Modeling Risk Weighted Assets and the Risk Sensitivity of Related Capital Requirements

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Abstract

The use of internal Bank models for meeting capital requirements has been approved for some time. Regulators then face issues of model approval necessitating some public domain analysis of model performance. This paper presents a new approach to risk model evaluation using forward looking risk neutral probabilities. In addition to VaR and $CVaR$ we analyse a new measure termed here $RWAVaR$ that was proposed in Carr, Madan and Vicente Alvarez (2011). The new measure is a formalization of the popular concept of risk weighted assets used in the Basel approach to capital requirements. The formalization allows for a possible leveraging of information contained in bid and ask prices and the study reports on the potential of this approach. Capital measures using $RWAVaR$ are observed to be sensitive to volatility, the volatility of volatility, the skewness of return distributions and the volatility spread across maturities. Movements in bid ask spreads also strongly influence capital requirements. Additionally there is a potential for some procyclicality to be built into the requirements, particularly when one adapts requirements to movements in liquidity spreads.

1 Introduction

Risk Weighted Assets (RWA) have been a critical component of capital requirements of the Basel system since its inception in December 1992. Though initially set up to absorb systemic losses and attain cross country consistency the system moved in June 2004 to Basel II towards intertemporal consistency by allowing the use of internal risk based models. The models could in principle measure changes in risk levels and adjust capital requirements accordingly. There was and still is a concern related to "risk-weighted asset optimization" or the choice of models that minimize the risk held. There were reports in November 2011 (FT Wednesday November 9, 2011) of banks in Europe reducing capital requirements by declaring assets held to be less risky based on new models. However, the models employed must be approved by bank supervisors.

Consequently, there is an important interest in understanding the internal models, their properties and the precise risk sensitivities captured. In this regard one notes that even though Value at Risk or VaR has been criticised for its mathematical properties (Artzner, Delbaen, Eber and Heath (1999)) and for its destablizing potential, Danielsson et. al. (2001), Basak and Shapiro (2001), Leippold, Trojani and Vanini (2006), it remains the reference measure for market risk. In the U.S. the SEC requires mandatory VaR disclosures. The destablizing potential arises from countercyclicality in capital requirements that feed a boom with lower requirements and enhance a bust with higher requirements. It is then a regulatory interest that capital requirements be procyclical if possible.

Little is known about the internal VaR models used in banks and approval is granted on the basis of accuracy estimated on past statistical data. Perignon Deng and Wang (2006) observe that “Berkowitz and O’Brien (2002) published the first direct evidence on the performance of US banks’ internal VaR models.” Internal models have access to information on exact trading positions while external assessments must work with benchmark portfolios. Nonetheless a performance evaluation of models in the public domain is critical to informed regulatory decisions on model approval.

A particular interest in evaluating risk model performance are a deliniation of the precise risk sensitivities driving capital requirement variations? The covariations of regulatory capital requirements across time and entities with macro and micro economic factors are a further focus of study. A secondary interest relates to procyclicality of capital requirements that may reduce leverage at the height of a boom while relaxing the same in a down turn. To answer these questions we investigate the behaviour of capital requirments supporting stylized risk positions. The capital requirements themselves are are derived from a variety of models. Next we proceed to evaluate the risk sensitivities of the derived required capital.

The determination of required capital necessitates an identification of the data to be used along with a broad understanding about the purpose of such requirements. In consistency with the Basel accords we take the objectives to be an assessment of RWA that seek to determine the prevailing loss exposure. Given the focus on loss exposures we entertain the use of forward looking option price data with a view to complementing historical time series data that is more commonly used in the industry for model validation. This focus on option prices and risk neutral probabilities is further motivated by the need to ensure sufficient capital that makes risk positions acceptable to the general economy. From this perspective risk positions failing to earn required risk compensations reflected in risk neutral pricing are viewed as unacceptable. A simple calculation presented later shows that the loss exceedances at a prespecified risk neutral VaR quantile, will when evaluated on the historical measure, be far below the exceedances at a comparable historical quantile. In fact a typical risk neutral formulation would explain the results of Pérignon, Deng and Wang (2006), who observed 2 exceedances when 74 were historically expected.

An second and important reason for exploiting option prices is the relative wealth of this data for some entities. Additionally option data reflects many new risk features relevant for valuation which are not available in other data sources. Historically we get just one new observation per day. On the option surface we get a read on a multiplicity of risk neutral distributions with well over a hundred new observations each day. One gets access to changes in risk neutral skewness that on occasion may arise during a prolonged boom as investors buy downside puts to cover the risks related to an expected bust. Capital requirements based on option data can potentially leverage such skewness changes to increase capital requirements in a boom, thereby inducing some procyclicality in RWA computations. Of even greater interest is the use of data on explicit bid and ask prices. The bid ask spread can widen in a boom to discourage the accumulation of a rising inventory of downside puts. Both the skews and the bid ask spreads are sources for some procyclicality that may be difficult to access in historical return time series.

