an exponential function is, it requires to extend the existing theory on the solution of parabolic equations in the proper way by introducing weighted spaces. Therefore after summarizing some basic properties of the underlying stochastic processes in Section 2, we first define and study in Section 3 exponentially weighted Schwartz as well as exponentially weighted Sobolev–Slobodeckii spaces. Pseudo-differential operators on these weighted spaces are discussed in Section 4. In Section 5 we prove the basic existence and uniqueness result for the solution of the parabolic equation under appropriate assumptions on the symbol of the associated pseudo-differential operator. Using Fourier transformation an explicit solution in the homogeneous case is derived in Section 6. This explicit solution of the parabolic equation presents itself exactly in the form which is well-known from the competing Fourier based approach to compute option prices which are given as conditional expectations. The Feynman–Kac relation is an immediate consequence. In order to illuminate the scope of this approach we discuss in Section 7 a number of examples of processes and options. We conclude with a numerical example.

2. Time-inhomogeneous Lévy processes, infinitesimal generator and symbol

This section provides the basic notation and preliminary results on the symbol of time-inhomogeneous Lévy processes. Lévy processes are adapted stochastic processes with càdlàg paths with stationary and independent increments. The wider class of time-inhomogeneous Lévy processes, also called PIIAC (process with independent increments and absolutely continuous characteristics), consists of those adapted stochastic processes with càdlàg paths, that have independent increments, compare Eberlein, Jacod and Raible (2005). This class of processes is closely related to the class of additive processes, in particular, every time-inhomogeneous Lévy process is an additive process, see Sato (1999) and Kluge (2005, Lemma 1.3).

An introduction to Lévy processes is provided in Sato (1999), Bertoin (1996), Kyprianou (2006), and Applebaum (2009). Details on time-inhomogeneous Lévy processes can e.g. be found in Kluge (2005).

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a stochastic basis. The distribution of an $\mathbb{R}^d$-valued time-inhomogeneous Lévy process L is determined by the characteristic functions of the distributions of

L_t for $t \geq 0$,

$$E\,\mathrm{e}^{i\langle \xi, L_t\rangle} = \mathrm{e}^{\int_0^t \theta_s(i\xi)\,\mathrm{d}s}\,, \tag{1}$$

where the cumulant function θ_s for any fixed $s \geq 0$ equals

$$\theta_s(i\xi) = -\frac{1}{2}\langle \xi, \sigma_s \xi\rangle + i\langle \xi, b_s\rangle + \int\limits_{\mathbb{R}^d} \Big(\mathrm{e}^{i\langle \xi, y\rangle} - 1 - i\langle \xi, h(y)\rangle\Big)\, F_s(\mathrm{d}y) \tag{2}$$

for a truncation function $h : \mathbb{R}^d \to \mathbb{R}$. A bounded measurable function $h : \mathbb{R}^d \to \mathbb{R}$ with compact support is called a *truncation function*, if $h(x) = x$ in a neighborhood of 0.

Here $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^d$. Furthermore, for every $s > 0$, σ_s is a symmetric, positive semi-definite $d \times d$-matrix, $b_s \in \mathbb{R}^d$, and F_s is a Lévy measure, i.e. a Borel measure on $\mathbb{R}^d$ with $\int (|x|^2 \wedge 1) F_s(\mathrm{d}x) < \infty$. The maps $s \mapsto \sigma_s$, $s \mapsto b_s$ and $s \mapsto \int (|x|^2 \wedge 1) F_s(\mathrm{d}x)$ are Borel-measurable with

$$\int\limits_0^T \Big(|b_s| + \|\sigma_s\|_{\mathcal{M}(d\times d)} + \int\limits_{\mathbb{R}^d} (|x|^2 \wedge 1) F_s(\mathrm{d}x) \Big)\, \mathrm{d}s < \infty$$

for every $T > 0$, where $\|\cdot\|_{\mathcal{M}(d\times d)}$ is a norm on the vector space formed by the $d \times d$-matrices. For Lévy processes L, the identity (1) is the Lévy–Khintchine formula, and in this case the quantities b, σ, F and θ do not depend on s.

The canonical representation of the process is, according to Jacod and Shiryaev (1987, Theorem II.2.34), given by

$$L = \int\limits_0^{\cdot} b_s\,\mathrm{d}s + L^c + h * (\mu - \nu) + \big(x - h(x)\big) * \mu\,,$$

where L^c denotes the continuous martingale part of L, μ is the random measure of jumps of the process L, and ν is the predictable compensator of μ. The integral process with respect to μ is defined as $\big(x - h(x)\big) * \mu_t(\omega) := \int_0^t \int_{\mathbb{R}^d} \big(x - h(x)\big)\mu(\omega, \mathrm{d}s, \mathrm{d}x)$, moreover, $h*(\mu-\nu)$ denotes the stochastic integral of h with respect to $(\mu - \nu)$. It is defined as any purely discontinuous local martingale M such that the jumps of M, ΔM, and $\Delta L \mathbb{1}_{\Delta L \neq 0}$ are indistinguishable. The continuous martingale part L^c can be written in the form $L^c = \int_0^{\cdot} \sigma_s^{1/2}\,\mathrm{d}W_s$ with a standard Brownian motion W with values in $\mathbb{R}^d$, see Karatzas and Shreve (1991, Theorem 3.4.2). Choosing the truncation function $h(x) := x\mathbb{1}_{\{|x|\leq 1\}}(x)$, one obtains the more explicit representation

$$L = \int\limits_0^{\cdot} b_s\,\mathrm{d}s + \int\limits_0^{\cdot} \sigma_s^{1/2}\,\mathrm{d}W_s + h * (\mu - \nu) + \sum_{s \leq t} \Delta L_s \mathbb{1}_{\{|\Delta L_s| > 1\}}\,.$$

In case L is a special semimartingale, we can choose h to be the identity, $h(x) = x$, which leads to the more convenient representation

$$L = \int\limits_0^{\cdot} b_s\,\mathrm{d}s + \int\limits_0^{\cdot} \sigma_s^{1/2}\,\mathrm{d}W_s + x * (\mu - \nu)$$

with different coefficients b_s, see Jacod and Shiryaev (1987, Corollary II.2.38).

Of special interest in the next sections is the infinitesimal generator $\mathcal{G}_s$ of time-inhomogeneous Lévy processes L, that is

$$\begin{aligned}\mathcal{G}_s\varphi(x) &= \frac{1}{2}\sum_{j,k=1}^{d}\sigma_s^{j,k}\frac{\partial^2\varphi}{\partial x_j\partial x_k}(x)+\sum_{j=1}^{d}b_s^j\frac{\partial\varphi}{\partial x_j}(x)\\ &\quad+\int_{\mathbb{R}^d}\Big(\varphi(x+y)-\varphi(x)-\sum_{j=1}^{d}\frac{\partial\varphi}{\partial x_j}(x)(h(y))_j\Big)F_s(\mathrm{d}y)\end{aligned} \tag{3}$$

for $\varphi\in C_0^2(\mathbb{R}^d,\mathbb{R})$, compare e.g. Dynkin (1965).

We define $\mathcal{A}_t := -\mathcal{G}_t$ for every $t\geq 0$. It turns out that $\mathcal{A}_t$ can be written in the form

$$\mathcal{A}_t u(x)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathrm{e}^{-i\langle\xi,x\rangle}A_t(\xi)\mathcal{F}(u)(\xi)\,\mathrm{d}\xi\qquad\text{for all } u\in C_0^\infty(\mathbb{R}^d,\mathbb{R})\,.$$

For short we write

$$\mathcal{A}_t u=\mathcal{F}^{-1}\big(A_t\mathcal{F}(u)\big)\qquad\text{for all } u\in C_0^\infty(\mathbb{R}^d,\mathbb{R})\,, \tag{4}$$

where

$$\begin{aligned}A_t(\xi) &:= \frac{1}{2}\langle\xi,\sigma_t\xi\rangle+i\langle\xi,b_t\rangle-\int_{\mathbb{R}^d}\Big(\mathrm{e}^{-i\langle\xi,y\rangle}-1+i\langle\xi,h(y)\rangle\Big)\,F_t(\mathrm{d}y)\\ &= -\theta_t(-i\xi)\qquad\qquad(\xi\in\mathbb{R}^d)\,,\end{aligned} \tag{5}$$

$\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ its inverse. It is standard to show that for each $t\geq 0$ there exists a constant $C_t>0$ such that

$$\big|A_t(\xi)\big|\leq C_t\big(1+|\xi|\big)^2\qquad\text{for all } \xi\in\mathbb{R}^d\,. \tag{6}$$

As a consequence, the Fourier inversion $\mathcal{F}^{-1}$ in (4) is well defined. (4) shows that $\mathcal{A}_t$ is a so-called pseudo differential operator (PDO) with symbol A_t. In analogy to the symbol of a Lévy process, compare Jacob (2001), we call the family $(A_t)_{t\in[0,T]}$ the symbol of the time-inhomogeneous Lévy process.

We outline in the following remarks and lemmas some properties of the symbol of time-inhomogeneous Lévy processes, where we focus on its analytic extension which allows to interpret the operator $\mathcal{A}$ as a continuous linear operator between exponentially weighted Sobolev–Slobodeckii spaces (see Section 4). In the sequel we restrict ourselves to a finite time horizon $[0,T]$ and we will concentrate on an analytic extension of the symbol to the domain

$$U_{-\eta}=U_{-\eta^1}\times\cdots\times U_{-\eta^d}\,, \tag{7}$$

which is defined for $\eta=(\eta^1,\ldots,\eta^d)\in\mathbb{R}^d$ by the strips $U_{-\eta^j}:=\mathbb{R}-i\,\mathrm{sgn}(\eta^j)[0,|\eta^j|)$ in the complex plane for $\eta^j\neq 0$. For $\eta^j=0$ we define $U_{-\eta^j}=U_0:=\mathbb{R}$. We denote by R_η the d-dimensional cuboid $R_\eta=\mathrm{sgn}(\eta^1)[0,|\eta^1|]\times\cdots\times\mathrm{sgn}(\eta^d)[0,|\eta^d|]$.

The following lemma generalizes Lemma 25.17 (ii) and (iii) in Sato (1999) from Lévy processes to time-inhomogeneous Lévy processes. In particular, we show that an analytic extension of the symbol to the domain $U_{-\eta}$ exists, if a related exponential moment condition is satisfied. For z, $w \in \mathbb{C}^d$ we define

$$\langle z, w \rangle := \sum_{j=1}^{d} z_j w_j \,.$$

Note that this is not the Hermetian scalar product in $\mathbb{C}^d$.

Lemma 2.1. *Let $\eta \in \mathbb{R}^d$.*

(a) $E\,\mathrm{e}^{\langle \eta, L_t \rangle} < \infty$ *for every $0 \le t \le T$ if and only if*

$$\int_0^T \int_{|x|>1} \mathrm{e}^{\langle \eta, x \rangle} \, F_s(\mathrm{d}x) \, \mathrm{d}s < \infty \,.$$

(b) *If $E\,\mathrm{e}^{\langle \eta, L_t \rangle} < \infty$ for every $0 \le t \le T$, then*

$$E\,\mathrm{e}^{\langle i\xi + \eta, L_t \rangle} = \mathrm{e}^{\int_0^t \theta_s(i\xi+\eta)\,\mathrm{d}s} = \mathrm{e}^{-\int_0^t A_s(-\xi+i\eta)\,\mathrm{d}s}$$

for every $t \in [0,T]$ and $\xi \in \mathbb{R}^d$.

(c) *If $E\,\mathrm{e}^{\langle \eta', L_t \rangle} < \infty$ for every $0 \le t \le T$ and every $\eta' \in R_\eta$, then the maps $z \mapsto A_s(-z)$ for every $s \ge 0$ as well as*

$$z \mapsto E\,\mathrm{e}^{i\langle z, L_t \rangle} = \mathrm{e}^{\int_0^t \theta_s(iz)\,\mathrm{d}s} = \mathrm{e}^{-\int_0^t A_s(-z)\,\mathrm{d}s}$$

have a continuous extension to the domain $\overline{U_{-\eta}}$ which is analytic in the interior $\mathring{U}_{-\eta}$.

Proof. Parts (a) and (b) are straightforward extensions of Theorem 25.17 in Sato (1999) as shown more explicitly in Lemma 6 and formula (14) in Eberlein and Kluge (2006).

In order to prove (c), let γ_j be an arbitrary (compact) triangle, that lies inside the strip $U_{\eta^j} = \mathbb{R} + i\,\mathrm{sgn}(\eta^j)[0, |\eta^j|)$, and let $w = (w_1, \ldots, w_d) \in \mathbb{C}^d$ for fixed $w_k \in U_{\eta^k}$ for every $k \in \{1, \ldots, d\} \setminus \{j\}$.

We shall derive

$$\int_{\partial\gamma_j} A_t(w_1, \ldots, w_{j-1}, w_j, w_{j+1}, \ldots, w_d) \, \mathrm{d}w_j = 0 \,.$$

Then, the analyticity of the mapping $w_j \mapsto A_t(w_1, \ldots, w_{j-1}, w_j, w_{j+1}, \ldots, w_d)$ in the interior of U_{η^j} follows from the theorem of Morera. Since this is true for every coordinate $j \in \{1, \ldots, d\}$, the analyticity of the map $w \mapsto A_t(w)$ in $\mathring{U}_\eta$ follows.

