Maximally Acceptable Portfolios

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Abstract

Portfolios are selected in non-Gaussian contexts to maximize a Cherny and Madan index of acceptability. Analytical gradients are developed for the purpose of optimizing portfolio searches on the unit sphere. It is shown that though an acceptability index is not a preference ordering, many utilities will concur with acceptability maximization. A stylized economy illustrates the advantages from the perspective of acceptability of nonlinear securities and options. In sample results for the year 2008 indicate that maximizing the acceptability index can lead to portfolios that second order stochastically dominate their Gaussian counterparts. Backtests over the period 1997 to 2008 reflect gains to maximizing acceptability over holding a maximal Sharpe ratio portfolio.

Portfolio rebalancing is now possible and is being executed at much higher frequencies than has been possible in the past. Some algorithms trade every five to fifteen minutes a fairly large number of stocks ranging from a thousand stocks upward. It has been known for some time now that at such short horizons returns are extremely non-Gaussian displaying significant levels of skewness and excess kurtosis. Additionally modern economies directly provide access to nonlinear cash flows via the markets for options and variance swaps. Optimal portfolio selection in such non-Gaussian contexts is expected to diverge from the multivariate Gaussian model that essentially focuses on maximizing the Sharpe ratio. This is primarily due to the recognition that investors are not indifferent to other aspects of a return distribution and ceteris paribus they prefer positive skewness and peakedness and dislike tailweightedness. Kurtosis, as noted in Eberlein and Madan (2009), is a preferentially confused statistic as it combines both peakedness and tailweightedness.

The choice of criterion on which to base portfolio selection is then a critical issue and many alternatives have been formulated in the literature. We refer to Biglova, Ortobelli, Rachev and Stoyanov (2004) for a survey and application of a number of these criteria that all take the form of ratios using the expected return in the numerator and a suitably chosen risk measure in the denominator. Here we propose to follow the generalization of Sharpe ratios to arbitrage consistent performance measures developed in Cherny and Madan (2009). These measures are not based on ratios and they do not separate risk from reward. Instead they attempt to directly measure the quality of cash flow distributions accessed at zero cost. First, by construction nonnegative cash flows accessed at zero cost are considered to be infinitely good as they are arbitrages. For other cash flows that are exposed to losses, one computes a stressed expectation and the quality of the cash flow is proportional to the level of stress that it can withstand. The measures are termed acceptability indices and the higher the index the smaller is the set of cash flow distributions acceptable at this level. At all levels the set of acceptable cash flows forms, as random variables, a convex set containing all the non-negative cash flows.

We develop in this paper fast algorithms for maximizing the acceptability index attained by a portfolio and show how to operationalize and implement the optimization procedure. When working with a single underlier we explicitly introduce nonlinear payoffs and options. For multiple assets the exercise requires the specification of the non-Gaussian joint law of asset returns and we recognize that there are numerous ways to do this. The algorithm we develop requires that one be able to simulate the joint law for the assets of interest and many researchers would like to work with their favorite specifications in this domain. Since our focus is on explaining and operationalizing the maximization of an index of acceptability we adopt a fairly simple yet adequate formulation of the joint law for our purposes. We thereby leave refinements in this direction to future investigations. In formulating the joint law we follow the suggestions of Malevergne and Sornette (2005) and merely compute a covariance matrix after transformation of marginals to a standard normal variate by passing through the composition of the distribution function and inverse normal cumulative distribution function. Malevergne and Sornette (2005) estimate marginals in a modified Weibull family but as we construct samples from the joint law with some frequency on 50 stocks we just employ the empirical distribution function of our samples. Refinements associated with estimating and simulating more general and more complicated densities preferably associated with limit laws or self-decomposable random variables can easily be entertained in extensions. One may also reestimate a few parameters at each rebalance while reestimating the whole set at a lower frequency. These are considerations that must be analysed in developing an industrial strategy but are not essential for the initial exposition of procedures devoted to designing maximally acceptable trades.

The outline of the rest of the paper is as follows. Section 1 provides basic details on indices of acceptability. The algorithm for constructing maximally acceptable portfolios is developed in Section 2. Section 3 presents a stylized economy in which we study the advantages offered by nonlinear cash flows and options from the perspective of enhancing acceptability. Section 4 applies this algorithm to recent data covering the volatile period of the year ending in December 5, 2008. Section 5 presents a backtest rebalancing maximally acceptable portfolios every 5 days compared with a maximal Sharpe ratio investment. Section 6 concludes.