These risk neutral considerations suggest a new approach to measuring RWA that may be termed $RWAVaR$ for Risk Weighted Asset Value at Risk. The specific proposal follows Carr, Madan and Vicente Alvarez (2011) and is related to two price equilibrium theory as developed in Madan (2012). In this theory all market participants trade with an abstract market as their counterparty. By way of example, if a market participant acquires a positive random cash flow X then the markets position $-X$ is nonpositive and not formally acceptable to the market. Acceptability to the market is formally modeled by a positive expectation under a set of test measures. These test probabilities approve market acceptability when all test valuations are positive. A participant acquiring $X \geq 0$ must offer a price u making $u - X$ acceptable. The smallest such u turns out to be the supremum of all test valuations of X and is an upper valuation $u(X)$. The acquisition may be financed and not all of the acquisition price needs to be part of the equity capital. However, the valuation by the market of X as an asset requires a lower valuation l making $X - l$ market acceptable. This lower valuation is the infimum of all test valuations, $l(X) < u(X)$. Financing is limited to $l(X)$ and consequently the required equity capital can be chosen as the difference $u(X) - l(X)$. The same argument may be equally applied to all random cash flows X that need not be restricted to being nonnegative. In the two price economy the only prices observable in the market are $u(X)$ and $l(X)$ for all X and the law of one price is abandoned. The spread between the upper and lower prices depends on the size and the variety of the measures being used to test for acceptability and the law of one price is observed only for cash flows that happen to have an identical valuation under all test probabilities. We define $RWAVaR$ by the spread between the upper and lower valuations of such a two price equilibrium. We show here that when the upper and lower valuations are constructed from a single underlying risk neutral distribution then this capital charge is both an integral of $CVaR$ and an integral of VaR across the quantiles of this risk neutral distribution. Consequently we end up working with the industry benchmark which is applied risk neutrally.

Markets do provide access to bid and ask prices on numerous traded securities and one may consider their use as candidates for the lower and upper valuations of a two price economy. The risk charge or capital required would then be this difference. However, these market spreads probably do not reflect reasonable capital charges for holding a risk position for any but the shortest time spans as they are more closely related to turning over positions in relatively liquid markets as opposed to actually holding positions over a substantive time interval. Upper and lower valuations derived from a wide range of test probabilities would deliver spreads of an order of magnitude above market bid and ask spreads that can better reflect capital charges for holding such positions. In order to exploit and leverage data on market bid and ask prices we calibrate parametric models for valuation spreads to such market data but then we enhance the set of test probabilities used to construct the upper and lower valuations for the $RWAVaR$ computations.

The $RWAVaR$ approach is implemented first at an aggregate level using options on the $S\&P500$ index for the period 1996 to 2012 using only midquote option prices. At a disaggregated level we implement the model for four bank stocks over the period 2007 to 2012, using now the bid and ask prices separately. Finally we report on the risk sensitivities and cyclicality of the resulting $RWAVaR$ computations. At both the aggregate and disaggregate levels we observe some procyclicality. This paper presents an exploration of an important field that is in need of much greater attention at a rigorous level.

The alternative approach offered by $RWAVaR$ has the benefit of leveraging information in options markets to help monitor financial leverage, or at least it provides a complementary observation on the issue. There is a disadvantage related to breadth of securities for which we have information on option prices. Some of these disadvantages may be addressed by attempting to employ exchange traded funds as spanning factors. Many of these funds have option data but how well they span other relevant risks remains to be investigated. Instead of using a stock option prices, one could also use factor spanning methods coupled with option prices for the spanning factors.

The outline of the rest of the paper is as follows. Section 2 introduces the concept of $RWAVaR$ and its relationship to VaR. Section 3 presents the results applied at the aggregate level. In Section 4 we present results on $RWAVaR$ at the disaggregated level of positions in options on four banks over the period 2007 to 2011. Section 5 remarks on our use of risk neutral probabilities as compared with the physical measure. Section 6 concludes. The Appendix provides further technical details.

2 RWAVaR

Risk weighted assets provide a way to evaluate values that may be lost. The computation includes a write down of asset values coupled with liability add ons. The latter are associated in particular with derivative liabilities that accomodate an allowance for a possible unfavorable liability unwind. A model for assets which takes risk weights into account is then a dual model for the asset write down and the liability add on. Cherny and Madan (2010) proposed a model for such lower and upper valuations that incorporates model uncertainty using a Choquet capacity (Huang, Iancu Petrik and Subramanian (2011)). Madan and Schoutens (2011) and Madan (2012) formulated two price equilibrium models where the law of one price fails in equilibrium and the only price information available from markets is that of two nonlinear lower and upper prices. Carr, Madan and Vicente Alvarez (2011) use these two prices to define capital as the spread between the upper and lower prices. It represents the cost of potentially entering and exiting the market on unfavorable terms in both directions.

The nonlinearity of upper and lower valuations recognizes that the lower price for the sum of two positions is above the sum of the two lower prices taken separately. Similarly the upper price for the sum is below the sum of two upper prices. As a result the capital for the aggregate position, defined by the difference in upper and lower valuations, is smaller than the sum of component capitals. As a consequence it is tempting to package risks in capital determination. A conservative approach is to not allow for packaging and this is the position taken in current RWA computations, with some exceptions when deemed appropriate.

