Jump processes

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Although historically models in mathematical finance were based on Brownian motion and thus are models with continuous price paths, jump processes play now a key role across all areas of finance (see e.g. [5]). One reason for this move into a new class of processes is that because of their distributional properties diffusions in many cases cannot provide a realistic picture of empirically observed facts. Another reason is the enormous progress which has been made in understanding and handling jump processes due to the development of semimartingale theory on one side and of computational power on the other side.

The simplest jump process is a process with just one jump. Let T be a random time – actually a stopping time with respect to an information structure given by a filtration $(\mathcal{F}_t)_{t\geq 0}$ – then

$$X_t = 1\!\!1_{\{T\leq t\}} \quad (t \geq 0) \tag{1}$$

has the value 0 until a certain event occurs and 1 then. As simple as this process looks like, as important it is in modeling credit risk, namely as the process which describes the time of default of a company. The next step are processes which are integer-valued with positive jumps of size 1 only, so-called *counting processes* $(X_t)_{t\geq 0}$. X_t describes the number of events which have occurred between time 0 and t. This could be the number of defaults in a large credit portfolio or of claims customers report to an insurance company. The standard case is a *Poisson process* $(N_t)_{t\geq 0}$ where the distribution of X_t is given by a Poisson distribution with parameter λt. Equivalently one can describe this process by requiring that the waiting times between successive jumps are independent, exponentially distributed random variables with parameter λ.

The natural extension in a *compound Poisson process* $(X_t)_{t\geq 0}$, i.e. a process with stationary independent increments where the jump size is no longer 1, but given by a probability law μ. Let $(Y_k)_{k\geq 1}$ be a sequence of independent random variables with distribution $\mathcal{L}(Y_k) = \mu$ for all $k \geq 1$. Denote by $(N_t)_{t\geq 0}$ a standard Poisson process with parameter $\lambda > 0$ as above which is independent of $(Y_k)_{k\geq 1}$, then we can represent $(X_t)_{t\geq 0}$ in the form

$$X_t = \sum_{k=1}^{N_t} Y_k. \tag{2}$$

A typical application of compound Poisson processes is to model the cumulative claim size up to time t in a portfolio of insurance contracts where the individual claim size is distributed according to μ. For the sake of analytical tractability it is often useful to *compensate* this process, i.e. to substract the average claim size $E[X_t]$. Assuming that μ has a finite expectation and using stationarity and independence we conclude $E[X_t] = tE[X_1]$ and therefore get the representation

$$X_t = tE[X_1] + (X_t - E[X_t]). \tag{3}$$

The compensated process $(X_t - E[X_t])_{t\geq 0}$ is a martingale and therefore (3) is a decomposition of the process in a linear drift $E[X_1] \cdot t$ and a martingale. Representation (3) motivates the definition of a general *semimartingale* as a process which is adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$, has paths which are right-continuous and have left limits (càdlàg paths) and allows a decomposition

$$X_t = X_0 + V_t + M_t \quad (t \geq 0) \tag{4}$$

where $V = (V_t)_{t\geq 0}$ is an adapted, càdlàg process of finite variation and $M = (M_t)_{t\geq 0}$ is a local martingale. There exist processes which are not semimartingales. An important class of examples are fractional Brownian motions with the exception of usual Brownian motion which is a semimartingale. We do not go beyond semimartingales in this discussion mainly because for semimartingales there is a well-developed theory of stochastic integration, a fact which is crucial for modeling in finance.