1 Acceptability Indices

We present here the essential details leading to the operational indices of acceptability defined in Cherny and Madan (2008). For this purpose, we model the financial outcomes of trading as zero-cost terminal cash flows seen as random variables on a probability space $(\Omega, \mathcal{F}, P)$. A short review of the development of acceptability indices and its links to more classical ideas may be helpful. For an expected utility maximizing investor, with utility function u, with a given random initial position W the set of zero cost random variables acceptable to this investor is given by the set of all random variables X such that $E[u(W + X)] \geq E[u(W)]$, or the classical better than set. This is typically a convex set containing the nonnegative cash flows. If one is interested in cash flows acceptable to many investors then one must intersect all such convex sets, but the result will remain a convex set containing the nonnegative cash flows. If we now shift attention to cash flows that move marginally in the direction X, leaving issues of size to other considerations like market depth or impact, then one may model the acceptable cash flows by the smallest convex cone containing all the classical better than convex sets.

Such a formulation for acceptable cash flows was axiomatized and adopted in Artzner, Delbaen, Eber, and Heath (1999) and studied further for its asset pricing implications in Carr, Geman, and Madan (2001). Such cones of acceptable cash flows are supported by a set of probability measures and cash flows are acceptable just if they have a positive expectation under all the supporting probability measures. It follows that the larger is the set of supporting measures the smaller is the cone of acceptability. Cherny and Madan (2008) went on to index a decreasing sequence of cones by a real valued level of acceptability with the property that the higher the level of acceptability, the larger the set of supporting measures. Cash flows with a positive expectation are acceptable at level zero while arbitrages are infinitely acceptable. They then constructed a performance measure for cash flows as the highest level of acceptability attained by a potential cash flow. Such performance measures based on indices of acceptability are a generalization of the Sharpe ratio and the Gain-Loss ratio of Bernardo and Ledoit (2000) and like them are scale invariant, but improve on the associated economic properties.

The construction of operational cones of acceptability led Cherny and Madan (2008) to consider law invariant cones of acceptability. Here the decision on the acceptability of a cash flow depends only on the distribution function. This property, though not ideal, is shared with expected utility, and all the various ratios used in risk analysis and mentioned earlier by reference to Biglova, Ortobelli, Rachev and Stoyanov (2004). Such law invariant operational cones of acceptability are related to a sequence of concave distortions $\Psi^{\gamma}(y)$ also studied in Eberlein and Madan (2009). Each function $\Psi^{\gamma}(y)$ is a concave distribution function defined on the unit interval with values in the unit interval that is pointwise increasing in the level of the distortion γ. A random variable X with distribution function $F(x)$ is acceptable at level γ just if its expectation under such a distortion is nonnegative or that

$$\int_{-\infty}^{\infty} x d\Psi^{\gamma}(F(x)) \geq 0.$$

The acceptability index of X, $\gamma^*(X)$ is then given by

$$\gamma^*(X) = \sup \left\{ \gamma \mid \int_{-\infty}^{\infty} x d\Psi^{\gamma}(F(x)) \geq 0 \right\}.$$

It may be tempting to think of the level of acceptability as a degree of risk aversion but this is not correct. A few remarks address the important differences. First, risk aversions may be increased to arbitrarily high levels depending on the preferences being represented. Levels of acceptability can not be increased in the same way as there is a market determined limit to the highest level possibly attainable. Second, we observe that increases in γ amount to a further distortion of probability and do not introduce greater concavity in utility. In fact there is no distortion of wealth, comparable to its utility, occurring in the definition of an acceptability index. We refer the reader to Jin and Zhou (2008) for a deeper discussion of all these distortions. However, we note by way of comparison to utility considerations that expectations under concave distortions are also expectations under a change of measure as, supposing the existence of the density $f(x)$ of the distribution function $F(x)$, we have that

$$\begin{aligned} \int_{-\infty}^{\infty} x d\Psi^{\gamma}(F(x)) &= \int_{-\infty}^{\infty} x \Psi^{\gamma\prime}(F(x)) f(x) dx \\ &= E^{Q}[X] \end{aligned}$$

where the change of measure is

$$\frac{dQ}{dP} = \Psi^{\gamma\prime}(F(X)). \tag{1}$$

Note that the measure change depends explicitly on the cash flow X as indicated in expression (1). We note that increased risk aversion introduces greater concavity and nonlinearity in the measure change and the same applies to increasing γ but as already noted, there are market determined limits to how far γ may be increased but no such limits apply to risk aversion.

Critical to the various levels of acceptability are the measures supporting acceptability at this level. Fortunately there is a clear understanding of these measures provided in Cherny (2006). One has to first construct the conjugate dual Φ^{γ} to the distortion defined by

$$\Phi^{\gamma}(x) = \sup_{0 \leq y \leq 1} \left(\Psi^{\gamma}(y) - xy \right)$$

and the supporting set of measures has densities Z with respect to P satisfying

$$E\left[(Z-c)^{+}\right] \leq \Phi^{\gamma}(c), \quad c \geq 0.$$

Cherny and Madan (2008) provide four examples of useful concave distortions. The first termed $MINVAR$ is given by

$$\Psi^{\gamma}(y) = 1-(1-y)^{1+\gamma}.$$

An expectation under this distortion for integral γ is easily seen to be the expectation of the minimum of $(1+\gamma)$ independent draws from the distribution function. Hence more generally we say that X is $MINVAR$ acceptable at level γ if the minimum of $1+\gamma$ independent draws has a positive expectation. A simple computation shows that the measure change (1) does not reweight large losses, when $F(x)$ is near zero, to arbitrarily high levels and hence the economic dissatisfication with this distortion. A similar critique accompanies the Gain-Loss ratio.