The risk weighting formulas in the Basel capital regulations, in particular, do not allow for portfolio effects Specifically the capital required in support of loans depends only on the risk of the loan and not on the portfolio to which it is added (Gordy, 2003). The approach of adding up singular capital charges has the disadvantage of ignoring diversification benefits and is generally justified only under comonotonicity or a strong single factor model. In general one should adopt a portfolio approach whenever there is demonstrable understanding of the joint probability laws or correlations involved. In the absence of such confidence in joint laws or the correlations involved, the practice of adding singular charges provides a conservative upper bound. For simplicity and with a view to avoiding issues of modeling joint returns we report results based on the additivity approach.

The literature on modeling the lower and upper prices in two price equilibrium economies begins by modeling acceptable risks thought of as random cash flows X by requiring such risks to have a positive expectation under a whole host of test probability measures $Q \in \mathcal{M}$. We term these Q scenario probabilities. The set of acceptable risks then forms a convex set $\mathcal{A}$ of random variables that contains the nonnegative random variables. The latter are always acceptable. The set of acceptable risks $\mathcal{A}$ is defined as

$$\mathcal{A} = \left\{ X \mid E^Q[X] \geq 0, \quad \text{all } Q \in \mathcal{M} \right\}$$

The set of test probabilities $\mathcal{M}$ may be thought of as the set of measures approving acceptability or membership in $\mathcal{A}$ of a random variable X.

The two prices, lower $l(X)$ and upper $u(X)$, in equilibrium (Madan (2012), Cherny and Madan (2010), Madan and Schoutens (2011)) are then given by infima and suprema across test probabilities. Formally we have

$$\begin{aligned} l(X) &= \inf_{Q \in \mathcal{M}} E^Q[X] \\ u(X) &= \sup_{Q \in \mathcal{M}} E^Q[X]. \end{aligned}$$

When acceptability is determined solely by the probability distribution function $F_X(x)$ of the random variable X then the lower and upper prices have a simpler representation. Before this may be presented we need to identify the distribution function involved.

For this purpose consider the classical complete markets context when the law of one price prevails for all X. In this case we have $l(X) = u(X)$ for all X and $\mathcal{M}$ has a single element given by an equilibrium risk neutral measure. Acceptability in the classical case reduces to positive risk neutral expectation or a positive alpha trade. In order to ensure that the acceptable risks in a two price economy are strictly smaller than the classically acceptable positive alpha trades we begin with the risk neutral distribution function. Such a choice ensures that acceptable trades for the two price economy continue to earn classical risk compensations.

For acceptability defined in terms of this risk neutral distribution function, it is shown in Cherny and Madan (2009) that there exists a concave distribution function $\Psi(u)$ defined on the unit interval $0 \leq u \leq 1$ such that

$$l(X) = \int_{-\infty}^{\infty} x d\Psi(F_X(x)) \tag{1}$$

$$u(X) = -\int_{-\infty}^{\infty} x d\Psi\left(1 - F_X(x)\right). \tag{2}$$

It is shown in Cherny and Madan (2010) that, in this context, the set of test measures Q approving acceptability, are given by all densities $Z(u)$ on the unit interval with antiderivatives L, satisfying $L' = Z$, and $L \leq \Psi$. This approach is related to Choquet integrals (see e.g. Schmeidler (1989) and Föllmer and Schied (2004)).

Further, it is shown in Carr, Madan and Vicente Alvarez (2011) that capital measured by the spread is an integral against the differential of the inverse distribution $G_X(u)$ defined by $F_X(G_X(u)) = u$. Specifically

$$u(X) - l(X) = \int_0^1 \left(\Psi(u) + \Psi(1-u) - 1\right) dG_X(u).$$

According to the arguments above we set

$$RWAVaR(X) = u(X) - l(X).$$

The following proposition relates $RWAVaR$ to both VaR and $CVaR$. First we note that

$$\begin{aligned} VaR(u) &= -G_X(u) \\ CVAR(u) &= -\frac{1}{u}\int_{-\infty}^{G_X(u)} x dF_X(x). \end{aligned}$$

Proposition 1 *$RWAVaR$ may be related to $CVaR$ and VaR by*

$$\begin{aligned} RWAVaR &= -\int_0^1 u\left(\Psi''(u) + \Psi''(1-u)\right) CVaR(u) du \\ &= \int_0^1 VaR(u) \int_u^1 -\left(\Psi''(v) + \Psi''(1-v)\right) dv\ du \end{aligned}$$

Proof. *See Appendix.* ■

We note that both $CVaR$ and $RWAVaR$ are integrals with respect to VaR across quantiles with some weight function. Traditionally capital is related to VaR under the historical probability. Recommendations have been made to improve the theoretical properties of the risk measure by using instead $CVaR$ under the historical probability. The recommendation flowing from two price equilibrium economies is to use instead $RWAVaR$ evaluated under a risk neutral probability. Though $RWAVaR$ may be computed directly without reference to VaR or $CVaR$ proposition 1 is useful in establishing the link between the concepts and justifies the nomenclature $RWAVaR$.