Consider the symbol as given in (5). The mapping $w_j \mapsto \frac{1}{2}\langle w, \Sigma_t w \rangle + i\langle w, b_t \rangle$ is analytic in $\mathbb{C}$. The same is true for the mapping $w_j \mapsto \big(\mathrm{e}^{-i\langle w, y \rangle} - 1 + i\langle w, h(y) \rangle\big)$. An application of the theorem of Fubini and the lemma of Goursat yields

$$\begin{aligned}\int_{\partial\gamma_j} & A_t(w_1,\dots,w_{j-1},w_j,w_{j+1},\dots,w_d)\,\mathrm{d}w_j \\ &= \int_{\partial\gamma_j}\Big(\frac{1}{2}\langle w,\Sigma_t w\rangle + i\langle w,b_t\rangle\Big)\,\mathrm{d}w_j - \int_{\partial\gamma_j}\int_{\mathbb{R}^d}\Big(\mathrm{e}^{-i\langle w,y\rangle} - 1 + i\langle w,h(y)\rangle\Big)\,F_t(\mathrm{d}y)\,\mathrm{d}w_j \\ &= -\int_{\mathbb{R}^d}\int_{\partial\gamma_j}\Big(\mathrm{e}^{-i\langle w,y\rangle} - 1 + i\langle w,h(y)\rangle\Big)\,\mathrm{d}w_j\,F_t(\mathrm{d}y) \\ &= 0\,.\end{aligned}$$

To justify the use of Fubini's theorem, we derive an upper bound of $\Big|\mathrm{e}^{-i\langle w,y\rangle} - 1 + i\langle w,h(y)\rangle\Big|$ in $L^1(F_t(\mathrm{d}y),\mathbb{R}^d)$ which does not depend on w_j.

For $|y|\le 1$ we have

$$\Big|\mathrm{e}^{-i\langle w,y\rangle} - 1 + i\langle w,h(y)\rangle\Big| \le \frac{1}{2}|\langle w,y\rangle|^2\,\mathrm{e}^{\langle\Im(w),y\rangle} \le \frac{c'}{2}|w|^2|y|^2 \le \frac{c'}{2}\Big(\max_{w_j\in\partial\gamma_j}|w|^2\Big)|y|^2$$

with the positive constant $c' := \sup_{|y|\le 1, w_j\in\partial\gamma_j}\mathrm{e}^{\langle\Im(w),y\rangle}$. For $|y|>1$, and the choice $h(y) = y\mathbb{1}_{|y|\le 1}$ we have

$$\Big|\mathrm{e}^{-i\langle w,y\rangle} - 1 + i\langle w,h(y)\rangle\Big| = \Big|\mathrm{e}^{-i\langle w,y\rangle} - 1\Big| \le \mathrm{e}^{\langle\Re(-iw),y\rangle} + 1 = \mathrm{e}^{\langle\Im(w),y\rangle} + 1\,,$$

where we denote by $\Re(w)$ (resp. $\Im(w)$) the vector of the real parts (resp. the imaginary parts) of the components of the vector $w\in\mathbb{C}^d$. By assumption $\Im(w)$ belongs to $R_{-\eta}$. This is also the case for

$$v^1 := \Big(\Im(w_1),\dots,\Im(w_{j-1}),\max_{w_j\in\partial\gamma_j}\Im(w_j),\Im(w_{j+1}),\dots,\Im(w_d)\Big)$$

and for

$$v^2 := \Big(\Im(w_1),\dots,\Im(w_{j-1}),\min_{w_j\in\partial\gamma_j}\Im(w_j),\Im(w_{j+1}),\dots,\Im(w_d)\Big).$$

It follows that

$$\int_{|y|>1}\Big|\mathrm{e}^{-i\langle w,y\rangle} - 1 + i\langle w,h(y)\rangle\Big|F_t(\mathrm{d}y) \le \int_{|y|>1}\Big(\mathrm{e}^{\langle v^1,y\rangle} + \mathrm{e}^{\langle v^2,y\rangle} + 1\Big)\,F_t(\mathrm{d}y) < \infty\,.$$

For every fixed choice of complex numbers $w_k\in\overline{U_{\eta^k}}$ for $k\in\{1,\dots,d\}\setminus\{j\}$ and every fixed $y\in\mathbb{R}^d$, the mapping $w_j\mapsto\mathrm{e}^{-i\langle w,y\rangle} - 1 + i\langle w,h(y)\rangle$ is continuous. Furthermore, the following estimate is valid,

$$\begin{aligned}\Big|\mathrm{e}^{-i\langle w,y\rangle} - 1 + i\langle w,h(y)\rangle\Big| \le\;& \frac{c'}{2}\Big(\max_{w_j\in\partial\gamma_j}|w|^2\Big)|y|^2\mathbb{1}_{\{|y|\le 1\}}(y) \\ &+ \Big(\mathrm{e}^{\langle v^1,y\rangle} + \mathrm{e}^{\langle v^2,y\rangle} + 1\Big)\,\mathbb{1}_{\{|y|>1\}}(y),\end{aligned} \tag{8}$$

which is an upper bound in $L^1(F_t(\mathrm{d}y),\mathbb{R}^d)$, from where the continuity of the mapping $w\mapsto A_t(w)$ on $\overline{U_\eta}$ follows. □

The next lemma collects further elementary properties of $\big(A_t(\cdot - i\eta)\big)_{t\in[0,T]}$.

Lemma 2.2. *Let L be a PIIAC with characteristic triplet $(b_t, \sigma_t, F_t)_{t\in[0,T]}$ and with symbol $(A_t)_{t\in[0,T]}$. If*

$$\int_0^T \int_{|x|>1} \mathrm{e}^{-\langle \eta', x\rangle} F_t(\mathrm{d}x) < \infty \qquad \text{for all } \eta' \in R_\eta\,,$$

then the following holds.

(a) *For every $\eta' \in R_\eta$ we have*

$$A_t(\xi - i\eta') = A_t(-i\eta') + A_t^{L^{-\eta'}}(\xi)\,,$$

where $A^{L^{-\eta'}}$ is the symbol of a time-inhomogeneous Lévy process $L^{-\eta'}$ which is given by the characteristic triplet $(b_t^{-\eta'}, \sigma_t, F_t^{-\eta'})_{t\geq 0}$ with

$$\begin{aligned} b_t^{-\eta'} &= b_t - \sigma_t \eta' + \int \left(\mathrm{e}^{-\langle \eta', x\rangle} - 1\right) h(x) F_t(\mathrm{d}x) \quad \text{and} \\ F_t^{-\eta'}(\mathrm{d}x) &= \mathrm{e}^{-\langle \eta', x\rangle} F_t(\mathrm{d}x)\,. \end{aligned}$$

(b) *For every $\eta' \in R_\eta$ there is equivalence between*

$$A_t(-i\eta') = 0 \qquad \text{for all } t \in [0,T]\big)$$

and the martingale property of the process $\left(\mathrm{e}^{-\langle \eta', L_t\rangle}\right)_{t\geq 0}$.

(c) *For every $\xi \in \mathbb{R}^d$ we have*

$$\begin{aligned} \Re\big(A_t(\xi - i\eta')\big) &= A_t(-i\eta') + \frac{1}{2}\langle \xi, \sigma_t \xi\rangle - \int \big(\cos(\langle \xi, x\rangle) - 1\big) F_t^{-\eta'}(\mathrm{d}x) \\ &\geq A_t(-i\eta')\,. \end{aligned}$$

Proof. The derivation of the decomposition (a) follows in a straightforward way, compare Glau (2010, Satz II.15), as does (c).
To show (b), we notice that

$$E\,\mathrm{e}^{-\langle \eta', L_t\rangle} = \mathrm{e}^{\int_0^t \theta_s(-\eta')\,\mathrm{d}s} = \mathrm{e}^{-\int_0^t A_s(-i\eta')\,\mathrm{d}s}$$

and for $s \leq t$ the equality

$$E\left(\mathrm{e}^{-\langle \eta', L_t\rangle} \,\big|\, \mathcal{F}_s\right) = \mathrm{e}^{-\langle \eta', L_s\rangle}\, E\,\mathrm{e}^{-\langle \eta', L_t - L_s\rangle} = \mathrm{e}^{-\langle \eta', L_s\rangle}\, \mathrm{e}^{-\int_s^t A_u(-i\eta')\,\mathrm{d}u}$$

follows. Hence $\mathrm{e}^{-\langle \eta', L\rangle}$ is a martingale, if and only if $A_t(-i\eta') = 0$ holds for every $t \in [0,T]$. □

3. Exponentially weighted Sobolev–Slobodeckii spaces

We consider so-called weighted Sobolev–Slobodeckii spaces with weight functions of the form $x \mapsto \mathrm{e}^{\langle \eta, x\rangle}$ with a vector $\eta \in \mathbb{R}^d$. We study only Sobolev–Slobodeckii spaces with exponential weight functions and define these spaces analogously to the definition of Sobolev–Slobodeckii spaces based on Fourier transformed functions in Wloka (1987). The main reason besides the benefits of an appropriate access via the symbol is the result which will be given in Theorem 3.4, that the dual space $\big(H_\eta^s(\mathbb{R}^d)\big)'$ of $H_\eta^s(\mathbb{R}^d)$ is isometrically isomorphic to the space $H_\eta^{-s}(\mathbb{R}^d)$. This property of the Sobolev space is necessary for the interpretation in Section 5 of the PDOs $\mathcal{A}_t$ associated with the symbols A_t as linear operators from the Hilbert space $H_\eta^s(\mathbb{R}^d)$ to its dual space $\big(H_\eta^s(\mathbb{R}^d)\big)'$.

We denote by $L^2_\eta(\mathbb{R}^d)$ the Hilbert space of complex-valued square integrable functions

$$L^2_\eta(\mathbb{R}^d) := \left\{u \in L^1_{\mathrm{loc}}(\mathbb{R}^d) \,\middle|\, x \mapsto u(x)\,\mathrm{e}^{\langle \eta, x\rangle} \in L^2(\mathbb{R}^d)\right\} \tag{9}$$

with scalar product

$$\langle u, v\rangle_{L^2_\eta} := \int_{\mathbb{R}^d} u(x)\overline{v(x)}\,\mathrm{e}^{2\langle \eta, x\rangle}\,\mathrm{d}x \quad \text{for all } u,\ v \in L^2_\eta(\mathbb{R}^d)\,. \tag{10}$$

The crucial step for a definition of the Sobolev spaces via Fourier transforms is based on Parseval's identity,

$$\langle u, v\rangle_{L^2} = \int_{\mathbb{R}^d} u(x)\overline{v(x)}\,\mathrm{d}x = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{u}(\xi)\overline{\hat{v}(\xi)}\,\mathrm{d}\xi\,. \tag{11}$$

In order to derive the analogous identity for functions in the space $L^2_\eta(\mathbb{R}^d)$, we denote

$$\begin{aligned} u_\eta(x) &:= u(x)\,\mathrm{e}^{\langle \eta, x\rangle} \\ \hat{u}(\xi - i\eta) &:= \int \mathrm{e}^{i\langle \xi, x\rangle}\,u(x)\,\mathrm{e}^{\langle \eta, x\rangle}\,\mathrm{d}x = \mathcal{F}(u_\eta)(\xi) \end{aligned} \tag{12}$$

for functions $u : \mathbb{R}^d \to \mathbb{C}$ with $\int |u(x)|\,\mathrm{e}^{\langle \eta, x\rangle}\,\mathrm{d}x < \infty$. Let us further notice the equality

$$\langle u, v\rangle_{L^2_\eta} = \frac{1}{(2\pi)^d}\int \hat{u}(\xi - i\eta)\overline{\hat{v}(\xi - i\eta)}\,\mathrm{d}\xi\,, \tag{13}$$

for functions $u\,,\ v \in L^2_\eta(\mathbb{R}^d)$. This leads to the following characterization of the space $L^2_\eta(\mathbb{R}^d)$.

Remark 3.1. The space $L^2_\eta(\mathbb{R}^d)$ is isometrically isomorphic to the space

$$\left\{u \in L^1_{\mathrm{loc}}(\mathbb{R}^d) \,\middle|\, \mathcal{F}(u_\eta) \in L^2(\mathbb{R}^d)\right\}.$$

Furthermore the space $\left(L^2_\eta(\mathbb{R}^d), \langle\cdot,\cdot\rangle_{L^2_\eta}\right)$ is a separable Hilbert space. The space $C_0^\infty(\mathbb{R}^d, \mathbb{C})$ of complex functions with compact support which have derivatives of any order is a dense subspace.

For a consistent definition of the Sobolev–Slobodeckii spaces with exponential weights, we first define the analogue of the Schwartz space $S(\mathbb{R}^d)$ and of the generalized functions.