The second distortion termed $MAXVAR$ is given by

$$\Psi^{\gamma}(y) = y^{\frac{1}{1+\gamma}}.$$

Here large losses are reweighted up to infinity but the gains are not discounted to zero. Expectation under this distortion is from the distribution function of a random variable that is so bad that one has to make $1+\gamma$ independent draws and take the maximum outcome to get to the original distribution being evaluated. The other two combine these in two ways. We shall here work with $MINMAXVAR$ for which

$$\Psi^{\gamma}(y) = 1-(1-y^{\frac{1}{1+\gamma}})^{1+\gamma}$$

and we note that in this case both, large losses and large gains, are respectively reweighted up to infinity and down to zero. This property also holds for $MAXMINVAR$ for which

$$\Psi^{\gamma}(y) = \left(1-(1-y)^{1+\gamma}\right)^{\frac{1}{1+\gamma}}.$$

Given that an index of acceptability is a performance measure, like the Sharpe ratio, and not a preference ordering for an investor, the question arises as to why one should consider maximizing this index of acceptability. We recognize that though Sharpe ratios have been maximized in practice, we have been forewarned in numerous studies and we cite Goetzmann, Ingersoll, Spiegel and Welch (2002) and Agarwal and Naik (2004) about how such strategies may be preferentially inferior. It is well recognized that outside a Gaussian framework, one may for example increase the Sharpe ratio by accessing negative skewness on selling downside puts but actually take positions that decrease expected utilities.

When managing money for a single investor, expected utility is a well established and sound criterion, notwithstanding its more modern critique from the considerations of behavioral finance. One of the motivations behind acceptability is the recognition that money is often managed on behalf of large groups of individuals and here one would like to maximize the consent of a sizable set of economically sensible supporting kernels. Certainly an arbitrage would have the full consent of all rational kernels. We also recognize that if a random variable X second order stochastically dominates Y then it has a higher acceptability level. This is not true for many performance measures but it does hold for an index of acceptability. However, the implication does not go in the reverse direction though we shall encounter occasions where we are able to associate with a higher acceptability level a situation of second order stochastic dominance, in which case we have carried all preference orderings along.

Unlike the situation with Sharpe ratios, one has a much clearer understanding of all the preference orderings that will concur with a particular trade in a direction enhancing an index of acceptability. If the random variable X with distribution function F is acceptable at level γ for a distortion Ψ then we have that

$$\int_{-\infty}^{\infty} x d\Psi^{\gamma}(F(x)) \geq 0. \tag{2}$$

We also know that such a trade is marginally acceptable to a utility function u at a random initial wealth W provided

$$E\left[u'(W)X\right] \geq 0. \tag{3}$$

Now define by

$$\Lambda(x) = E\left[u'(W)|X = x\right]$$

and write

$$E\left[u'(W)X\right] = \int_{-\infty}^{\infty} x\Lambda\left(x\right) f(x)dx.$$

We now note that on the provision

$$\left(\Lambda(x) - \Psi^{\gamma\prime}(F(x))\right) x \geq 0$$

we have that (2) implies (3). Hence for investors whose expected marginal utility does not rise on losses beyond $\Psi^{\gamma\prime}(F(x))$ and does not fall on gains beyond $\Psi^{\gamma\prime}(F(x))$ a positive acceptability receives their consent. The importance of having $\Psi^{\gamma\prime}$ go to infinity and zero at the two extremes of zero and unity is now even clearer as we do expect marginal utilities to behave this way for a wide class of utility functions. We recognize that we will not necessarily carry all utilities but there is a large class that comes along. As mentioned earlier we shall have occasion to associate with a particular enhancement in acceptability a second order stochastic dominance and then we do carry all utility functions.

Acceptability is thus considerably differentiated from utility and in particular one does not have to specify a degree of risk aversion in working with acceptability as an objective. The acceptability level γ^* will be endogeneously determined through the optimization and unlike risk aversion, it is not an input that needs to be specified. One may then wonder what happens to investor preferences in this approach. They essentially go into the choice of distortions. For example the distortion $MINVAR$ is relatively lenient towards large losses with a maximal reweighting of losses capped at $1+\gamma$. Such a distortion will not carry many utility functions along with its decisions as the expected marginal utility $\Lambda(x)$ for losses will easily rise above this bound of $1+\gamma$. This is why the use of $MINMAXVAR$ is more conservative. However, once one has chosen a distortion that has a derivative rising sufficiently fast for losses and falling sufficiently fast for gains, its decisions will satisfy a sufficiently large number of utilities and one can concentrate on improving the quality of cash flows for wide collections of investors simultaneously, by maximizing acceptability and leaving issues of risk aversion aside.