The representation (4) is not unique in general. It becomes unique with a *predictable* process V if we consider *special semimartingales*. A semimartingale can be made special by taking the big jumps away, e.g. jumps with absolute jump size bigger than 1. This follows from the well-known fact that a semimartingale with bounded jumps is special [11, I.4.24]. Denote by $\Delta X_t = X_t - X_{t-}$ the jump at time t if there is any, then

$$X_t - \sum_{s \le t} \Delta X_s 1\!\!1_{\{|\Delta X_s|>1\}} \tag{5}$$

has bounded jumps. Further let us note that any local martingale M (with $M_0 = 0$) admits a unique (orthogonal) decomposition into a local martingale with continuous paths M^c and a purely discontinuous, local martingale M^d ([11, I.4.18]). Assuming $X_0 = 0$ we got the following unique representation for semimartingales

$$X_t = V_t + M^c + M^d + \sum_{s \le t} \Delta X_s 1\!\!1_{\{|\Delta X_s|>1\}}. \tag{6}$$

In order to analyse M^d in more detail we introduce the *random measure of jumps*

$$\mu^X(\omega; dt, dx) = \sum_{s>0} 1\!\!1_{\{\Delta X_s(\omega) \neq 0\}} \varepsilon_{(s, \Delta X_s(\omega))}(dt, dx) \tag{7}$$

where ε_a denotes as usual the unit mass in a. Thus μ^X is a random measure which for ω fixed places a point mass of size 1 on each pair $(s, \Delta X_s(\omega)) \in \mathbb{R}_+ \times \mathbb{R}$ if for this ω the process has a jump of size $\Delta X_s(\omega)$ at time s. Expressed differently for any Borel subset $B \subset \mathbb{R}$, $\mu^X(\omega; [0,t] \times B)$ counts the number of jumps with size in B which can be observed along the path $(X_s(\omega))_{0 \le s \le t}$. With this notation (5) can be written as

$$X_t - \int_0^t \int_{\mathbb{R}} x 1\!\!1_{\{|x|>1\}} \mu^X(ds, dx). \tag{8}$$

The purely discontinuous local martingale M^d, i.e. the process of *compensated jumps* of absolute size less than 1, has then the form

$$M_t^d = \int_0^t \int_{\mathbb{R}} x 1\!\!1_{\{|x| \leq 1\}} (\mu^X - \nu^X)(ds, dx), \tag{9}$$

where ν^X is another random measure, the (*predictable*) *compensator* of μ^X. Whereas μ^X counts the exact number of jumps, ν^X roughly stands for the expected, i.e. the average number of jumps. The integral with respect to $\mu^X - \nu^X$ in (9) in general cannot be separated in an integral with respect to μ^X and another one with respect to ν^X. This is because the sum of the small jumps

$$\sum_{s \leq t} \Delta X_s 1\!\!1_{\{|\Delta X_s| \leq 1\}} = \int_0^t \int_{\mathbb{R}} x 1\!\!1_{\{|x| \leq 1\}} \mu^X(ds, dx) \tag{10}$$

does not converge in general.

For many applications it is sufficient to reduce generality and to consider the subclass of *Lévy processes*, i.e. processes with stationary and independent increments. The components in (6) and (9) are then

$$\begin{aligned} V_t &= bt \quad (t \geq 0) \\ M_t^c &= \sqrt{c} W_t \quad (t \geq 0) \\ \nu^X([0,t] \times B) &= tK(B), \end{aligned} \tag{11}$$

where b, c are real numbers, $c \geq 0$, $(W_t)_{t \geq 0}$ is a standard Brownian motion, and K, the *Lévy measure*, is a (possible infinite) measure on the real line which satisfies $\int (1 \wedge x^2) K(dx) < \infty$. The law of X is completely determined by the *triplet of local characteristics* (b, c, K) since these are the parameters which appear in the classical Lévy–Khintchine formula. This formula expresses the *characteristic function* $\varphi_{X_t}(u) = E[\exp(iuX_t)]$ in the form

$$\varphi_{X_t}(u) = \exp(t\psi(u)) \tag{12}$$

with the characteristic exponent

$$\psi(u) = iub - \frac{1}{2}u^2 c + \int \left(\mathrm{e}^{iux} - 1 - iux 1\!\!1_{\{|x| \leq 1\}}\right) K(dx). \tag{13}$$

The *truncation function* $h(x) = x\mathbb{1}_{\{|x|\leq 1\}}$ could be replaced by other versions of truncation functions, e.g. smooth functions which are identical to the identity in a neighborhood of the origin and go to 0 outside of this neighborhood. Changing h affects the drift parameter b, but neither c nor K. All the information on the jump behaviour of the process $(X_t)_{t\geq 0}$ is contained in K. The frequency of large jumps, expressed by the weight K puts on the tails, determines finiteness of the moments of the process as the following result states (for proofs of the Propositions see [15])