The particular distortion function $\Psi(u)$ which we employ in the paper was introduced in Cherny and Madan (2009, 2010). It is called minmaxvar and is defined by

$$\Psi(u) = 1 - \left(1 - u^{\frac{1}{1+\eta}}\right)^{1+\eta}, \qquad 0 \le u \le 1,\ \eta \ge 0$$

where the parameter η determines the level of concavity. When we take for η an integer value then this distortion combines the operations of evaluating the expectation of the minimum of $\eta + 1$ draws from the cash flow distribution function with that of evaluating the expectation from a distribution function such that the maximum of $\eta + 1$ draws has a distribution function matching the cash flow distribution function. The greater the value of η, the more concave is the distortion, making the set of acceptable risks $\mathcal{A}$ smaller. The choice of this distortion is motivated by the desire to reweight large losses upwards and large gains downwards. Expectation under concave distortion is also an expectation under a change of measure with the measure change being defined by the derivative of the distortion computed at the quantile. We therefore want $\Psi'(u)$ to go to infinity near zero and to go to zero near unity. This is accomplished by minmaxvar.

3 VaR, CVaR and RWAVaR at the aggregate level

For a hypothetical aggregate risk that may be referred to in making judgements about capital requirements in the economy we consider first the risk exposure of holding a position in the S&P 500 index for say three months, with the position initiated on a particular day. From our perspective this requires access to the risk neutral distribution of the index three months out. Such information is embedded in the prices of index options and given their relative liquidity we work here with just the mid quotes of option prices. We shall consider liquidity issues and the specific use of bid and ask prices separately when we consider risks at the level of individual stocks. At the level of individual stocks credit issues also play a part and strategies for incorporating credit risks in capital charges will also be addressed. For the aggregate index we initially put aside liquidity and credit considerations. What remains by way of data are mid quote option prices.

There are many prices at any given time covering a relatively random set of strikes depending on how the index has moved for a fixed number of maturities, that are themselves time varying. However, even though there are some hundred prices, there are many constraints on these prices imposed by the absence of static arbitrage as shown for example in Carr and Madan (2005). The number of degrees of freedom in movement is therefore much smaller than the number of option prices. Informed opinion from market participants and the literature places the number of degrees near four. A lot of information about the set of option prices varying across strikes and maturities at a particular point in calendar time is captured by four entities. They are i) level of at-the-money volatility, ii) the implied volatility skew, iii) the implied volatility convexity and iv) the spread of at-the-money volatilities at two maturities. The four entities may be mapped into the four parameters of the Sato process based on the variance gamma process. Carr, Geman, Madan and Yor (2007) showed that this process adequately synthesizes a static option surface at a given point in time.

The particular model chosen is not that critical and a number of models can serve as adequate synthesizers or interpolators for a static surface provided they are sufficiently rich in their parametric structure. Once calibrated they reproduce the market risk neutral distribution that is essentially unique between the smallest and highest calibrating strike. Outside this range there is little market information and the distributions reflect model tail extrapolations. However, many models calibrating the option prices also have similar tail decay rates and are expected to give similar results for the upper and lower prices studied here. The particular model employed here was termed the VGSSD model (see Appendix 2.) as the Sato process leverages the property of the variance gamma law at unit time. This law is self decomposable and one obtains the marginal distribution at all relevant maturities by scaling the self decomposable variable. In addition to the three parameters of the variance gamma law capturing volatility, skewness and kurtosis there is an additional fourth scaling parameter that calibrates the spread of at-the-money volatilities. The VGSSD model is employed as an interpolator summarizing mid quote option prices into four parameters which subsequently allows for the construction of risk distributions at any specific maturity.

The VGSSD model is based on a variance gamma law (Madan and Seneta (1990), Carr, Madan and Chang (1998)) for the logarithm of the stock price at unit time. The variance gamma law is the law of the random variable $X = \theta G + \sigma\sqrt{G}Z$, where G is a gamma variate with unit mean and variance ν while Z is an independent standard normal variable. Conditional on the gamma random variable G the logarithm of the stock is normally distributed with mean θG and standard deviation $\sigma\sqrt{G}$. The volatility ν of the gamma random variable then serves as a volatility of volatility parameter and captures the kurtosis of the distribution. Skewness is controlled by θ while the base volatility is σ. In the $VGSSD$ model the distribution of the logarithm of the stock at maturities t is given in distribution by the law of $t^{\gamma}X$ for a scaling parameter γ. With the addition of this scaling parameter γ that controls the spread of volatilities between two maturities we have a four parameter model for all option prices across strikes and maturities at market close for each day, where the parameters are σ, ν, θ, and γ.

At market close on each day from the start of January 1996 to the end of December 2011 we estimate the four parameters of the VGSSD model using maturities between one month and six months. Thus we obtain time series σ_t,

ν_t, θ_t and γ_t. Anticipating that any practical procedure would employ some parameter smoothing we construct smoothed parameter values in the various capital candidate computations. The unnormalized weight applied at time t to a parameter value at time $t-s$, for $s \geq 0$ is chosen as λ^s for $\lambda = 0.9$. The weight for a monthly delay is around 11%.