2 Constructing Maximally Acceptable Portfolios

We develop in this section an efficient algorithm for constructing portfolios that are maximally acceptable over a prespecified finite set of potential stock investments. We envisage the investment as being on day t to be unwound either the next day or a few days later. The use of such a short horizon is predicated on the belief that we are unable to describe adequately multivariate return possibilities over long horizons using statistical data on recent daily returns. We may not be able to describe the possibilities over the short horizon either but we suspend our disbelief in this proposition and entertain a statistical approach to such short term investment.

Our first task is to describe the joint law for daily returns on n selected assets that we denote by $R=(R_1,\cdots,R_n)$. We suppose the marginal distribution function of the i^{th} return is $F_i(r)$. In constructing the joint law we follow Malevergne and Sornette (2005) and define standard Gaussian random variates Z_i by

$$Z_i = N^{-1}(F_i(R_i))$$

where $N(x)$ is the distribution function of a standard normal variate. We postulate that the variables Z_i are correlated with a correlation matrix C. They have unit variance and zero means by construction. The non-Gaussian nature of our returns is captured in the nonlinear transformation back with

$$R_i = F_i^{-1}(N(Z_i)).$$

We wish to construct a portfolio with h_i dollars invested long or short in asset i with the portfolio return

$$Y = h'R.$$

We wish to find the portfolio weights h with a view to maximizing the level of acceptability of the cash flow Y. The optimization will be conducted on a simulated sample space where we generate M readings on the n joint returns that are stored in the n by M matrix A. The portfolio returns on this sample space are then given by the vector

$$c = h'A.$$

We sort the vector c in increasing order to construct

$$s_i = c_{k(i)}$$

where s_1 is the smallest element and s_M is the largest element of the vector c. The acceptability index for the vector c, $\gamma(c)$ is implicitly defined by the equation

$$\sum_{i=1}^{M} s_i \left(\Psi^{\gamma} \left(\frac{i}{M} \right) - \Psi^{\gamma} \left(\frac{i-1}{M} \right) \right) = 0 \tag{4}$$

Given that acceptability indices are scale invariant by construction, the search for the optimal h may be restricted to the surface of the sphere in dimension n defined by $h'h = 1$. The search algorithm is then fairly simple once we have the gradient

$$\gamma_h = \frac{\partial \gamma}{\partial h}.$$

We merely follow the gradient to the point $h + \gamma_h$ which we renormalize to unit length and stop when the renormalized point equals the original point h. Hence for implementation we need an explicit gradient computation of γ_h.

Taking the total differential of (4) we get that

$$\begin{aligned} & \sum_{i=1}^{N} ds_i \left(\Psi^{\gamma}(\frac{i}{N}) - \Psi^{\gamma} \left(\frac{i-1}{N} \right) \right) \\ + & \sum_{i=1}^{N} s_i \frac{\partial}{\partial \gamma} \left[\Psi^{\gamma}(\frac{i}{N}) - \Psi^{\gamma} \left(\frac{i-1}{N} \right) \right] d\gamma \\ = & \ 0 \end{aligned}$$

From which it follows that

$$\frac{d\gamma}{ds_i} = - \frac{\Psi^{\gamma}(\frac{i}{N}) - \Psi^{\gamma}\left(\frac{i-1}{N}\right)}{\sum_{i=1}^{N} s_i \frac{\partial}{\partial \gamma} \left[\Psi^{\gamma}(\frac{i}{N}) - \Psi^{\gamma}\left(\frac{i-1}{N}\right) \right]}$$

For $MINMAXVAR$ we have that

$$\Psi^{\gamma}(y) = 1 - (1 - y^{\frac{1}{1+\gamma}})^{1+\gamma}$$

$$\begin{aligned} \frac{\partial}{\partial \gamma} \Psi^{\gamma}(y) &= - \left(1 - y^{\frac{1}{1+\gamma}} \right)^{1+\gamma} \ln \left(1 - y^{\frac{1}{1+\gamma}} \right) \\ & \quad - \left(1 - y^{\frac{1}{1+\gamma}} \right)^{\gamma} y^{\frac{1}{1+\gamma}} \frac{\ln(y)}{1+\gamma}. \end{aligned}$$

For the other distortions we have for $MINVAR$

$$\frac{\partial}{\partial\gamma}\Psi^{\gamma}(y) = -(1-y)^{1+\gamma}\ln(1-y)$$

For $MAXVAR$ we have

$$\frac{\partial}{\partial\gamma}\Psi^{\gamma}(y) = -y^{\frac{1}{1+\gamma}}\frac{\ln(y)}{(1+\gamma)^2}$$

Finally for $MAXMINVAR$ we have

$$\begin{aligned}\frac{\partial}{\partial\gamma}\Psi^{\gamma}(y) &= -\left(1-(1-y)^{1+\gamma}\right)^{\frac{1}{1+\gamma}}\frac{\ln\left(1-(1-y)^{1+\gamma}\right)}{(1+\gamma)^2} \\ &\quad -\frac{1}{1+\gamma}\left(1-(1-y)^{1+\gamma}\right)^{-\frac{\gamma}{1+\gamma}}(1-y)^{1+\gamma}\ln(1-y)\end{aligned}$$

To construct the partial of the acceptability index γ with respect to h_j we must evaluate

$$\frac{\partial\gamma}{\partial h_j} = \sum_i \frac{d\gamma}{ds_i}R_{jk(i)}.$$

We employ this gradient computation in a search restricted to the surface of the sphere $h'h = 1$ to find the portfolio that maximizes the acceptability index.