Definition 3.2 (Exponentially weighted Schwartz space)**.** *For* $\eta \in \mathbb{R}^d$ *let*

$$S_\eta(\mathbb{R}^d) := \left\{u \in C^\infty(\mathbb{R}^d, \mathbb{C}) \,\middle|\, \|u\|_{m,\eta} < \infty \text{ for all } m \geq 0\right\}$$

with

$$\|\varphi\|_{m,\eta} := \|\varphi\,\mathrm{e}^{\langle \eta, \cdot\rangle}\,\|_m$$

and we denote by $S'_\eta(\mathbb{R}^d)$ *the dual space of* $S_\eta(\mathbb{R}^d)$*.*

For every integer $m \geq 0$ the norms $\|\cdot\|_m$ are defined as usual by

$$\|\varphi\|_m := \sup_{|p| \leq m}\sup_{x \in \mathbb{R}^d}\left(1 + |x|^2\right)^m |D^p\varphi(x)|\,,$$

compare e.g. (Rudin, 1973, Section 7.3). In the following remark, we define a Fourier transform $\mathcal{F}_\eta$ for functions in the weighted Schwartz space which is the analogue of the Fourier transform $\mathcal{F}$ on the Schwartz space.

Remark 3.3. [Fourier transform with weights] The mappings

$$\mathcal{F}_\eta(\varphi) := \mathrm{e}^{-\langle \eta,\cdot\rangle}\,\mathcal{F}\big(\varphi\,\mathrm{e}^{\langle \eta,\cdot\rangle}\big) \qquad (\varphi \in S_\eta(\mathbb{R}^d) \quad \text{resp.} \quad \varphi \in L^2_\eta(\mathbb{R}^d))$$

and

$$\mathcal{F}^{-1}_\eta(\varphi) := \mathrm{e}^{-\langle \eta,\cdot\rangle}\,\mathcal{F}^{-1}\big(\varphi\,\mathrm{e}^{\langle \eta,\cdot\rangle}\big) \qquad (\varphi \in S_\eta(\mathbb{R}^d) \quad \text{resp.} \quad \varphi \in L^2_\eta(\mathbb{R}^d))$$

are continuous bijections, $\mathcal{F}_\eta : S_\eta(\mathbb{R}^d) \to S_\eta(\mathbb{R}^d)$ resp. $\mathcal{F}_\eta : L^2_\eta(\mathbb{R}^d) \to L^2_\eta(\mathbb{R}^d)$.

It follows similarly that the transformation $\mathcal{F}_\eta : S'_\eta(\mathbb{R}^d) \to S'_\eta(\mathbb{R}^d)$, defined by the Parseval identity

$$\mathcal{F}_\eta(u)(\varphi) := (2\pi)^d u\big(\mathcal{F}^{-1}_\eta(\varphi)\big) \quad \big(u \in S'_\eta(\mathbb{R}^d),\, \varphi \in S_\eta(\mathbb{R}^d)\big)$$

resp.

$$u(\varphi) = \frac{1}{(2\pi)^d}\mathcal{F}_\eta(u)\big(\mathcal{F}_\eta(\varphi)\big) \quad \big(u \in S'_\eta(\mathbb{R}^d),\, \varphi \in S_\eta(\mathbb{R}^d)\big) \tag{14}$$

is continuous and bijective. The weighted Sobolev–Slobodeckii spaces for $s \in \mathbb{R}$ and $\eta \in \mathbb{R}^d$ are defined via

$$H^s_\eta(\mathbb{R}^d) := \big\{u \in S'_\eta(\mathbb{R}^d) \,\big|\, \|\,\mathrm{e}^{\langle \eta,\cdot\rangle}\,\mathcal{F}_\eta(u)\|_{\widehat{H}^s} < \infty\big\}$$

with the scalar product

$$\langle u, v\rangle_{H^s_\eta} := \langle \mathcal{F}_\eta(u), \mathcal{F}_\eta(v)\rangle_{\widehat{H}^s_\eta} := \langle \mathrm{e}^{\langle \eta,\cdot\rangle}\,\mathcal{F}_\eta(u), \mathrm{e}^{\langle \eta,\cdot\rangle}\,\mathcal{F}_\eta(v)\rangle_{\widehat{H}^s} \tag{15}$$

where

$$\langle \varphi, \psi\rangle_{\widehat{H}^s} := \int \varphi(\xi)\overline{\psi(\xi)}\,\big(1 + |\xi|\big)^{2s}\,\mathrm{d}\xi\,. \tag{16}$$

For the scalar product of the weighted space, this entails

$$\langle u, v\rangle_{H^s_\eta} = \int \mathcal{F}_\eta(u)(\xi)\overline{\mathcal{F}_\eta(v)(\xi)}\,\big(1 + |\xi|\big)^{2s}\,\mathrm{e}^{2\langle \eta,\xi\rangle}\,\mathrm{d}\xi\,. \tag{17}$$

The space of Fourier transforms of functions in $H^s_\eta(\mathbb{R}^d)$ is given by

$$\widehat{H}^s_\eta(\mathbb{R}^d) := \big\{\mathcal{F}_\eta(u) \,\big|\, u \in H^s_\eta(\mathbb{R}^d)\big\}$$

with scalar product $\langle\cdot,\cdot\rangle_{\widehat{H}^s_\eta}$. Inserting the notation $u_\eta = u\,\mathrm{e}^{\langle \eta,\cdot\rangle}$ and $v_\eta = v\,\mathrm{e}^{\langle \eta,\cdot\rangle}$ yields

$$\langle u, v\rangle_{H^s_\eta} = \big\langle\, \mathrm{e}^{\langle \eta,\cdot\rangle}\,\mathcal{F}_\eta(u), \mathrm{e}^{\langle \eta,\cdot\rangle}\,\mathcal{F}_\eta(v)\big\rangle_{\widehat{H}^s} = \big\langle \mathcal{F}(u_\eta), \mathcal{F}(v_\eta)\big\rangle_{\widehat{H}^s} = \langle u_\eta, v_\eta\rangle_{H^s}\,. \tag{18}$$

In particular for $\eta = 0$ the weighted space $H^s_0(\mathbb{R}^d)$ is the Sobolev–Slobodeckii space $H^s(\mathbb{R}^d)$ as it is e.g. defined in Wloka (1987) and the norms coincide as well, $\|u\|_{H^s_0} = \|u\|_{H^s}$.

Theorem 3.4. *The dual space* $\big(H^s_\eta(\mathbb{R}^d)\big)'$ *of* $H^s_\eta(\mathbb{R}^d)$ *is isometrically isomorphic to* $H^{-s}_\eta(\mathbb{R}^d)$.

Proof. We argue similarly to Eskin (1981, p. 62–63). Let $l \in \big(H^s_\eta(\mathbb{R}^d)\big)'$. From the representation theorem of Riesz, we conclude the unique existence of a function $v \in H^s_\eta(\mathbb{R}^d)$ with $\|v\|_{H^s_\eta} = \|l\|_{(H^s_\eta(\mathbb{R}^d))'}$, such that by equation (18) and (15)

$$l(\varphi) \quad = \quad \langle v, \varphi\rangle_{H^s_\eta} = \int (1 + |\xi|)^{2s}\,\hat{v}(\xi - i\eta)\overline{\hat{\varphi}(\xi - i\eta)}\,\mathrm{d}\xi$$

for every $\varphi \in H^s_\eta(\mathbb{R}^d)$. If we then define $w := \mathrm{e}^{-\langle \eta,\cdot\rangle}\,\mathcal{F}^{-1}\big(\,(1+|\cdot|)^{2s}\,\hat{v}(\cdot - i\eta)\big)$, we obtain

$$\begin{aligned}\|w\|^2_{H^{-s}_\eta} &= \int_{\mathbb{R}^d} \big(1+|\xi|\big)^{-2s}\Big|\mathcal{F}\big(\,\mathrm{e}^{\langle \eta,\cdot\rangle}\,w\big)(\xi)\Big|^2\,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^d} \big(1+|\xi|\big)^{2s}\big|\hat{v}(\xi - i\eta)\big|^2\,\mathrm{d}\xi \\ &= \|v\|^2_{H^s_\eta}\end{aligned}$$

and hence $w \in H^{-s}_\eta(\mathbb{R}^d)$ with $\|w\|_{H^{-s}_\eta} = \|v\|_{H^s_\eta} = \|l\|_{(H^s_\eta(\mathbb{R}^d))'}$. Hence $l \mapsto w$ defines an isometry from $\big(H^s_\eta(\mathbb{R}^d)\big)'$ to the space $H^{-s}_\eta(\mathbb{R}^d)$. Since the Riesz mapping $l \mapsto v$ and the mapping defined by $v \mapsto w := \mathrm{e}^{-\langle \eta,\cdot\rangle}\,\mathcal{F}^{-1}\Big(\big(1+|\cdot - i\eta|^2\big)^s\,\hat{v}(\cdot - i\eta)\Big)$ are both bijective maps, their composition defines the desired isomorphism. ☐

4. Symbol and Pseudo-Differential operator

Let $\mathcal{A}$ be a PDO with symbol A as in (5), i.e.

$$\mathcal{A}u = \mathcal{F}^{-1}\big(A\mathcal{F}(u)\big) \qquad \text{for all } u \in S(\mathbb{R}^d)$$

with $A : \mathbb{R}^d \to \mathbb{C}$ measurable and satisfying

$$\big|A(\xi)\big| \le c\big(1+|\xi|\big)^\alpha \qquad \text{for all } \xi \in \mathbb{R}^d$$

for an $\alpha \in \mathbb{R}$ and a constant $c \ge 0$. The latter estimate guarantees that the Fourier inversion operator $\mathcal{F}^{-1}$ is well defined. In (Eskin, 1981, Lemma 4.4) it is shown that

$$\|\mathcal{A}u\|_{H^{s-\alpha}} \le c\|u\|_{H^s} \qquad \text{for all } u \in S(\mathbb{R}^d)\,.$$

As a consequence, $\mathcal{A}$ has a unique extension to a continuous linear operator $\mathcal{A} : H^s(\mathbb{R}^d) \to H^{s-\alpha}(\mathbb{R}^d)$.

In this section we derive conditions on the symbol, that allow the interpretation of $\mathcal{A}$ as a continuous linear operator

$$\mathcal{A} : H^s_\eta(\mathbb{R}^d) \to H^{s-\alpha}_\eta(\mathbb{R}^d)\,.$$

Let $U_{-\eta}$ be given as in (7). We denote by $S_\alpha(-\eta)$ the set of symbols A that have a continuous extension $A : \overline{U_{-\eta}} \to \mathbb{C}$ that is analytic in the interior of $U_{-\eta}$, and further satisfies the *continuity condition*

$$\big|A(z)\big| \le C_\eta\big(1+|z|\big)^\alpha \quad \big(\text{for all } z \in U_{-\eta}\big)\,. \tag{19}$$

Note that as a consequence of the identity theorem for holomorphic functions, the extension is unique on $U_{-\eta}$. By continuity, the extension is unique on the closure $\overline{U_{-\eta}}$.

Let us observe that by definition of the Fourier transform $\mathcal{F}_\eta$ and its inverse and by the estimate (19), it is obvious that

$$u \mapsto \mathcal{F}^{-1}_\eta\big(A(\cdot - i\eta)\mathcal{F}_\eta(u)\big)$$

is a linear continuous mapping from $H^s_\eta(\mathbb{R}^d)$ to $H^{s-\alpha}_\eta(\mathbb{R}^d)$. We prove the following consistency result.

Theorem 4.1. *Let $\mathcal{A}$ be a PDO with symbol $A \in S_\alpha(-\eta)$ (see (19)) for an index $\alpha \in \mathbb{R}$ and a weight index $\eta \in \mathbb{R}^d$. Then*

$$\mathcal{A}\,u = \mathcal{F}^{-1}\big(A\mathcal{F}(u)\big) = \mathcal{F}_\eta^{-1}\Big(A(\cdot - i\eta)\mathcal{F}_\eta(u)\Big) \qquad \text{for all } u \in C_0^\infty(\mathbb{R}^d, \mathbb{C})$$

and there exists a constant $c(\eta) > 0$ with

$$\big\|\mathcal{A}\,u\big\|_{H_\eta^{s-\alpha}(\mathbb{R}^d)} \le c(\eta)\|u\|_{H_\eta^s(\mathbb{R}^d)} \qquad \text{for all } u \in C_0^\infty(\mathbb{R}^d, \mathbb{C})\,.$$

Moreover, the operator $\mathcal{A}$ can be extended to a linear continuous operator $\mathcal{A} : H_\eta^s(\mathbb{R}^d) \to H_\eta^{s-\alpha}(\mathbb{R}^d)$ in a unique way.