We then build the risk neutral distribution of the stock at precisely three months into the future from date t. From this distribution one may evaluate the VaR_t and $CVaR_t$ for a long position in the index at market close for each date. Additionally on choosing a concave distribution function for a distortion one may also evaluate the $RWAVaR_t$. The particular distortion employed was minmaxvar introduced in Cherny and Madan (2009) with a distortion parameter η that controls the degree of concavity and thereby reduces the set of acceptable risks for higher levels of stress or concavity η. The Appendix provides further details. For the index we work with a fixed level for η. Later in the paper when working with individual stocks we estimate η directly from data on bid and ask prices of options.

In order to better understand the relationship between our candidates for capital requirements, VaR_t, $CVaR_t$ and $RWAVaR_t$ and the risks as they are captured in the option surface we regress these measures on the surface parameters $\sigma_t, \nu_t, \theta_t$ and γ_t. With a view to addressing cyclicality we include the level of the index lagged some two months on the view that the real economy lags the financial markets by at least a few months. Given the potential nonlinearities present between VaR and its various integrals and the index level, we also include the squared index level. The choice of two months though somewhat arbitrary, is motivated by the recognition that stock markets are leading indicators of turns in the business cycles of the non-financial sector. We felt that the lag of one month was too short and three months was probably too long and so settled for two months.

The results for the full period 1996-2011 are presented in Table 1, with t-statistics reported below coefficients to two decimal places. The risk dimensions of volatility, skew, kurtosis and the vol spread as represented by $\sigma, \nu, \theta, \gamma$ are all significant, though VaR and $CVaR$ are positively related to ν while $RWAVaR$ is negatively related. Hence kurtosis appears to reflect tailweightedness for VaR and $CVaR$ while it may be representing peakedness for $RWAVaR$.

Interestingly, for both VaR and $CVaR$ the lagged index and its square are not significant indicating that these measures are not related to the economic cycle. $RWAVaR$ on the other hand is positively related to the lagged index with a negative coefficient on the squared index. The general relationship is that of procyclicality as $RWAVaR$ rises with the index with the turnaround occuring in a region where the market has more than doubled.

To get a further appreciation of the nature of these relationships we partioned the data at the end of the tech bubble as observed in a trough for the index. This gives two periods $1996-2004$ and $2004-2011$. Tables 2 and 3 report results for these two periods. The results for the first subperiod compare with the full period with regard to the risk dimensions. All three capital indices reflect procyclicality as observed by significant and positive t-statistics for the index in Table 2.

For the second subperiod kurtosis is not significant for $RWAVaR$ otherwise the results compare with the full period with regard to the risk dimensions. With regard to cyclicality however, all three capital candidates appear to be comparably countercyclical. We address the possibility of attaining procyclicality if we also leverage movements in bid ask spreads in this second subperiod. These results are reported in the next section after we explain the bid ask pricing model in greater detail.

The tech boom did take the market to new levels in the first period with fears of the bust possibly leading to increased skews reflected in higher capital requirements as the boom roared along. The second period can be better characterized by a recovery than by a boom and at this level of market performance procyclicality of the measures did not set in. This conclusion is reversed in section 4 when the effects of spread movements are included.

4 RWAVaR at the disaggregated level

The analysis of capital requirements at the level of positions in single stocks should possibly take account of movements in bid and ask prices or liquidity spreads in addition to the risk dimensions already observed at the aggregate level. In this section we develop the application of equations (1) and (2) with a view towards adapting capital requirements to information revealed by changes in spreads on single name options. The implementation of these equations necessitates a specification of the distortion to be employed in constructing the bid and ask prices and here we follow Cherny and Madan (2009, 2010) and use the minmaxvar distortion.

In addition to the parameters $\sigma, \nu, \gamma, \theta$ seen at the aggregate level, we have the additional parameter η that will essentially calibrate or synthesize the observed bid and ask prices. One hopes that some variations in bid and ask prices during booms may get related to sell side inventories building up on down side protection that may then serve as a source for some procyclicality in capital requirements. Analytical formulas relating a risk neutral distribution function for the stock price to bid and ask prices of put and call options are developed in Cherny and Madan (2010), and are employed in the calibrations conducted.

For this particular distortion one may relate $RWAVaR$ to an expectation of $CVaR$ taken at a random quantile related to the $Beta$ distribution.

Proposition 2 *For the distortion minmaxvar we may express $RWAVaR$ as*

$$RWAVaR = E\left[CVaR\left(B^{1+\eta}\right) + \left(\frac{1}{B^{1+\eta}} - 1\right) CVaR\left(1 - B^{1+\eta}\right)\right]$$

where B is $Beta(1, \eta)$ distributed.