3 Nonlinearity and Acceptability in Economies

We consider in this section a stylized economy and the role played by nonlinear securities like variance swaps and options in enhancing the acceptability of cash flows that may be accessed in markets. The distortion employed is $MINMAXVAR$. Consider a two date one period economy with a single risky asset and a zero interest rate. The risky asset is assumed to be lognormally distributed with a mean rate of return of $\mu = .15$ and a volatility $\sigma = .35$. The final asset value is

$$S = \exp\left(\mu + \sigma Z - \frac{\sigma^2}{2}\right)$$

where Z is a standard normal variate. The initial price of this risky asset is unity and the pricing kernel or measure change is given by the measure change for the Black-Scholes economy with

$$\frac{dQ}{dP} = \exp\left(\alpha Z - \frac{\alpha^2}{2}\right)$$

for $\alpha = -\mu/\sigma$.

The first zero cost cash flow available to investors is the risky return

$$R = S - 1.$$

The level of acceptability of this cash flow using $MINMAXVAR$ is .2624.

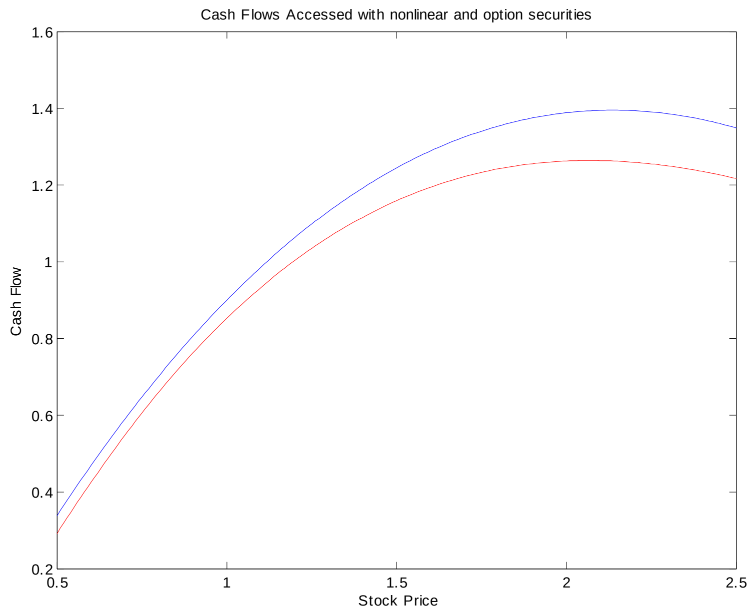

Figure 1: Cash flows accessed using squared and cubic securities in blue. Cash flows with options and nonlinear securities in red.

We now successively introduce nonlinear securities into this economy with cash flows given by R^2, R^3 and two out of the money options, a put on S struck at the 5% level and a call, struck at the 95% level. The specific strikes are .6141 and 1.9715. We price these securities using the measure change and the zero discount rate to get the prices .1724, .1258, .0056 and .0108 respectively.

Now on just introducing the squared return the level of acceptability rises to .2946 and the trade direction on the unit circle is .9186 shares and $-.3951$ units of the squared return. If we now introduce the claim paying R^3 the acceptability rises to .2971 and the trade direction is $(.9003, -.4346, .0226)$.

We next introduce the put option and then the call option. The levels of acceptability rise to .3001 and .3021 respectively. The final trade direction is $(.8528, -.5160, .0728, .0314, .00013)$ reflecting investment in the risky asset, shorting the squared return and buying skewness and some out of the money puts and calls. We present in Figure (1) the cash flow accessed with squared and cubic assets, and then the final cash flow including the options.

4 In sample application to portfolios constructed for the year 2008

It is well recognized that the year 2008 was very volatile with significant possibilities for departure from Gaussian returns. In the next section we shall consider backtesting over a much longer period starting in 1997 and finishing in December 2008. For this longer period we obtained data on 771 stocks that were continuously quoted among the top 1500 names over the whole period. In this section we consider three portfolios of 50 stocks made up of those with the top 50 realized means over the year, the second 50 and third 50 realized means. For each of these three sets of 50 stocks we first construct the benchmark Gaussian investment by normalizing to the unit sphere the vector

$$\begin{aligned} a &= V^{-1}m \\ g &= \frac{a}{\sqrt{a'a}} \end{aligned}$$

where V is the covariance matrix of the 50 returns over the year and m is vector of realized means over the year.