Proof. For $u \in C_0^\infty(\mathbb{R}^d, \mathbb{C})$ we have

$$\mathcal{A}\,u(x) \;=\; \mathcal{F}^{-1}\big(A\mathcal{F}(u)\big)(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{-i\langle\xi,x\rangle}\, A(\xi)\mathcal{F}(u)(\xi)\,\mathrm{d}\xi \qquad \text{for all } x \in \mathbb{R}^d\,.$$

The map $\xi \mapsto \mathrm{e}^{-i\langle\xi,x\rangle}\, A(\xi)\mathcal{F}(u)(\xi)$ is continuous on $\overline{U_{-\eta}}$ and holomorphic in the interior $\mathring{U}_{-\eta}$. The continuity of A on $\overline{U_{-\eta}}$ entails

$$|A(z)| \le C_\eta\big(1 + |z|\big)^\alpha \qquad \text{for all } z \in \overline{U_{-\eta}}\,. \tag{20}$$

Furthermore, for $\eta' := (\eta'_1, \ldots, \eta'_d)$ with $\eta'_j \in \operatorname{sgn}(\eta^j)[0, |\eta^j|]$, we obtain

$$\begin{aligned}\big|\,\mathrm{e}^{-i\langle\xi - i\eta',x\rangle}\, A(\xi - i\eta')\hat{u}(\xi - i\eta')\big| \;&=\; \mathrm{e}^{-\langle\eta',x\rangle}\,\big|A(\xi - i\eta')\hat{u}(\xi - i\eta)\big| \\ &\le\; \mathrm{e}^{-\langle\eta',x\rangle}C_\eta\big(1 + |\xi - i\eta'|\big)^\alpha C_N \frac{\mathrm{e}^{R|\eta'|}}{\big(1 + |\xi - i\eta'|\big)^N}\end{aligned}$$

with a constant C_N for arbitrary $N \in \mathbb{N}_0$, if the support of the function u is inside of the open ball with radius $R \in \mathbb{R}_+$, $tr(u) \subset B_R(0)$. This is a direct consequence of $A \in S_\alpha(-\eta)$ and the Paley–Wiener–Schwartz theorem, compare Jacob (2001, Theorem 3.4.6).

Now define $f(\xi - i\eta') := \mathrm{e}^{-i\langle\xi - i\eta',x\rangle}\, A(\xi - i\eta')\hat{u}(\xi - i\eta')$ for $x \in \mathbb{R}^d$ fixed, then for any $N > \alpha + d + 1$ this yields in particular

$$\big|f(\xi - i\eta')\big| \le C_\eta C_N\,\mathrm{e}^{-\langle\eta',x\rangle}\,\mathrm{e}^{R|\eta'|}\,\frac{1}{(1 + |\xi - i\eta'|)^{N-\alpha}} \le C(x, \eta, N, R)\big(1 + |\xi|\big)^{-(d+1)} \tag{21}$$

with a constant $C(x, \eta, N, R)$ independent of ξ and η' for all $(\xi - i\eta') \in U_{-\eta}$ and

$$f(\xi - i\eta') \to 0 \quad \text{for } |\xi| \to \infty\,,$$

which shows assumption (i) of Lemma A.1. The integrability assumption (ii) in the same lemma is obviously satisfied, hence we can apply the version of Cauchy's theorem, that we provide in Lemma A.1, from where we obtain

$$\begin{aligned}\mathcal{A}\,u(x) \;&=\; \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{-i\langle\xi,x\rangle}\, A(\xi)\hat{u}(\xi)\,\mathrm{d}\xi \\ &=\; \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{-i\langle\xi - i\eta,x\rangle}\, A(\xi - i\eta)\hat{u}(\xi - i\eta)\,\mathrm{d}\xi\,.\end{aligned}$$

We then insert the definition of $\mathcal{F}_\eta$, compare equation (12) and Remark 3.3 to obtain

$$\begin{aligned}\mathcal{A}\,u(x) &= \mathrm{e}^{-\langle \eta,x\rangle}\,\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{-i\langle \xi,x\rangle}\,A(\xi-i\eta)\,\mathrm{e}^{\langle \eta,\xi\rangle}\,\mathrm{e}^{-\langle \eta,\xi\rangle}\,\hat{u}(\xi-i\eta)\,\mathrm{d}\xi\\ &= \mathrm{e}^{-\langle \eta,x\rangle}\,\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{-i\langle \xi,x+i\eta\rangle}\,A(\xi-i\eta)\mathcal{F}_\eta(u)(\xi)\,\mathrm{d}\xi\,. \end{aligned}\tag{22}$$

Furthermore we show that the mapping $\xi\mapsto A(\xi-i\eta)\mathcal{F}_\eta(u)(\xi)$ belongs to $L^1_\eta(\mathbb{R}^d)$ where the latter space is defined in an analogous way to (9). Using the definition of $\mathcal{F}_\eta$ and (21) we get

$$\begin{aligned}\big\|A(\cdot-i\eta)\mathcal{F}_\eta(u)(\cdot)\big\|_{L^1_\eta(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \big|A(\xi-i\eta)\mathcal{F}_\eta(u)(\xi)\big|\,\mathrm{e}^{\langle \eta,\xi\rangle}\,\mathrm{d}\xi\\ &= \int_{\mathbb{R}^d} \big|A(\xi-i\eta)\big|\big|\hat{u}(\xi-i\eta)\big|\,\mathrm{d}\xi\\ &\le C(x,\eta,N,R)\int_{\mathbb{R}^d}\big(1+|\xi|\big)^{-(d+1)}\,\mathrm{d}\xi\\ &< \infty\,.\end{aligned}$$

It follows from the last line in (22) that

$$\mathcal{A}\,u(x) = \mathcal{F}_\eta^{-1}\Big(A(\cdot-i\eta)\mathcal{F}_\eta(u)\Big)(x)\,.$$

In order to prove the continuity property, we choose $u\in C_0^\infty(\mathbb{R}^d,\mathbb{C})$, and inserting (17) we estimate

$$\begin{aligned}\big\|\mathcal{A}\,u\big\|^2_{H^{s-\alpha}_\eta(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \big|\mathcal{F}_\eta\big(\mathcal{A}\,u\big)\big|^2\big(1+|\xi|\big)^{2(s-\alpha)}\,\mathrm{e}^{2\langle \eta,\xi\rangle}\,\mathrm{d}\xi\\ &= \int_{\mathbb{R}^d} \big|A(\xi-i\eta)\big|^2\big|\mathcal{F}_\eta(u)(\xi)\big|^2\big(1+|\xi|\big)^{2(s-\alpha)}\,\mathrm{e}^{2\langle \eta,\xi\rangle}\,\mathrm{d}\xi\\ &\le C_\eta^2(1+|\eta|)^{2\alpha}\int_{\mathbb{R}^d}\big|\mathcal{F}_\eta(u)(\xi)\big|^2\big(1+|\xi|\big)^{2s}\,\mathrm{e}^{2\langle \eta,\xi\rangle}\,\mathrm{d}\xi\\ &= c(\eta)\|u\|^2_{H^s_\eta(\mathbb{R}^d)}\,.\end{aligned}$$

Since $C_0^\infty(\mathbb{R}^d,\mathbb{C})$ is dense in the space $H^s_\eta(\mathbb{R}^d)$ the operator has a unique continuous extension $\mathcal{A}:H^s_\eta(\mathbb{R}^d)\to H^{s-\alpha}_\eta(\mathbb{R}^d)$. □

5. Parabolic Equation

In Glau (2011) a Sobolev index is introduced, and it is shown that the evolution problem associated with a Lévy process with Sobolev index α has a unique weak solution in the Sobolev–Slobodeckii space $H^{\alpha/2}$. In this section, we generalize the results obtained in Glau (2011) to the case of weighted Sobolev–Slobodeckii spaces, and we examine $\mathbb{R}^d$-valued time-inhomogeneous Lévy processes instead of genuine Lévy processes.

Let L be an $\mathbb{R}^d$-valued PIIAC with local characteristics $\big(b_s,\sigma_s,F_s\big)$ for $s\ge 0$. Let us consider the following assumptions on the symbol A of the process as defined in (5).

(A1) *Assume for some fixed time horizon $T > 0$ that*

$$\int_0^T \int_{|x|>1} \mathrm{e}^{-\langle \eta', x\rangle} F_s(\mathrm{d}x)\,\mathrm{d}s < \infty \qquad \forall \eta' \in R_\eta .$$

(A2) *There exists a constant $C_1 > 0$ uniform in time, with*

$$\big|A_t(z)\big| \le C_1\big(1+|z|\big)^\alpha$$

for all $z \in U_{-\eta}$ and for all $t \in [0,T]$. *(Continuity condition)*

(A3) *There exist constants $C_2 > 0$ and $C_3 \ge 0$ uniform in time, such that for a certain $0 \le \beta < \alpha$*

$$\Re\big(A_t(z)\big) \ge C_2\big(1+|z|\big)^\alpha - C_3\big(1+|z|\big)^\beta$$

for all $z \in U_{-\eta}$ and for all $t \in [0,T]$. *(Gårding condition)*

The set $U_{-\eta} = \{z \in \mathbb{C}^d | \Im(z) \in -\operatorname{sgn}(\eta^j)[0, |\eta^j|) \text{ for } j = 1,\dots,d\}$ was defined in (7).
Let us make the following remarks.

Remark 5.1. (i) For Lévy processes with Brownian part the conditions (A2) and (A3) are valid for $\alpha = 2$ and those $\eta \in \mathbb{R}$ that satisfy assumption (A1). In particular the Brownian motion (with drift) satisfies the assumptions for every $\eta \in \mathbb{R}$.
(ii) Conditions (A1)–(A3) are for example satisfied for CGMY-processes with parameters C, G, $M > 0$ and $Y \in [1,2)$ with $\alpha = Y$ and $\eta \in (-M, G)$.

See also Section 7 where further examples are studied.

Remark 5.2. Conditions (A2) and (A3) are actually not necessary assumptions of Theorem 5.3 about the existence and uniqueness of the weak solution of the corresponding PIDE in a weighted Sobolev–Slobodeckii space. We choose this set of assumptions, since usually the symbol is well known for real arguments and it is hence convenient to extend the polynomial growth conditions to the complex domain $U_{-\eta}$. Moreover by Theorem 4.1, this approach allows us to work in a unique framework for the PDO $\mathcal{A}$ associated with exponentially weighted Sobolev–Slobodeckii spaces $H^s_\eta(\mathbb{R}^d)$ with different weights η.

However, it is instead also possible to assume the growth conditions (A2) and (A3) only for the function $x \mapsto A_t(x - i\eta)$ where η is fixed. Instead of (A1) one would then have to assume $\int_0^T \int_{|x|>1} \mathrm{e}^{-\langle \eta, x\rangle} F_s(\mathrm{d}x)\,\mathrm{d}s < \infty$.

Under assumptions (A1) and (A2), we conclude from Theorem 4.1, that for every fixed $t \in [0,T]$ the operator $\mathcal{A}_t|_{C_0^\infty(\mathbb{R}^d,\mathbb{C})} : C_0^\infty(\mathbb{R}^d,\mathbb{C}) \to C^\infty(\mathbb{R}^d,\mathbb{C})$ associated with the symbol A_t has a unique linear and continuous extension

$$\mathcal{A}_t : H^{\alpha/2}_\eta(\mathbb{R}^d) \to H^{-\alpha/2}_\eta(\mathbb{R}^d)$$

with $\mathcal{A}_t u = \mathcal{F}^{-1}_\eta\big(A_t(\cdot - i\eta)\mathcal{F}_\eta(u)\big)$ for all $u \in H^{\alpha/2}_\eta(\mathbb{R}^d)$. Since the Hilbert spaces $H^{-\alpha/2}_\eta(\mathbb{R}^d)$ and $\big(H^{\alpha/2}_\eta(\mathbb{R}^d)\big)'$ are isomorphic, the operators $\mathcal{A}_t$ can be identified with continuous linear operators

$$\mathcal{A}_t : H^{\alpha/2}_\eta(\mathbb{R}^d) \to \Big(H^{\alpha/2}_\eta(\mathbb{R}^d)\Big)'.$$

Let us further define the family of bilinear forms a_t by

$$a_t(\varphi,\psi) := \big(\mathcal{A}_t\varphi\big)(\psi) \quad \text{for } \varphi,\, \psi \in H^{\alpha/2}_\eta(\mathbb{R}^d) \tag{23}$$

for every $t \in [0,T]$. Inserting Parseval's equality (11), we obtain for every $\varphi,\, \psi \in H^{\alpha/2}_\eta(\mathbb{R}^d)$ the equality

$$\begin{aligned} a_t(\varphi,\psi) &= \frac{1}{(2\pi)^d}\langle \mathcal{F}_\eta(\mathcal{A}_t\varphi), \mathcal{F}_\eta(\psi)\rangle_{L^2_\eta(\mathbb{R}^d)} \\ &= \frac{1}{(2\pi)^d}\int A_t(\xi - i\eta)\mathcal{F}_\eta(\varphi)(\xi)\,\overline{\mathcal{F}_\eta(\psi)(\xi)}\,\mathrm{e}^{2\langle\eta,\xi\rangle}\,\mathrm{d}\xi \\ &= \frac{1}{(2\pi)^d}\int A_t(\xi - i\eta)\hat{\varphi}(\xi - i\eta)\overline{\hat{\psi}(\xi - i\eta)}\,\mathrm{d}\xi \end{aligned}$$

for each $t \in [0,T]$.