Proof. See Appendix. ■

Employing market close data on bid and ask prices of options on single stocks one may estimate five parameters each day for each stock. For stock i, we then have a time series of estimates for $\sigma_{it}, \nu_{it}, \theta_{it}, \gamma_{it}, \eta_{it}$. Again we take smoothed estimates for the computation of $RWAVaR_{itk}$ for a position in each of a set of K specified options for $k = 1, \cdots, K$. The sum over k is the value $RWAVaR_{it}$. The calculations are illustrated for a hypothetical portfolio in options. For the spot at 100, the option strikes are $80, 90, 100, 110$ and 120. The option maturities are .25, .5 and 1.0. Interest rates and dividend yields are set at zero. The underlying stock tickers are BAC, GS, JPM and WFC. Recognizing that regulatory acceptability is stricter than what would be reflected by market spreads we multiplied the estimated stress parameter η by 10 in the computation of capital requirements based on $RWAVaR$. Other multiples could easily be entertained and one may attempt to calibrate the multiple to Basel levels of risk weights for different asset classes. We leave for future research a finer determination of such multiples.

Table 4 presents the results for the four banks. The coefficient for excess kurtosis or the parameter ν, is positive reflecting tail weightedness for all excepting JPM where it is negative and possibly associated with peakedness. Vol spreads given by the parameter γ have a negative effect on $RWAVaR$ in all four cases. Procyclicality is observed for BAC and GS with a positive coefficient for the lagged index while JPM and WFC are countercyclical, with a negative coefficient for the lagged index.

For the purpose of studying the effects of variations in bid ask spreads on capital requirements we take option data on the exchange traded fund SPY that tracks the SPX and has bid and ask price data on many options daily. The average number of options is over 500 in the moneyness range for strikes within 30% of the spot for maturities between a month and two and a half years. We estimated the five parameter model that now also includes the stress parameter η_t each day. We then applied the same procedure for capital computations for a long position in the SPX index and then regressed the capital requirements on the five parameters, the lagged SPX and its square. The time period covered was October 22 2007 to December 23 2011. The R^2 of this regression was 98.84% and the coefficients and t-statistics are presented in Table 5. We observe the significance of procyclicality.

TABLE 5
SPX capital requirement regression results

	Parameters as Explanatory Variables							
	$const.$	σ	ν	θ	γ	η	SPX	SPX^2
Coefficient	−31.23	16.66	1.89	−49.85	−3.04	5.98	27.67	−11.85
t-statistic	−18.98	4.82	6.18	−15.67	−2.18	85.46	13.33	−13.01

5 Remark on Risk Neutral vs Historical VaR

One may compare risk neutral VaR exceedances with those under the physical or real world probability. Under a simple model of Gaussian log returns with a mean return of μ and volatility σ the physical VaR_q for a log asset level of x is given by

$$\begin{aligned} N\left(\frac{x-\mu+\frac{\sigma^2}{2}}{\sigma}\right) &= q \\ VaR_q &= e^{\mu}-e^{x} \end{aligned} \qquad (3)$$

while for a risk neutral volatility of $\widetilde{\sigma}$ and a log asset level $\widetilde{x}$ we have

$$\begin{aligned} \widetilde{VaR}_{\widetilde{q}} &= 1-e^{\widetilde{x}} \\ N\left(\frac{\widetilde{x}+\frac{\widetilde{\sigma}^2}{2}}{\widetilde{\sigma}}\right) &= \widetilde{q}. \end{aligned}$$

So

$$\begin{aligned} VaR_q &= \exp(\mu)-\exp\left(\mu-\frac{\sigma^2}{2}+\sigma N^{-1}(q)\right) \\ &\quad \text{and} \\ \widetilde{VaR}_{\widetilde{q}} &= 1-\exp\left(-\frac{\widetilde{\sigma}^2}{2}+\widetilde{\sigma}N^{-1}(\widetilde{q})\right) \end{aligned}$$

For the physical parameter setting $\mu = .06, \sigma = .15$ and with risk neutral $\widetilde{\sigma} = .25$ we have

$$\begin{aligned} VaR_{.01} &= .3212 \\ \widetilde{VaR}_{.01} &= .4582 \end{aligned}$$

Proceeding risk neutrally we observe that the one percent risk neutral level is $x = -.4582$ or a loss of 45.82 on a 100 dollar asset level. The physical probability of exceeding this loss level computed in accordance with equation (3) is $q = .00036$ or 3.6 in ten thousand. In the study conducted by Pérignon, Deng and Wang (2006) there was an expected exceedance of 74 at a one percent level with approximately 7400 days covered in the sample period. The observed exceedances are more in line with the use of risk neutral as opposed to the physical loss points. The observed exceedance was only 2 reflecting a probability of just 2.7 in ten thousand. This probability is closer to the risk neutral probability of 3.6 in ten thousand than it is to the physical one percent level, suggesting that markets may be determining their loss points using risk neutral calculations in place of the physical measure. In fact if the risk neutral volatility is raised to 0.26 we have $\widetilde{VaR}_{.01} = 47.20$ and the physical probability of this exceedance is 2.6 in ten thousand or within a point of what was observed. Hence we contend that it is the use of risk neutral probabilities and volatilities that explains the observed difference in exceedances.