Next we transform to standard Gaussian variates using the empirical distribution function constructed from daily returns over the past year, (252 observations), we then compute the correlation matrix of these transformed variates. Finally we generate 10000 draws from a multivariate Gaussian model with this correlation matrix and transform back via $F_i^{-1}(N(x))$ to get 10000 joint readings on our 50 stocks. This gives us three sets of 50 by 10000, potential A matrices for which we implement the search procedure to find the maximally acceptable portfolio h for the distortion $MINMAXVAR$.

We then construct, for each of the three sets separately, the returns $g'A$ and $h'A$ and present in figures (2 to 4) the empirical densities for the Gaussian and maximally acceptable portfolios.

We observe that for the top 50 means there is a clear domination by the maximally acceptable portfolio of the Gaussian portfolio. To investigate this further we constructed the double integral of the empirical density or the integral of the distribution function to find that the Gaussian distribution function integral lies above the maximally acceptable distribution function integral for both, the top 50 and second 50, sets of portfolios. This suggests that the maximally acceptable portfolios second order stochastically dominate the Gaussian portfolios in these two cases. In this case all utility functions would prefer the maximally acceptable portfolio to the Gaussian one. We present in Figures (5 to 7) the integrals of these distribution functions.

We see clearly that for the third 50 stocks this domination is lost and we dominate only for utility functions that are strictly concave for large positive returns but are linear for small positive and negative returns.

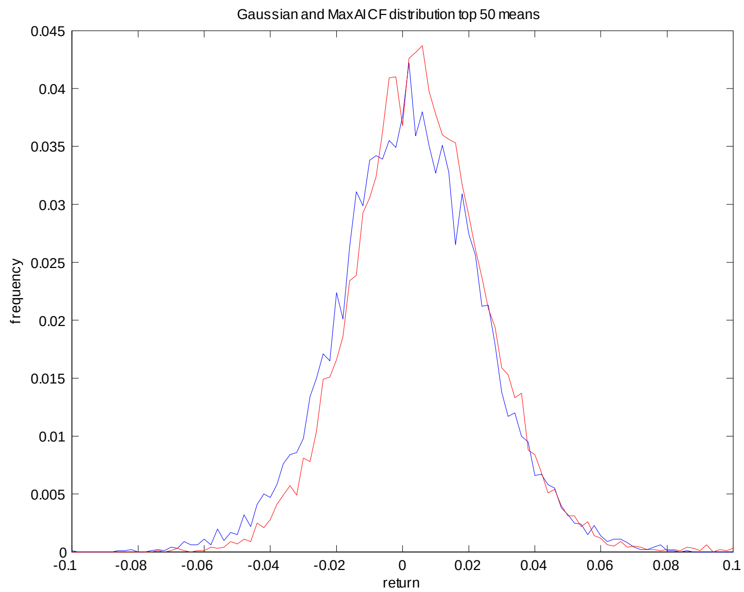

Figure 2: Gaussian empirical density in blue and maximally acceptable density in red for the stocks with the top 50 realized annual mean returns.

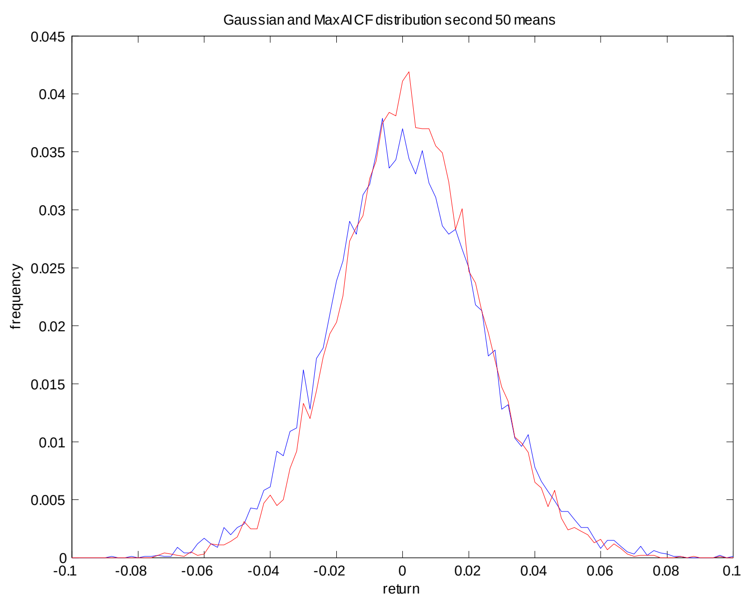

Figure 3: Gaussian empirical density in blue and maximally acceptable density in red for the stocks with the second 50 realized annual mean returns.