Theorem 5.3. *Let L be an $\mathbb{R}^d$-valued PIIAC with local characteristics $\big(b_t, \sigma_t, F_t\big)$ for $t \geq 0$ and symbol $A = (A_t)_{t\in[0,T]}$ and associated pseudo differential operators $(\mathcal{A}_t)_{t\in[0,T]}$. If the assumptions (A1)–(A3) are satisfied, then the parabolic equation*

$$\begin{aligned} \partial_t u + \mathcal{A}_t u &= \quad f \\ u(0) &= \quad g\,, \end{aligned} \tag{24}$$

with real-valued $f \in L^2\big(0,T; H^{-\alpha/2}_\eta(\mathbb{R}^d)\big)$ and real-valued initial condition $g \in L^2_\eta(\mathbb{R}^d)$ has a unique weak solution $u \in W^1\big(0,T; H^{\alpha/2}_\eta(\mathbb{R}^d), L^2_\eta(\mathbb{R}^d)\big)$, and the estimate

$$\|u\|_{W^1(0,T;H^{\alpha/2}_\eta(\mathbb{R}^d),L^2_\eta(\mathbb{R}^d))} \leq C(T)\big(\|f\|_{L^2(0,T;H^{-\alpha/2}_\eta(\mathbb{R}^d))} + \|g\|_{L^2_\eta(\mathbb{R}^d)}\big)$$

with a constant $C(T) > 0$, only depending on T, is satisfied.

The space $W^1\big(0,T; H^{\alpha/2}_\eta(\mathbb{R}^d), L^2_\eta(\mathbb{R}^d)\big)$ consists of those functions $u \in L^2\big(0,T; H^{\alpha/2}_\eta(\mathbb{R}^d)\big)$ that have a derivative with respect to time $\partial_t u$ in a distributional sense that belongs to the space $L^2\big(0,T; \big(H^{\alpha/2}_\eta(\mathbb{R}^d)\big)'\big)$. For a Hilbert space H, the space $L^2\big(0,T;H\big)$ denotes the space of functions $u : [0,T] \to H$, that are weakly measurable and that satisfy $\int_0^T \|u(t)\|_H^2\,\mathrm{d}t < \infty$. For the definition of weak measurability and for a detailed introduction of the space $W^1\big(0,T; H^{\alpha/2}(\mathbb{R}^d), L^2(\mathbb{R}^d)\big)$ that relies on the Bochner integral, we refer to Wloka (1987).

Proof. To apply the classical result on existence and uniqueness of solutions of linear parabolic equations in Hilbert spaces, see e.g. Wloka (1982, Satz 25.5, p. 381), it is at this point sufficient to verify the Gårding inequality of the bilinear form $a : [0,T] \times H^{\alpha/2}_\eta(\mathbb{R}^d) \times H^{\alpha/2}_\eta(\mathbb{R}^d) \to \mathbb{R}$ uniformly in $t \in [0,T]$. For $\varphi \in H^{\alpha/2}_\eta(\mathbb{R}^d)$, we conclude

$$\Re\big(a_t(\varphi,\varphi)\big) = \frac{1}{(2\pi)^d}\int \Re\big(A_t(\xi - i\eta)\big)\big|\mathcal{F}_\eta(\varphi)(\xi)\big|^2\,\mathrm{e}^{2\langle\eta,\xi\rangle}\,\mathrm{d}\xi\,.$$

From the Gårding condition (A3) and an elementary calculation the Gårding inequality

$$\Re\big(a_t(\varphi,\varphi)\big) \geq C_2\|\varphi\|^2_{H^{\alpha/2}_\eta(\mathbb{R}^d)} - C_3'\|\varphi\|^2_{L^2_\eta(\mathbb{R}^d)}\,.$$

with constants $C_2 > 0$ and $C_3' \geq 0$ follows uniformly in $t \in [0,T]$. □

6. Explicit solution of the Fourier transformed equation and a Feynman–Kac formula

In Theorem 5.3, we showed the existence of a unique solution of the parabolic equation (24) under (A1)–(A3) which are defined on page 15. We will now look for a more explicit form of this solution in the homogeneous case $f \equiv 0$. Since we are moreover interested in a stochastic representation of the solution that usually corresponds to an evolution problem with given terminal condition, we replace the operator $\mathcal{A}_t$ in equation (24) with $\mathcal{A}_{T-t}$. Thus we consider

$$\begin{aligned} \partial_t u + \mathcal{A}_{T-t} u = & \quad 0 \\ u(0) = & \quad g\,, \end{aligned} \tag{25}$$

with real-valued initial condition $g \in L^2_\eta(\mathbb{R}^d)$. It will turn out that the weak solution has an explicit Fourier transform. Furthermore it is smooth.

In order to derive the Fourier representation, let us notice that a function u that belongs to the space $W^1\big(0,T; H^\alpha_\eta(\mathbb{R}^d), L^2_\eta(\mathbb{R}^d)\big)$ is a solution of the linear parabolic equation (25), if and only if

$$\mathcal{F}_\eta(\partial_t u) + \mathcal{F}_\eta(\mathcal{A}_{T-t} u) = 0 \quad \text{in } L^2\big(0,T; \widehat{H}^{-\alpha}(\mathbb{R}^d)\big) \tag{26}$$

and

$$\mathcal{F}_\eta\big(L^2_\eta - \lim_{t\downarrow 0} u(t)\big) = \mathcal{F}_\eta(g)\,. \tag{27}$$

It is a consequence of the continuity of the Fourier transform $\mathcal{F}_\eta$ with respect to the L^2_η-norm, that equation (27) is equivalent to $L^2_\eta - \lim_{t\downarrow 0} \mathcal{F}_\eta(u(t)) = \mathcal{F}_\eta(g)$. Furthermore the equality $\mathcal{F}_\eta\big(\partial_t u\big) = \partial_t \mathcal{F}_\eta(u)$ can be derived inserting the definition of the Bochner integral: For every $\psi \in C_0^\infty((0,T))$ the following chain of equalities for elements in the Hilbert space $\big(H^\alpha_\eta(\mathbb{R}^d)\big)'$ holds,

$$\begin{aligned} \int_0^T \mathcal{F}_\eta\big(\partial_s u(s)\big)\big(\psi(s)\big)\,\mathrm{d}s &= \mathcal{F}_\eta\left(\int_0^T \big(\partial_s u(s)\big)\big(\psi(s)\big)\,\mathrm{d}s\right) = -\mathcal{F}_\eta\left(\int_0^T \big(u(s)\big)\big(\partial_s\psi(s)\big)\,\mathrm{d}s\right) \\ &= -\int_0^T \mathcal{F}_\eta\big(u(s)\big)\big(\partial_s\psi(s)\big)\,\mathrm{d}s = \int_0^T \big(\partial_s \mathcal{F}_\eta\big(u(s)\big)\big)\big(\psi(s)\big)\,\mathrm{d}s\,. \end{aligned}$$

From Theorem 4.1 we conclude

$$\mathcal{F}_\eta(\mathcal{A}_t v) = A_t(\cdot - i\eta)\mathcal{F}_\eta(v) \quad \text{for all } v \in H^s_\eta(\mathbb{R}^d)\,.$$

Altogether, we have $u \in W^1\big(0,T; H^{\alpha/2}_\eta(\mathbb{R}^d), L^2_\eta(\mathbb{R}^d)\big)$ is a solution of equation (25), iff $\mathcal{F}_\eta(u)$ belongs to the space $W^1\big(0,T; \widehat{H}^{\alpha/2}_\eta(\mathbb{R}^d), L^2_\eta(\mathbb{R}^d)\big)$ and $\mathcal{F}_\eta(u)$ solves the ordinary differential equation (ODE)

$$\begin{aligned} \partial_t \mathcal{F}_\eta(u) + A_{T-t}(\cdot - i\eta)\mathcal{F}_\eta(u) = 0 \\ \mathcal{F}_\eta(u)(t=0) = \mathcal{F}_\eta(g)\,. \end{aligned}$$

Theorem 6.1. *Assume (A1)–(A3). The function $u \in W^1\big(0,T; H^{\alpha/2}_\eta(\mathbb{R}^d), L^2_\eta(\mathbb{R}^d)\big)$ is a weak solution of equation* (25)*, iff $\mathcal{F}_\eta(u) \in W^1\big(0,T; \widehat{H}^{\alpha/2}_\eta(\mathbb{R}^d), L^2_\eta(\mathbb{R}^d)\big)$ and the Fourier transform*

$\mathcal{F}_\eta(u)$ solves the ODE

$$\begin{aligned} \partial_t \mathcal{F}_\eta\big(u(t)\big)(\xi) + A_{T-t}(\xi - i\eta)\mathcal{F}_\eta(u(t))(\xi) &= 0 \qquad \text{in } (0,T) \text{ for a.e. } \xi \in \mathbb{R}^d \\ \mathcal{F}_\eta\big(u(t=0)\big) &= \mathcal{F}_\eta(g)\,. \end{aligned} \tag{28}$$

The solution of (28) *is given by*

$$\mathcal{F}_\eta\big(u(t)\big)(\xi)) = \mathcal{F}_\eta(g)(\xi)\,\mathrm{e}^{-\int_{T-t}^T A_s(\xi - i\eta)\,\mathrm{d}s} \tag{29}$$

and hence

$$u(t,x) = \frac{\mathrm{e}^{-\langle \eta, x\rangle}}{(2\pi)^d} \int\limits_{\mathbb{R}^d} \mathrm{e}^{-i\langle \xi, x + i\eta\rangle}\,\mathcal{F}_\eta(g)(\xi)\,\mathrm{e}^{-\int_{T-t}^T A_s(\xi - i\eta)\,\mathrm{d}s}\,\mathrm{d}\xi \tag{30}$$

is the weak solution of equation (25)*. If furthermore the mapping $t \mapsto A_t(\xi - i\eta)$ is continuous for every fixed $\xi \in \mathbb{R}^d$, then we have $u \in C^1\big((0,T); H^m_\eta(\mathbb{R}^d)\big)$ for every $m \in \mathbb{N}$ and hence $u \in C^1\left((0,T), C^m(\mathbb{R}^d)\right)$ for every $m \geq 0$ is the point wise solution of the equation* (25)*.*

As a direct consequence, under the additional assumption that $g_\eta \in L^1$, we obtain the stochastic representation

$$u(T-t,x) = E\big(g(L^t_{T-t} + x)\big) \tag{31}$$

with $L^t_{T-t} := L_T - L_t$. We use this notation since the process $L^u := \big(L_{u+s} - L_u\big)_{s\geq 0}$ is a PIIAC as well, and the local characteristics of L^u, $(b^{L^u}_s, \sigma^{L^u}_s, F^{L^u}_s)$, with respect to the truncation function h are given by $\big(b_{u+s}, \sigma_{u+s}, F_{u+s}\big)$.

To show equation (31), we fix t and T and we set

$$U(x) := E\big(g(L^t_{T-t} + x)\big) = \mathrm{e}^{-\langle \eta, x\rangle}\,E\big(\,\mathrm{e}^{-\langle \eta, L^t_{T-t}\rangle}\,g_\eta(L^t_{T-t} + x)\big)\,,$$

then a short calculation based on Fubini's theorem provides

$$\mathcal{F}_\eta(U)(\xi) = \mathrm{e}^{-\langle \eta, \xi\rangle}\,\mathcal{F}(U_\eta)(\xi) = \mathcal{F}_\eta(g) E\big(\,\mathrm{e}^{i\langle L_T - L_t, i\eta - \xi\rangle}\,\big) = \mathcal{F}_\eta(g)\,\mathrm{e}^{-\int_t^T A_s(\xi - i\eta)\,\mathrm{d}s}\,.$$

Let us notice that the real-valued initial function g results in a real-valued solution u of the parabolic equation. This stems from the fact that $\mathcal{A}_t\varphi$ is real-valued, if $\varphi \in C_0^\infty(\mathbb{R}^d, \mathbb{R})$. Let us mention that this property of a PDO $\mathcal{A}$ can be translated to symmetry properties of the corresponding symbol, compare e.g. p. 206 in Glau (2010).

Remark 6.2. Theorem 6.1 illuminates the parallelism between Fourier and PIDE methods for option pricing. The PIDE for a European option is interpreted as a pseudo differential equation, then the Fourier transform is applied which results in an ordinary differential equation that can be solved explicitly. The solution leads to equation (30) which coincides with the famous convolution formula for option prices, derived independently in Carr and Madan (1999) and Raible (2000). See Eberlein, Glau and Papapantoleon (2010) for a derivation of the formula under conditions similar to (A1)–(A3).

Proof of Theorem 6.1: Our previous arguments show the equivalence of equations (25) and (28). Equations (29) and (30) are immediate consequences. Hence we are left to show that the function u defined by equation (29) resp. (30) satisfies $u \in C^1\left((0,T), H^m_\eta(\mathbb{R}^d)\right)$ and $u \in C^1\left((0,T), C^m(\mathbb{R}^d)\right)$ for every $m \geq 0$.

An elementary calculation provides that the Gårding inequality yields

$$\Re\big(A_s(\xi - i\eta)\big) \geq C_1|\xi|^\alpha - C_2 \qquad \big(s \in (0,T),\, \xi \in \mathbb{R}^d\big)\,.$$

with a strictly positive constant C_1 and $C_2 \geq 0$. Whence the inequality

$$\mathrm{e}^{-\int_s^t \Re(A_u(\xi - i\eta))\,\mathrm{d}u} \leq c_2\,\mathrm{e}^{-(t-s)C_1|\xi|^\alpha} \tag{32}$$

with a positive constant c_2 independent of $s \in [0,T]$.