6 Conclusion

Bank internal models for capital adequacy purposes have been approved for some time and regulators face issues of model approval. A public domain analysis of the performance of such models is thereby necessitated. This paper reports on an investigation conducted using forward looking risk neutral probabilities extracted from option prices at an aggregate level from options on the S&P 500 index and at a disaggregate level for a benchmark portfolio of options on four bank stocks.

At the aggregate level, in addition to VaR and $CVaR$ we analyse the new measure termed $RWAVaR$ that was proposed in Carr, Madan and Vicente Alvarez (2011). The new measure allows for a leveraging of information contained in bid and ask prices and the study at the disaggregated level reports on this potential. Capital measures along these risk neutral lines are observed to be sensitive to volatility, the volatility of volatility, the skewness of return distributions and the volatility spread across horizons. Movements in bid ask spreads also strongly influence capital requirements. Additionally there is a potential for procyclicality to be built into the requirements. Further avenues for enhancing procyclicality remains an important research question.

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7 Appendix

Proof of Proposition 1. It is shown in Carr, Madan, and Vicente Alvarez (2011) that

$$a(X) - b(X) = m + \int_0^1 K(u)dG(u)$$

where m is the median, G is the inverse distribution function and

$$K(u) = \Psi(u) + \Psi(1-u) - 1.$$

We wish to relate this capital to an integral of $CVaR_X$ that is defined as

$$CVaR_X(u) = -\frac{1}{u}\int_{-\infty}^{G(u)} xf(x)dx$$

for a random variable X with density $f(x)$.

Consider the integral

$$\begin{aligned}\int_0^1 L(u)CVaR_X(u)du &= -\int_0^1 L(u)\frac{1}{u}\int_{-\infty}^{G(u)} xf(x)dxdu \\ &= -\int_{-\infty}^{\infty} xf(x)\int_{F(x)}^1 \frac{L(u)}{u}dudx\end{aligned}$$

Define $H(u)$ by

$$H(u) = \int_0^u \int_v^1 \frac{L(w)}{w}dwdv$$

and observe that

$$\begin{aligned}\int_0^1 L(u)CVaR_X(u)du &= -\int_{-\infty}^{\infty} xd\left(H(F(x))\right). \\ &= -\int_0^1 G(u)d(H(u))\end{aligned}$$

We now write

$$\begin{aligned}\int_0^1 G(u)d(H(u)) &= \int_0^{1/2} G(u)d(H(u)) + \int_{1/2}^1 G(u)d\left(H(u)\right) \\ &= G(1/2)H(1/2) - \int_0^{1/2} H(u)dG(u) \\ &\quad + G(1/2)(1-H(1/2)) + \int_{1/2}^1 (1-H(u))dG(u) \\ &= m + \int_0^1 \left(\mathbf{1}_{u>1/2} - H(u)\right)dG(u)\end{aligned}$$

For $RWAVaR$ to be an integral of $CVaR$ we then require that

$$K(u) = -\left(\mathbf{1}_{u>1/2} - H(u)\right)$$

or

$$K'(u) = H'(u)$$

or

$$\begin{aligned}\Psi'(u) - \Psi'(1-u) &= H'(u) \\ &= \int_u^1 \frac{L(v)}{v}dv\end{aligned}$$

whereby we get that

$$\Psi''(u) + \Psi''(1-u) = -\frac{L(u)}{u}$$

or

$$L(u) = -u\left(\Psi''(u) + \Psi''(1-u)\right)$$

It follows that

$$\int_0^1 L(u)CVaR_X(u)du = -\int_0^1 u\left(\Psi''(u) + \Psi''(1-u)\right)CVaR_X(u)du$$

or that

$$RWAVaR = -\int_0^1 u\left(\Psi''(u) + \Psi''(1-u)\right)CVaR_X(u)du.$$

The expression for VaR follows on writing $CVaR$ in terms of VaR and changing the order of integration.

2. Details for the $VGSSD$ model. The risk neutral law for the stock at time t is given by

$$S(t) = S(0)\exp\left((r-q)\,t + \omega(t) + X(t)\right)$$

where

$$X(t) \overset{(d)}{=} t^{\gamma}X$$

and

$$X = \theta G + \sigma\sqrt{G}Z$$

for Z a standard normal variate and G an independent gamma variate with unit mean, variance ν and density

$$f(x) = \frac{1}{\Gamma(1/\nu)\,\nu^{\frac{1}{\nu}}}x^{\frac{1}{\nu}-1}\exp\left(-\frac{x}{\nu}\right).$$

The convexity correction $\omega(t)$ satisfies

$$\exp(-\omega(t)) = E[\exp(X(t))].$$

3. Closed forms for bid and ask prices of put and call options. It is shown in Cherny and Madan (2010) that bid and ask prices $C_b(K,t), C_a(K,t)$ for call options and $P_b(K,t), P_a(K,t)$ for put options for strike K on a stock with risk neutral distribution $F_t(s) = P(S(t) \leq s)$ may be obtained for a distortion $\Psi(u)$ as follows.