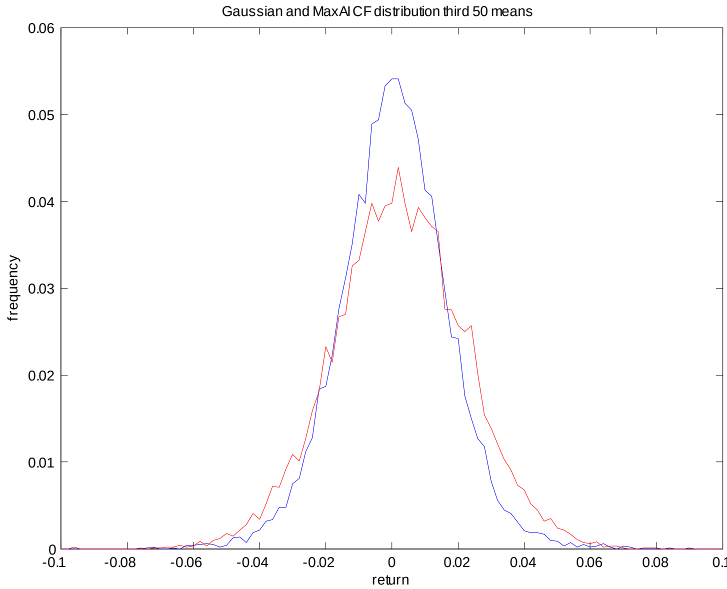

Figure 4: Gaussian empirical density in blue and maximally acceptable density in red for the stocks with the third 50 realized annual mean returns.

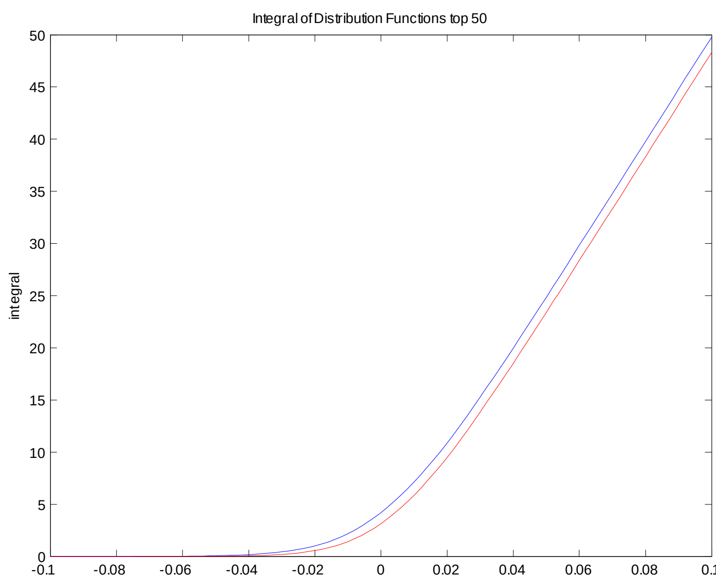

Figure 5: Distribution function integrals, Gaussian in blue and maximally acceptable in red, top 50

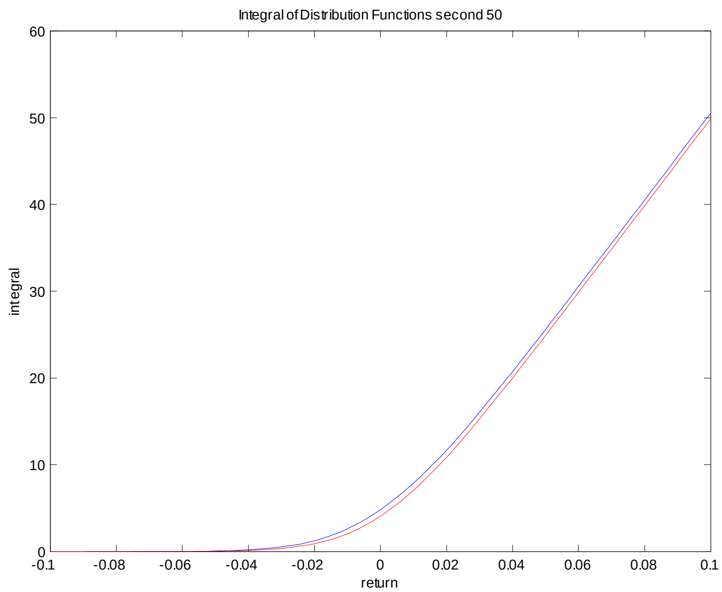

Figure 6: Distribution function integrals, Gaussian in blue and maximally acceptable in red, second 50