We derive successively for every $t \in (0,T)$ and for every $m \geq 0$

(i) $u(t) \in H^m_\eta(\mathbb{R}^d)$,
(ii) $\lim_{s\to t} \|u(t) - u(s)\|_{H^m_\eta(\mathbb{R}^d)} = 0$,
(iii) $\partial_t u(t) = \mathcal{F}_\eta^{-1}\Big(A_{T-t}(\cdot - i\eta)\mathcal{F}_\eta\big(u(t)\big)\Big) \in H^m_\eta(\mathbb{R}^d)$ and
(iv) $\lim_{s\to t} \|\partial_t u(t) - \partial_s u(s)\|_{H^m_\eta(\mathbb{R}^d)} = 0$

hence $u \in C^1\left((0,T), H^m_\eta(\mathbb{R}^d)\right)$ for every $m \geq 0$. In view of the smoothness of the weight function, we conclude from the Sobolev embedding theorem, compare e.g. Wloka (1982), that the function u also belongs to the space $C^1\big((0,T), C^m(\mathbb{R}^d)\big)$ for every $m \geq 0$.

Let us first estimate the norm of u,

$$\begin{aligned}
\|u(t)\|^2_{H^m_\eta(\mathbb{R}^d)} &= \int |\mathcal{F}_\eta(u(t))(\xi)|^2 (1+|\xi|)^{2m}\,\mathrm{e}^{2\langle\eta,\xi\rangle}\,\mathrm{d}\xi \\
&= \int |\mathcal{F}_\eta(g)(\xi)|^2 \big|\,\mathrm{e}^{-2\int_{T-t}^T A_s(\xi-i\eta)\,\mathrm{d}s}\,\big|(1+|\xi|)^{2m}\,\mathrm{e}^{2\langle\eta,\xi\rangle}\,\mathrm{d}\xi \\
&= \int |\mathcal{F}_\eta(g)(\xi)|^2\,\mathrm{e}^{-2\int_{T-t}^T \Re(A_s(\xi-i\eta)\,\mathrm{d}s}(1+|\xi|)^{2m}\,\mathrm{e}^{2\langle\eta,\xi\rangle}\,\mathrm{d}\xi \\
&\leq c_2 \int |\mathcal{F}_\eta(g)(\xi)|^2\,\mathrm{e}^{2\langle\eta,\xi\rangle}\,\mathrm{e}^{-2tC_1|\xi|^\alpha}(1+|\xi|)^{2m}\,\mathrm{d}\xi \\
&< \infty
\end{aligned}$$

for every $t > 0$ and every $m \geq 0$.

In order to derive (ii) we conclude

$$\begin{aligned}
&\|u(t) - u(s)\|^2_{H^m_\eta(\mathbb{R}^d)} \\
&= \left\|\mathcal{F}_\eta(g)\,\mathrm{e}^{-\int_{T-s}^T A_u(\cdot - i\eta)\,\mathrm{d}u}\,\big|\,\mathrm{e}^{-\int_{T-t}^{T-s} A_u(\cdot-i\eta)\,\mathrm{d}u} - 1\big|\right\|^2_{\widehat{H}^m_\eta(\mathbb{R}^d)} \\
&= \int_{\mathbb{R}^d} \big|\mathcal{F}(g_\eta)(\xi)\big|^2\,\mathrm{e}^{-2\int_{T-s}^T \Re(A_u(\xi-i\eta))\,\mathrm{d}u}\left|\,\mathrm{e}^{-\int_{T-t}^{T-s} A_u(\xi-i\eta)\,\mathrm{d}u} - 1\right|^2 \big(1+|\xi|\big)^{2m}\,\mathrm{d}\xi \\
&\to 0 \quad (s \to t)\,,
\end{aligned}$$

which follows by dominated convergence if $t > \epsilon > 0$ or $m = 0$, since $\big|\,\mathrm{e}^{-\int_{T-t}^{T-s} A_u(\xi-i\eta)\,\mathrm{d}u} - 1\big| \to 0$ for $s \to t$ and

$$\sup_{\xi\in\mathbb{R}^d} \big|\,\mathrm{e}^{-\int_{T-t}^{T-s} A_u(\xi-i\eta)\,\mathrm{d}u} - 1\big| \leq const\,.$$

In order to derive the explicit expression for the Fourier transform of the time derivative of u given in (iii) we consider

$$\begin{aligned}&\Big\|\frac{u(t)-u(s)}{t-s}-\mathcal{F}_\eta^{-1}\big(A_{T-t}(\cdot-i\eta)\mathcal{F}_\eta(u(t))\big)\Big\|_{H^m_\eta(\mathbb{R}^d)} \qquad (33)\\ &=\left\|\mathcal{F}_\eta(g)\left(\frac{\mathrm{e}^{-\int_{T-t}^T A_u(\cdot-i\eta)\,\mathrm{d}u}-\mathrm{e}^{-\int_{T-s}^T A_u(\cdot-i\eta)\,\mathrm{d}u}}{t-s}-A_{T-t}(\cdot-i\eta)\,\mathrm{e}^{-\int_{T-t}^T A_u(\cdot-i\eta)\,\mathrm{d}u}\right)\right\|_{\widehat{H}^m_\eta(\mathbb{R}^d)}.\end{aligned}$$

From the continuity of $t\mapsto A_t(\xi-i\eta)$ for every fixed $\xi\in\mathbb{R}^d$ we get

$$\frac{\mathrm{e}^{-\int_{T-t}^T A_u(\xi-i\eta)\,\mathrm{d}u}-\mathrm{e}^{-\int_{T-s}^T A_u(\xi-i\eta)\,\mathrm{d}u}}{t-s}\to A_{T-t}(\xi-i\eta)\,\mathrm{e}^{-\int_{T-t}^T A_u(\cdot-i\eta)\,\mathrm{d}u}\qquad\text{for } s\to t$$

for every fixed $\xi\in\mathbb{R}^d$. From inequality (32) and assumption (A2) it follows

$$\Big|A_{T-t}(\cdot-i\eta)\,\mathrm{e}^{-\int_{T-t}^T A_u(\cdot-i\eta)\,\mathrm{d}u}\Big|\le C_2\big(1+|\xi|\big)^\alpha\,\mathrm{e}^{-tC_1|\xi|^\alpha} \tag{34}$$

with a positive constant C_2. Because of the continuity of $t\mapsto A_t(\xi-i\eta)$ for every fixed $\xi\in\mathbb{R}^d$ the mean-value theorem moreover yields together with inequality (32) and assumption (A2)

$$\left|\frac{\mathrm{e}^{-\int_{T-t}^T A_u(\xi-i\eta)\,\mathrm{d}u}-\mathrm{e}^{-\int_{T-s}^T A_u(\xi-i\eta)\,\mathrm{d}u}}{t-s}\right|\le C_3\big(1+|\xi|\big)^\alpha\,\mathrm{e}^{-(t\wedge s)C_1|\xi|^\alpha} \tag{35}$$

with a constant $C_3>0$. Hence by dominated convergence we get that the term (33) vanishes for $s\to t$ for any $t\in(0,T)$, which shows $\partial_t u(t)=\mathcal{F}_\eta^{-1}\big(A_{T-t}(\cdot-i\eta)\mathcal{F}_\eta(u(t))\big)\in H^m_\eta(\mathbb{R}^d)$ for every $m\ge 0$.

The continuity of the time derivative as a function $[\epsilon,T]\to\widehat{H}^m_\eta(\mathbb{R}^d)$ i.e. assertion (iv) follows in a similar way. □

7. Examples

We provide examples of options and (time-inhomogeneous) Lévy processes that satisfy the assumptions (A1)–(A3).

7.1. **Examples of payoff functions.** We list some typical payoff functions g of options in terms of the logarithmic stock price together with the weight η such that g belongs to the weighted space $L^2_\eta(\mathbb{R}^d)=\{g|x\mapsto g(x)\,\mathrm{e}^{\langle\eta,x\rangle}\in L^2(\mathbb{R}^d)\}$. Notice that each of these payoffs requires a damping factor $\eta\neq 0$.

First we consider options on a single stock i.e. $d=1$.

- (Call). The payoff function of the call in logarithmic variables is given as $g(x)=(S_0\,\mathrm{e}^x-K)^+$ and $g\in L^2_\eta(\mathbb{R})$ for every $\eta<-1$.
- (Put). For the put option we have $g(x)=(K-S_0\,\mathrm{e}^x)^+$ with $g\in L^2_\eta(\mathbb{R})$ for every $\eta>0$.
- (Power option). The payoff function corresponding to a power call is $g(x)=\big((S_0\,\mathrm{e}^x-K)^+\big)^h$ for some constant $h>0$. Here, $g\in L^2_\eta(\mathbb{R})$ for every $\eta<-h$.
- (Digital option). A digital or cash-or-nothing up and out option with level B has a payoff function of the form $g(x)=\mathbb{1}_{x<b}$ with $b:=\log(B/S_0)$. We have that $g\in L^2_\eta(\mathbb{R})$ for every $\eta>0$.

- (Asset-or-nothing option). An asset-or-nothing down and out option with level B is a binary option with a payoff function of the form $g(x) = S_0 \,\mathrm{e}^x \,\mathbb{1}_{x>b}$ with $b := \log(B/S_0)$. We have that $g \in L^2_\eta(\mathbb{R})$ for every $\eta < -1$.
- (European asset-or-nothing option). An asset-or-nothing down and out option with level B on a call option has a payoff function of the form $g(x) = (S_0 \,\mathrm{e}^x - K)^+ \mathbb{1}_{x>b}$ with $b := \log(B/S_0)$. We have that $g \in L^2_\eta(\mathbb{R})$ for every $\eta < -1$.

We proceed with examples of options on several assets. We denote by $S_t = (S_t^1, \dots, S_t^d)$ the vector of d different assets.

- (Basket option). A basket option (put) with strike K pays out $\Big(K - \sum_{i=1}^d a_i S_T^i\Big)^+$ at maturity T, where a_i denotes certain nonnegative weights. The corresponding payoff function in logarithmic variables is $g(x_1, \dots, x_d) = \Big(K - \sum_{i=1}^d a_i S_0^i \,\mathrm{e}^{x_i}\Big)^+$. We have $g \in L_\eta(\mathbb{R}^d)$ for any $\eta = (\eta_1, \dots, \eta_d)$ such that every component is positive, i.e. $\eta_j > 0$ for every $j = 1, \dots, d$.
- (Worst-of call). The payoff function of a worst-of call is given as $g(x_1, \dots, x_d) = \max\Big(\min\big(S_0^1 \,\mathrm{e}^{x_1}, \dots, S_0^d \,\mathrm{e}^{x_d}\big) - K, 0\Big)$. Notice that $g \in L^2_\eta(\mathbb{R}^d)$ for every $\eta = (\eta_1, \dots, \eta_d)$ with negative components and such that the sum over the components is smaller than -1, i.e. $\eta_j < 0$ for every $j = 1, \dots, d$ and $\sum_{j=1}^d \eta_j < -1$.
- (Best-of put). The payoff of a best-of put is of the form $g(x_1, \dots, x_d) = \max\Big(K - \max\big(S_0^1 \,\mathrm{e}^{x_1}, \dots, S_0^d \,\mathrm{e}^{x_d}\big), 0\Big)$ and $g \in L_\eta(\mathbb{R}^d)$ for every $\eta = (\eta_1, \dots, \eta_d)$ with positive components, i.e. $\eta_j > 0$ for every $j = 1, \dots, d$.

7.2. **Examples of Lévy processes.**

Example 7.1 (σ positive definite)**.** For Lévy processes with a Brownian part and a positive definite covariance matrix σ, the assumptions (A2) and (A3) with $\alpha = 2$ are satisfied for every choice of $\eta \in \mathbb{R}^d$ such that assumption (A1) is satisfied. In particular, for the Brownian motion with or without drift the assumptions are satisfied for $\alpha = 2$ and every $\eta \in \mathbb{R}^d$.

In order to derive the Gårding condition, we conclude by Lemma 2.2 (c)

$$\Re\big(A(\xi - i\eta')\big) = A(-i\eta') + \frac{1}{2}\langle \xi, \sigma\xi\rangle - \int \big(\cos(\langle \xi, x\rangle) - 1\big) F^{-\eta'}(\mathrm{d}x)\,.$$

Since the integrand is nonpositive and σ is positive definite we get

$$\Re\big(A(\xi - i\eta')\big) \geq A(-i\eta') + \frac{1}{2}\underline{\sigma}|\xi|^2\,,$$

where $\underline{\sigma}$ denotes the smallest eigenvalue of the matrix σ. The Gårding condition follows, since $|A(-i\eta')|$ is bounded for all $\eta' \in R_\eta$ by some constant only depending on η which can be shown similarly to inequality (8) by summing up over all possible combinations of signs. The continuity condition can be derived in a similar way.