$$\begin{aligned}
C_b(K,t) &= \int_K^{\infty}(1-\Psi(F_t(s)))ds \\
C_a(K,t) &= \int_K^{\infty}\Psi(1-F_t(s))ds \\
P_b(K,t) &= \int_0^{K}(1-\Psi(1-F_t(s)))ds \\
P_a(K,t) &= \int_0^{K}\Psi(F_t(s))ds.
\end{aligned}$$

4. Proof of Proposition 2. By Proposition 1 we have that

$$RWAVaR = -\int_0^1 u\left(\Psi''(u) + \Psi''(1-u)\right) CVaR(u)du$$

For the specific distortion of minmaxvar we have that

$$-\Psi''(u) = \frac{\eta}{1+\eta}(1 - u^{\frac{1}{1+\eta}})^{\eta-1} u^{-\frac{\eta}{1+\eta}-1}$$

Substituting this expression for $\Psi''(u)$ and changing variables to $z = u^{\frac{1}{1+\eta}}$ yields

$$\begin{aligned} RWAVaR &= \int_0^1 \left[CVaR(z^{1+\eta}) + \left(\frac{1}{z^{1+\eta}} - 1\right) CVaR\left(1 - z^{1+\eta}\right)\right] \eta(1-z)^{\eta-1}dz \\ &= E\left[CVaR(B^{1+\eta}) + \left(\frac{1}{B^{1+\eta}} - 1\right) CVaR\left(1 - B^{1+\eta}\right)\right], \end{aligned}$$

for B distributed $Beta(1, \eta)$.

TABLE 1			
Regression Results of Capital Candidates on Risk and Lagged Index			
	Full Period Results 1996-2012		
	Capital Candidate		
	VaR	CVaR	RWAVaR
Constant	28.4121	31.8459	-0.6094
	63.21	108.43	-0.88
sigma	23.0184	16.1676	29.8543
	22.29	23.96	18.87
nu	0.4886	0.7274	-2.1425
	4.80	10.95	-13.75
theta	-13.9865	-12.2367	-5.7701
	-19.63	-26.28	-5.29
gamma	2.9787	2.4867	6.7900
	7.78	9.93	11.57
spx	1.2921	0.6532	12.8004
	1.42	1.10	9.17
spx^2	-0.5236	-0.3058	-6.0211
	-1.25	-1.12	-9.40
RSQ	0.8915	0.9214	0.7660

TABLE 2			
Regression Results of Capital Candidates on Risk and Lagged Index			
	First Sub Period Results 1996-2004		
	Capital Candidate		
	VaR	CVaR	RWAVaR
Constant	26.1541	30.5047	5.1744
	77.81	120.29	14.11
sigma	17.8004	12.4427	28.9694
	16.50	15.29	24.61
nu	0.8345	0.9702	-1.6255
	10.71	16.51	-19.12
theta	-12.7971	-11.7139	-3.2803
	-17.00	-20.63	-3.99
gamma	1.8245	1.6409	0.7543
	6.12	7.29	2.32
spx	8.3840	5.0895	6.9094
	11.64	9.36	8.79
spx^2	-3.4816	-2.1351	-2.8371
	-10.38	-8.43	-7.75
RSQ	0.9392	0.9438	0.9140

TABLE 3			
Regression Results of Capital Candidates on Risk and Lagged Index			
	Second Sub Period Results 2004-2011		
	Capital Candidate		
	VaR	CVaR	RWAVaR
Constant	30.8531	34.0521	11.7625
	22.52	38.32	11.09
sigma	0.6871	2.5783	5.1491
	0.31	1.82	3.04
nu	2.4548	1.9307	0.0460
	11.35	13.76	0.27
theta	-30.5133	-22.1237	-23.7059
	-21.42	-23.95	-21.50
gamma	5.9453	4.2539	3.7412
	9.33	10.29	7.58
spx	-6.0011	-4.8126	-7.3002
	-2.83	-3.51	-4.46
spx^2	2.0196	1.6236	2.2653
	2.30	2.85	3.34
RSQ	0.9186	0.9416	0.9469

Table 4				
RWAVaR Risk Analysis for Four Bank Option Portfolios				
	BAC	GS	JPM	WFC
constant	-1.4766	-98.5366	83.9068	42.8607
	-0.16	-10.37	10.34	2.90
sigma	19.2570	27.7649	30.9027	25.0823
	10.19	14.21	19.41	14.85
nu	8.1306	16.6155	-2.3431	17.0498
	9.47	18.54	-3.54	23.90
theta	-17.8014	-45.4986	-12.6240	-38.8834
	-36.77	-43.33	-19.17	-30.00
gamma	-55.7912	-49.8347	-27.4779	-31.2926
	-13.16	-8.62	-7.39	-6.19
eta	0.8969	0.8537	0.8766	0.8626
	78.94	150.10	109.85	125.22
spx	0.0378	0.1414	-0.1222	-0.0837
	2.82	9.41	-10.00	-4.01
spx^2	2.69E-05	6.24E-05	4.08E-05	2.11E-05
	-4.72	-9.78	7.95	2.49
Rsquare	0.9723	0.9824	0.9723	0.9893