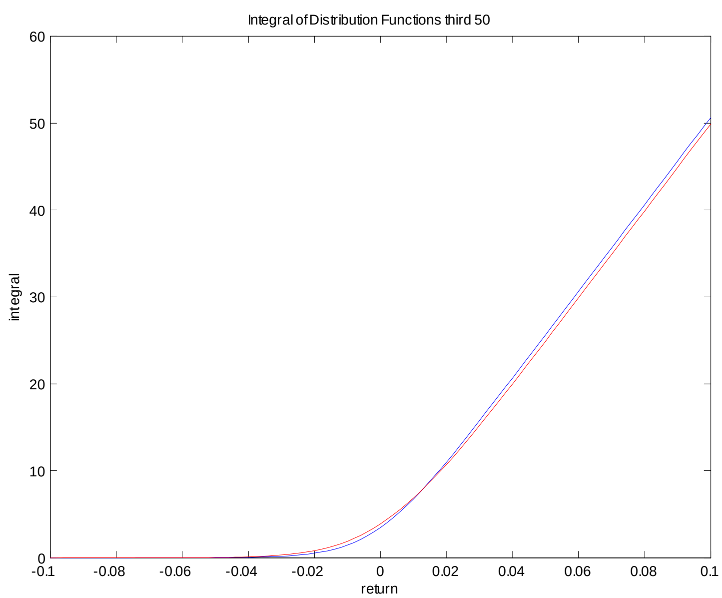

Figure 7: Distribution function integrals, Gaussian in blue and maximally acceptable in red, third 50

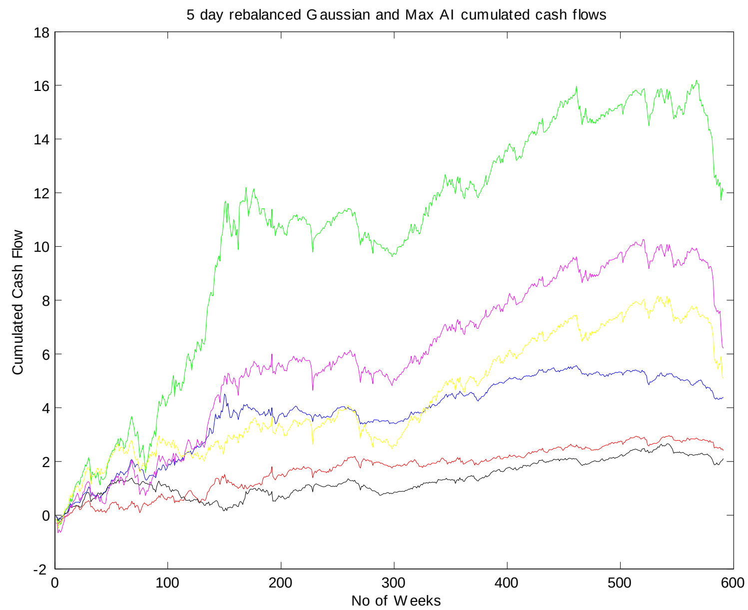

Figure 8: Cumulated cash flows maximally acceptable in green, magenta and yellow and Gaussian in blue, red and black for the top 50, second 50 and third 50 respectively.

5 Backtesting portfolio rebalancing from 1997 to 2008

We report in this section the results of a backtest where we start on March 10 1997 and end on November 28 2008, rebalancing portfolios every five days on the stocks with top 50, second 50 and third 50 realized mean returns over the past year. For each of these three sets of stocks we construct two portfolios, the straight Gaussian portfolio normalized to the unit sphere and the maximally acceptable one optimized on the unit sphere as per the construction described in section 2. Every five days we transform to standard Gaussians, draw from a suitably correlated Gaussian model 10000 joint return possibilities and maximize over the sphere for the portfolio h. Both the Gaussian and maximally acceptable portfolios are held for five days when they are unwound and a new portfolio is formed for the next five days.

There are in all six cash flows of length 591 for the 591 rebalancings that occurred over this period. They are the maximally acceptable and Gaussian results for the top, second and third 50 stocks for each rebalance day. We present in Figure (8) the backtested cumulated cash flows from these strategies.

We observe a clear domination of the top 50 over the second 50 and the third 50 for both strategies and a domination of the Gaussian by the maximally acceptable. The strategies took considerable losses towards the end of 2008, a phenomenon experienced by many strategies.

6 Conclusion

Portfolio selection in non-Gaussian environments is studied with a view towards maximizing an index of acceptability as defined in Cherny and Madan (2008). As the indices are scale invariant, optimal long short portfolios may be constructed by maximizing over the unit sphere. Analytical gradients are developed for the purpose of enhancing this search. The indices of acceptability are heuristically described as the maximum level of stress a potential cash flow can be subjected to before its stress distorted expectation turns negative. It is shown that though an acceptability index is not a preference ordering, it is related to preferences and certain well understood classes of utilities concur with its decisions. In fact, conditionally expected marginal utilities, conditional on the outcome, that rise less for losses and fall more for gains, than the derivative of the distortion taken at the cash flow quantile, agree with acceptability.

A stylized economy illustrates the acceptability enhancing features of non-linear securities and options. In sample results for the year 2008 indicate that some portfolios maximizing the acceptability index in fact second order stochastically dominate their Gaussian counterparts. Backtests over the period 1997 to 2008 reflect gains to maximizing acceptability over holding a maximal Sharpe ratio portfolio.

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