Example 7.2 (GH-processes)**.** Univariate GH-processes are real-valued Lévy processes with parameters $\lambda \in \mathbb{R}$, $\alpha' > 0$, β such that $-\alpha' < \beta < \alpha'$ and $\delta > 0$. The assumptions (A1)–(A3) are satisfied with index $\alpha = 1$ for every $\eta \in \mathbb{R}$ with

$$\beta - \alpha' < \eta < \beta + \alpha' .$$

Let us briefly derive that statement. It is shown in Raible (2000, Appendix A.1) that the characteristic function of the GH-distribution has an analytic extension for $z \in \mathbb{C}$ to the domain $-\alpha' < \beta - \Im(z) < \alpha'$. In particular this entails $E\,\mathrm{e}^{-\eta L_t} < \infty$ and hence $\int_{|x|>1} \mathrm{e}^{-\eta x} F(\mathrm{d}x) < \infty$ for $-\alpha' < \beta - \eta < \alpha'$. From Lemma 2.2 we obtain the following representation of the symbol A of the GH-process

$$A(\xi - i\eta) = A(-i\eta) + ib^{-\eta}\xi + \int \left(\mathrm{e}^{-i\xi x} - 1 - i\xi x\right) \mathrm{e}^{-\eta x} F^{GH}(\mathrm{d}x)\,,$$

where $b^{-\eta} = \mu + \int (\mathrm{e}^{-\eta x} - 1)\, x F_t^{GH}(\mathrm{d}x)$ and $(\mu, 0, F^{GH})$ are the local characteristics of the GH-process with respect to the truncation function $h(x) = x$.

Moreover, the Lévy measure F^{GH} has a Lebesgue density f^{GH}, whose behavior around the origin is explored in Raible (2000). The asymptotic behavior around the origin remains unaffected when multiplying with the term $\mathrm{e}^{-\eta\cdot}$. Therefore the statement can be proven as in the case $\eta = 0$ which is treated in Glau (2011).

CGMY processes can be discussed along the same lines. We now turn to a multivariate example.

Example 7.3 (Multivariate NIG-process)**.** Let L be an $\mathbb{R}^d$-valued NIG-process, i.e.

$$L_1 = (L_1^1, \dots, L_1^d) \sim \mathrm{NIG}_d(\alpha, \beta, \delta, \mu, \Delta),$$

with the following set of parameters: $\alpha, \delta \geq 0$, and $\beta, \mu \in \mathbb{R}^d$, and a symmetric positive definite matrix $\Delta \in \mathbb{R}^{d \times d}$ such that $\alpha^2 > \langle \beta, \Delta\beta \rangle$. Then the characteristic function of L_1 in $u \in \mathbb{R}^d$ is given by

$$E\,\mathrm{e}^{i\langle u, L_1\rangle} = \exp\left(i\langle u, \mu\rangle + \delta \left(\sqrt{\alpha^2 - \langle \beta, \Delta\beta\rangle} - \sqrt{\alpha^2 - \langle \beta + iu, \Delta(\beta + iu)\rangle} \right) \right),$$

where the square root of a complex number is uniquely specified by $\sqrt{z} := \sqrt{r}\,\mathrm{e}^{i\varphi/2}$ for $z = r\,\mathrm{e}^{i\varphi}$ with $r > 0$ and $\varphi \in [0, 2\pi)$.

The assumptions (A1)–(A3) are satisfied for the index $\alpha = 1$ for any $\eta \in \mathbb{R}^d$ such that $\alpha^2 > \langle \beta - \eta, \Delta(\beta - \eta)\rangle$.

Let us sketch the derivation of this statement. For

$$\begin{aligned} z :=& \alpha^2 - \langle \beta - iu, \Delta(\beta - iu)\rangle \\ =& \alpha^2 - \langle \beta, \Delta\beta\rangle + \langle u, \Delta u\rangle + i\langle \beta, \Delta u\rangle + i\langle u, \Delta\beta\rangle \end{aligned} \tag{36}$$

we obtain $|z| \geq \alpha^2 - \langle \beta, \Delta\beta\rangle - \langle u, \Delta u\rangle \geq 0$ and hence

$$\begin{aligned} \Re\big(A(u)\big) =& -\delta\sqrt{\alpha^2 - \langle \beta, \Delta\beta\rangle} + \delta\Re\big(\sqrt{z}\big) \\ =& \frac{\delta}{\sqrt{2}}\sqrt{|z| + \Re(z)} - \delta\sqrt{\alpha^2 - \langle \beta, \Delta\beta\rangle} \\ \geq& \delta\sqrt{\alpha^2 - \langle \beta, \Delta\beta\rangle + \langle u, \Delta u\rangle} - \delta\sqrt{\alpha^2 - \langle \beta, \Delta\beta\rangle} \\ \geq& \delta\sqrt{\lambda_{\min}}|u| - \delta\sqrt{\alpha^2 - \langle \beta, \Delta\beta\rangle}\,, \end{aligned}$$

where $\lambda_{\min}$ denotes the smallest Eigenvalue of the matrix Δ. Analogously, $|A(u)| \leq C(1 + |u|)$ for a positive constant C can be derived.

To conclude, we notice that inserting $u' := u - i\eta$ in (36) is equivalent to replacing β by $\beta - \eta$.

7.3. **Examples of time-inhomogeneous Lévy processes.** For time-inhomogeneous Lévy processes we make the following assumptions on the local characteristics $(b_t, \sigma_t, F_t)_{t\geq 0}$.

Assumption 7.4.

$$\sup_{t\in[0,T]} \Big\{ |b_t| + \|\sigma_t\|_{\mathcal{M}(d\times d)} + \int \big(|x|^2 \wedge 1\big) F_t(\mathrm{d}x) \Big\} < \infty \,.$$

Time-inhomogeneous Lévy processes with local characteristics $(b_t, \sigma_t, F_t)_{t\geq 0}$ that have a Brownian part with $(\sigma_t)_{t\in[0,T]}$ being uniformly positive definite and that satisfy an appropriate exponential moment condition, satisfy assumptions (A1)–(A3):

Example 7.5. Let L be a PIIAC with symbol $(A_t)_{t\geq 0}$, PDO $(\mathcal{A}_t)_{t\geq 0}$ and characteristic triplet $(b_t, \sigma_t, F_t)_{t\geq 0}$. If Assumption 7.4 is satisfied and

$$\sup_{t\in[0,T]} \int_{|x|>1} \mathrm{e}^{\langle \eta', x\rangle}\, F_t(\mathrm{d}x) < \infty \qquad \text{for all } \eta' \in U_{-\eta}\,,$$

which is a stronger condition than assumption (A1), and if furthermore the family of matrices $(\sigma_t)_{t\geq 0}$ is uniformly positive definite in the following sense,

$$\inf_{t\in[0,T]} \|\sigma_t\|_{\mathcal{M}(d\times d)} \geq \underline{\sigma} > 0\,,$$

then A satisfies the continuity and Gårding condition (A2) and (A3) with index $\alpha = 2$.

In order to show this, we conclude from Lemma 2.2

$$\big|A_t(\xi - i\eta)\big| \leq |A_t(-i\eta)| + \big|\langle b_t^{-\eta}, \xi\rangle\big| + \frac{1}{2}\big|\langle \xi, \sigma_t\xi\rangle\big| + \Big|\int \Big(\mathrm{e}^{-i\langle \xi, x\rangle} - 1 + i\langle \xi, h(x)\rangle\Big)\, F_t^{-\eta}(\mathrm{d}x)\Big|$$

with $b^{-\eta}$ and $F^{-\eta}$ as in the lemma. Inserting Assumption 7.4 yields

$$\begin{aligned} |A_t(-i\eta)| &\leq \sup_{t\in[0,T]} \left\{ |b_t||\eta| + \frac{1}{2}\|\sigma_t\|^2_{\mathcal{M}(d\times d)}|\eta|^2 + \Big|\int \Big(\mathrm{e}^{-\langle \eta, x\rangle} - 1 + \langle \eta, h(x)\rangle\Big)\, F_t(\mathrm{d}x)\Big| \right\} \\ &\leq c_1(\eta)\,, \end{aligned}$$

and together with

$$|b_t^{-\eta}| \leq |b_t| + \big|\sigma_t\eta'\big| + \int \big|\,\mathrm{e}^{-\langle \eta, x\rangle} - 1\big| h(x) F_t(\mathrm{d}x) \leq c_2(\eta)$$

and

$$\big|\langle b_t^{-\eta}, \xi\rangle\big| + \frac{1}{2}\big|\langle \xi, \sigma_t\xi\rangle\big| \leq \sup_{t\in[0,T]} |b_t^{-\eta}||\xi| + \sup_{t\in[0,T]} \|\sigma_t\|^2_{\mathcal{M}(d\times d)}|\xi|^2$$

we get the continuity condition

$$\begin{aligned} \big|A_t(\xi - i\eta)\big| &\leq |A_t(-i\eta)| + \big|\langle b_t^{-\eta}, \xi\rangle\big| + \frac{1}{2}\big|\langle \xi, \sigma_t\xi\rangle\big| + \Big|\int \Big(\mathrm{e}^{-i\langle \xi, x\rangle} - 1 + i\langle \xi, h(x)\rangle\Big)\, F_t^{-\eta}(\mathrm{d}x)\Big| \\ &\leq c(\eta)\big(1 + |\xi| + |\xi|^2\big) \end{aligned}$$

for some positive constants $c_1(\eta)$, $c_2(\eta)$ and $c(\eta)$. On the other hand we have

$$\Re\big(A_t(\xi - i\eta)\big) = \Re\big(A_t(-i\eta)\big) + \frac{1}{2}\langle \xi, \sigma_t \xi\rangle - \int \big(\cos\left(\langle \xi, x\rangle\right) - 1\rangle\big) F_t^{-\eta}(\mathrm{d}x)$$
$$\geq \min_{t\in[0,T]} \|\sigma_t\|^2_{\mathcal{M}(d\times d)} |\xi|^2 - \sup_{t\in[0,T]} \left|A_t(-i\eta)\right|,$$

whence the Gårding condition.

Example 7.6. A natural class of time-inhomogeneous Lévy processes is obtained through a time-dependent transformation of the symbol of a given Lévy process:

Let L be an $\mathbb{R}^d$-valued Lévy process and a special semimartingale with local characteristics (b, c, F) w.r.t. the truncation function $h(x) = x$ and with symbol A.

We define the process X as the time-inhomogeneous Lévy process whose symbol $(A_t^X)_{t\geq 0}$ is given by

$$A_t^X(\xi) := A\big(\xi f(t)\big) \tag{37}$$

for some measurable function $f : [0, \infty) \to \mathbb{R}_+$ which is bounded away from zero and from above on $[0, T]$, i.e. there exist constants $0 < f_*, f^*$ such that $f_* \leq f(t) \leq f^*$ for every $t \in [0, T]$. For the sake of a simple notation we choose f to be constant after T, i.e. $f(t) := f(T)$ for $t \geq T$.

Notice that the process X is well-defined and is a special semimartingale whose local characteristics $(b_t^X, c_t^X, F_t^X)_{t\geq 0}$ w.r.t. $h(x) = x$ are given by $b_t^X = f(t)b$, $c_t^X = f(t)c$ and

$$F_t^X(B) = \int_{\mathbb{R}^d} \mathbb{1}_B\big(f(t)x\big) F(\mathrm{d}x) \qquad \text{for every } B \in \mathcal{B}\left(\mathbb{R}^d \setminus \{0\}\right).$$

Assume that for the Lévy measure F and symbol A of the Lévy process L, the assumptions (A1)–(A3) are satisfied for a fixed index $\alpha \in (0, 2]$ and a certain $\eta \in \mathbb{R}^d$. Then, for the time-inhomogeneous process X i.e. for $(F_t^X)_{t\geq 0}$ and $(A_t^X)_{t\geq 0}$, the assumptions (A1)–(A3) are satisfied for the same index α and for $\tilde{\eta} := \eta/f^* = \big(\eta_1/f^*, \ldots, \eta_d/f^*\big)$.

Example 7.7 (Modified Sato processes)**.** Sato processes are time-inhomogeneous Lévy processes $(L_t)_{t\geq 0}$ such that the random variable $L_1 := X$ has a self-decomposable law and

$$E[\mathrm{e}^{i\langle u, L_t\rangle}] = E[\mathrm{e}^{i\langle ut^\gamma, X\rangle}]. \tag{38}$$

A probability measure μ on $\mathbb{R}^d$ is called self-decomposable, if for any $b > 1$ there exists a measure ρ_b on $\mathbb{R}^d$ such that

$$\hat{\mu}(z) = \hat{\mu}(b^{-1}z)\hat{\rho}_b(z).$$

It is shown in Corollary 15.11 in Sato (1999) that self-decomposable laws can be characterized as infinitely divisible laws on $\mathbb{R}^d$ whose Lévy measure F has a Lebesgue density $F(\mathrm{d}x) = \frac{k(x)}{|x|}\,\mathrm{d}x$ with a function $k(x) \geq 0$ that is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. In particular, Sato processes are a true subclass of the class of time-inhomogeneous Lévy processes obtained by the specification (38) with arbitrary infinitely divisible random variables X.

This is a construction similar to the previous example: Sato processes are obtained via the transformation (37) with $f(t) := t^\gamma$ for some positive exponent γ. However, such a time-change is excluded in Example 7.6 by the boundedness condition. The reason is that the related PIDE would degenerate at the final time point T i.e. for $\mathcal{A}_0$ since $f(0) = 0$ and hence

A_0 is constant. Such degenerations can cause severe problems both in the analysis of the equations and in developing efficient algorithms. See e.g. Reichmann (2012) where a similar type of degeneracy is studied.

Carr, Geman, Madan and Yor (2007) use Sato processes to model stock prices. These processes provide a superior approach to fit a surface of observed option prices along strikes and maturities if one has to consider a large scale of different maturities (for the specific behaviour at long maturities see Eberlein and Madan (2011)). With just one additional parameter, the γ, one gains considerable accuracy also in comparison with stochastic volatility models with many parameters. With the Sato process approach we observe a degeneration of the symbol for small times that causes problems in the analysis and the numerical treatment of the resulting PDEs. Therefore we suggest a modification of this approach by introducing a time-inhomogeneous process X by its symbol $(A_t)_{t\geq 0}$ which we set

$$A_t(\xi) := A\big(\xi f(t)\big)$$

for a smooth function $f : [0,\infty) \to [1,\infty)$ that is monotonically increasing and behaves asymptotically as t^γ for $t \uparrow \infty$. As in Example 7.6, under the assumptions (A1)–(A3) on the Lévy measure F and symbol A of a Lévy process and special semimartingale L for a fixed index $\alpha \in (0,2]$ and a certain $\eta \in \mathbb{R}^d$ we obtain the following:

For the time-inhomogeneous process X i.e. for $(F_t^X)_{t\geq 0}$ and $(A_t)_{t\geq 0}$, the assumptions (A1)–(A3) are satisfied for the same index $\alpha \in (0,2]$ and for $\tilde{\eta} := \eta/f^* = \eta/f(T)$.

8. Application to call prices

We consider a market of one stock driven by a PIIAC process, i.e. $S_t = S_0 \,\mathrm{e}^{L_t}$ for any $t \geq 0$, and a deterministic interest rate r_t that is continuously compounding, i.e. the discount factor is $\mathrm{e}^{-\int_0^t r_s \,\mathrm{d}s}$.

Let us price a call option with payoff $G(S_T) = (S_T - K)^+$ at maturity. The logarithmic price function is $g(x) := G(S_0 \,\mathrm{e}^x) = (S_0 \,\mathrm{e}^x - K)^+$. For any $\eta < -1$ we have $g \in L^2_\eta(\mathbb{R})$.

Furthermore let us be given the driving process L in the stock price model, $S_t = S_0 \,\mathrm{e}^{L_t}$ as a PIIAC with local characteristics $(b_t, c_t, F_t)_{t\geq 0}$ with respect to a truncation function h under a risk-neutral measure and in order to ensure that the model is arbitrage-free, the characteristics are assumed to satisfy the following drift condition

$$b_t = r_t - \frac{c_t}{2} - \int_{\mathbb{R}} \big(\mathrm{e}^x - 1 - h(x)\big) F_t(\mathrm{d}x).$$

Moreover, we assume that the local characteristics $(b_t, c_t, F_t)_{t\geq 0}$ and the symbol A satisfy the assumptions (A1)–(A3) for some fixed constants $\alpha \in [1,2]$, $0 \leq \beta < \alpha$ and $\eta < -1$. Using $d = 1$ and $\eta < -1$, these assumptions simplify to:

(A1) $\int_0^T \int_{|x|>1} \mathrm{e}^{-\eta x} F_s(\mathrm{d}x)\,\mathrm{d}s < \infty$ *(resp.* $\int_0^T \int_{x>1} \mathrm{e}^{|\eta| x} F_s(\mathrm{d}x)\,\mathrm{d}s < \infty$*).*

(A2) *There exists a constant* $C_1 > 0$ *uniformly in time, with*

$$\big|A_t(z)\big| \leq C_1\big(1 + |z|\big)^\alpha$$

for all $z \in \mathbb{C}$ *with* $0 \leq \Im(z) \leq |\eta|$ *and for all* $t \in [0,T]$. *(Continuity condition)*

(A3) *There exist constants* $C_2 > 0$ *and* $C_3 \geq 0$ *uniformly in time, such that for a certain* $0 \leq \beta < \alpha$

$$\Re\big(A_t(z)\big) \geq C_2\big(1 + |z|\big)^\alpha - C_3\big(1 + |z|\big)^\beta$$

for all $z \in \mathbb{C}$ *with* $0 \leq \Im(z) \leq |\eta|$ *and for all* $t \in [0,T]$. *(Gårding condition)*

In this framework, the discounted asset price $\widetilde{S}_t := \mathrm{e}^{-\int_0^t r_s\,\mathrm{d}s} S_t$ and the discounted fair price $\widetilde{\Pi}_t := \mathrm{e}^{-\int_0^t r_s\,\mathrm{d}s}\Pi_t$ are martingales where

$$\Pi_T = G(S_T) = g(L_T).$$

In order to apply Theorem 6.1 and the Feynman–Kac representation (31), we proceed through a change of variables to incorporate the discount factor into the equation.

Corollary 8.1. *The fair price satisfies*

$$\Pi_t = \mathrm{e}^{-\int_t^T r_s\,\mathrm{d}s} E\big[G(S_T)\big|\mathcal{F}_t\big] = \mathrm{e}^{-\int_t^T r_s\,\mathrm{d}s} E\big[g(L_T)\big|L_t\big] = U(T-t, L_t) \quad a.s.,$$

where U is the unique weak solution in $W^1\big(0,T;H_\eta^{\alpha/2}(\mathbb{R}), L^2_\eta(\mathbb{R})\big)$ of the equation

$$\begin{aligned} \partial_t U + \mathcal{A}_{T-t}U + r_{T-t}U &= 0 \\ U(0,\cdot) &= G \end{aligned} \tag{39}$$

with

$$\begin{aligned} \mathcal{A}_{T-t}\varphi(x) = & -\frac{c_{T-t}}{2}\partial^2_{xx}\varphi(x) - \Big(r_{T-t} - \frac{c_{T-t}}{2} - \int_{\mathbb{R}} (\mathrm{e}^y - 1 - h(y))F_{T-t}(\mathrm{d}y)\Big)\partial_x\varphi(x) \\ & - \int_{\mathbb{R}} \Big(\varphi(x+y) - \varphi(x) - h(y)\partial_x\varphi(x)\Big)F_{T-t}(\mathrm{d}y). \end{aligned} \tag{40}$$

Proof. Since the discounted asset price $\widetilde{S}_t := \mathrm{e}^{-\int_0^t r_s\,\mathrm{d}s} S_t$ and the discounted fair price $\widetilde{\Pi}_t := \mathrm{e}^{-\int_0^t r_s\,\mathrm{d}s}\Pi_t$ are martingales and the process L is PIIAC we have

$$\Pi_t = \mathrm{e}^{-\int_t^T r_s\,\mathrm{d}s} E\big[G(S_T)\big|\mathcal{F}_t\big] = \mathrm{e}^{-\int_t^T r_s\,\mathrm{d}s} E\big[g(L_T - L_t + y)\big]\big|_{y=L_t}.$$

For $\widetilde{U}(T-t,y) := E\big[g(L_T - L_t + y)\big]$ we conclude from Theorem 6.1 and the Feynman–Kac representation (31) that $\widetilde{U}$ is the unique weak solution $\widetilde{U} \in W^1\big(0,T;H_\eta^{\alpha/2}(\mathbb{R}), L^2_\eta(\mathbb{R})\big)$ of the equation

$$\begin{aligned} \partial_t\widetilde{U} + \mathcal{A}_{T-t}\widetilde{U} &= 0 \\ \widetilde{U}(0,\cdot) &= g\,. \end{aligned} \tag{41}$$

For the function

$$U(\tau,x) := \mathrm{e}^{-\int_{T-\tau}^T r_s\,\mathrm{d}s}\,\widetilde{U}(\tau,x)$$

we obtain on the one hand

$$U(T-t, L_t) = \mathrm{e}^{-\int_t^T r_s\,\mathrm{d}s}\,\widetilde{U}(T-t,L_t) = \Pi_t \tag{42}$$

almost surely. On the other side, U is the unique weak solution $U \in W^1\big(0,T;H_\eta^{\alpha/2}(\mathbb{R}), L^2_\eta(\mathbb{R})\big)$ of the equation

$$\begin{aligned} \partial_t U + \mathcal{A}_{T-t}U + r_{T-t}U &= 0 \\ U(0,\cdot) &= g\,. \end{aligned} \tag{43}$$

□

In order to demonstrate that efficient algorithms to solve the pricing PIDE (39) are available, we show in Figure 1 the call prices for varying initial stock prices in the CGMY-model which is a pure jump Lévy model. The code was developed in the working group of Prof. Schwab and is based on a wavelet-Galerkin scheme combined with a hp-discontinuous Galerkin method. We chose a level of 9 which corresponds to 2^{10} grid points. On a standard laptop the Matlab routine took 4.57 seconds.

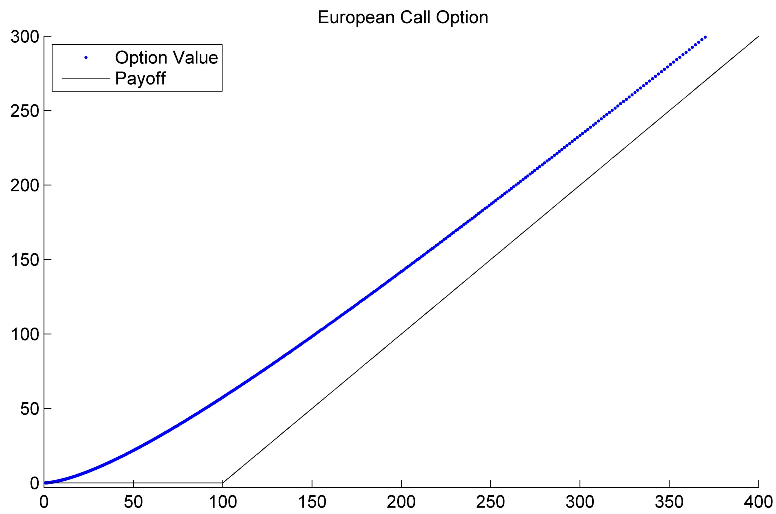

FIGURE 1. The figure shows the price of the call option in a pure jump CGMY model as a function of the initial stock value S_0 for a maturity of half a year, strike price $K = 100$, interest rate $r = 0.02$, and parameters $C = 2.0$, $G = 0.5$, $M = 1.2$ and $Y = 1.1$.

APPENDIX A. A MULTIVARIATE VERSION OF CAUCHY'S THEOREM

We state a special multivariate version of Cauchy's theorem which is used in the proof of Theorem 4.1.

Lemma A.1. *Let* $R_j := (-\infty, \infty) \times i[b_j, \beta_j]$ *with* $-\infty < b_j < \beta_j < \infty$ *for* $j = 1, \dots, d$ *and* $Q_d = R_1 \times \dots R_d$*. Let* $f : Q_d \to \mathbb{C}$ *be holomorphic in the interior* $\overset{\circ}{Q}_d$ *of* Q_d*, and continuous on* $\overline{Q_d} = Q_d$*. Further, we assume the following integrability and convergence properties.*

(i) *Assume*

$$f(z) \to 0 \quad \text{for } |\Re(z)| \to \infty \text{ and } \Im(z) \in [b_1, \beta_1] \times \dots \times [b_d, \beta_d]$$

with $z = (z_1, \dots, z_d)$.

(ii) *For* $z_j = x_j + iy_j$ *with* $x_j \in \mathbb{R}$ *and* $y_j \in [b_j, \beta_j]$ *for* $j = 1, \dots, d$ *we assume*

$$\left|f(z_1, \dots, z_d)\right| \leq h(x_1, \dots, x_d) \quad \text{uniformly for } y \in [b_1, \beta_1] \times \dots \times [b_d, \beta_d]$$

with a function $h \in L^1(\mathbb{R}^d)$*. For* $d = 1$ *let* $c > 0$ *be a constant such that*

$$\left|h(x)\right| \leq c \quad \text{for all } x \in \mathbb{R}\,.$$

In the case $d > 1$ we additionally assume for every $j \in \{1, \ldots, d\}$ the existence of a function $h_j \in L^1(\mathbb{R}^{d-1})$ such that

$$\big|h(x_1, \ldots, x_d)\big| \le h_j(x_1, \ldots x_{j-1}, x_{j+1}, \ldots, x_d)$$

for all $(x_1, \ldots, x_d) \in \mathbb{R}^d$.

Then the following is true

$$\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} f(x_1 + ib_1, \ldots, x_d + ib_d)\,\mathrm{d}x_1 \ldots \mathrm{d}x_d$$
$$= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} f(x_1 + iy_1, \ldots, x_d + iy_d)\,\mathrm{d}x_1 \ldots \mathrm{d}x_d\,.$$

for every $y \in [b_1, \beta_1] \times \ldots \times [b_d, \beta_d]$.

For the proof of this lemma we refer to Glau (2010).

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