Proposition 1. *Let $X = (X_t)_{t\geq 0}$ be a Lévy process with Lévy measure K, then $E[|X_t|^p]$ is finite for any $p \in \mathbb{R}_+$ if and only if $\int_{\{|x|>1\}} |x|^p K(dx) < \infty$.*

We note that if X_1 and consequently any X_t has finite expectation then one does not have to truncate in (13), i.e. $h(x) = x\mathbb{1}_{\{|x|\leq 1\}}$ can be replaced by $h(x) = x$.

The sum of the big jumps which is subtracted in (5) is finite since there are only finitely many of them from 0 to t for every path. The *fine structure* of the paths is determined by the frequency of the small jumps. A process is said to have *finite activity* if almost all paths have only a finite number of jumps along finite time intervals. The simplest examples are Poisson and compound Poisson processes. A process is said to have *infinite activity* if almost all paths have infinitely many jumps along any time interval of finite length.

Proposition 2. *Let $X = (X_t)_{t\geq 0}$ be a Lévy process with Lévy measure K. Then X has finite activity if $K(R) < \infty$ and has infinite activity if $K(R) = \infty$.*

Since a Lévy measure has a priori finite mass in the tails, i.e. $\int_{\{|x|>1\}} K(dx) < \infty$, the finiteness of $K(R)$ means finiteness of $\int_{\{|x|\leq 1\}} K(dx)$. Consequently having a finite or an infinite number of jumps along finite time intervals is determined by the mass of K around the origin. From the distribution of mass around the origin one can also see if the sum of (infinitely many) small jumps converges or does not. First let us recall that a standard Brownian motion has paths of infinite variation. Therefore a Lévy process has *infinite variation* as soon as it has a continuous martingale component, i.e. $c > 0$ in (11), but infinite variation can also come from the jumps.

Proposition 3. *Let $X = (X_t)_{t\geq 0}$ be a Lévy process with triplet (b, c, K), then almost all paths of X have finite variation if $c = 0$ and $\int_{\{|x|\leq 1\}} |x|K(dx) < \infty$. If this integral is infinite or $c > 0$ then almost all paths of X have infinite variation.*

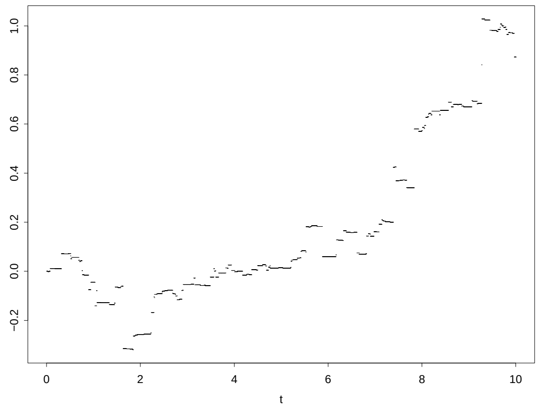

Fig. 1. Simulation of a purely discontinuous Lévy process with infinite activity and finite variation

Figure 1 shows a simulated path of a purely discontinuous, infinite activity process with finite variation, wheras Figure 2 shows a corresponding path with infinite variation.

Now we discuss some of the standard examples. The Poisson process with intensity parameter λ which we considered at the beginning has a finite number of jumps in any finite time interval and is constant between successive jumps. In terms of (11) it is characterized by $b = E[X_1] = \lambda$, $c = 0$ and $K = \lambda\varepsilon_1$. For the compound Poisson process (2), the unit mass ε_1 in K is replaced by a probability measure μ, the law of Y_1, i.e. $K = \lambda\mu$. For the drift parameter b one gets $\lambda E[Y_1]$. One gets a *Lévy jump diffusion* by adding a general drift term bt and a scaled Brownian motion to (2),

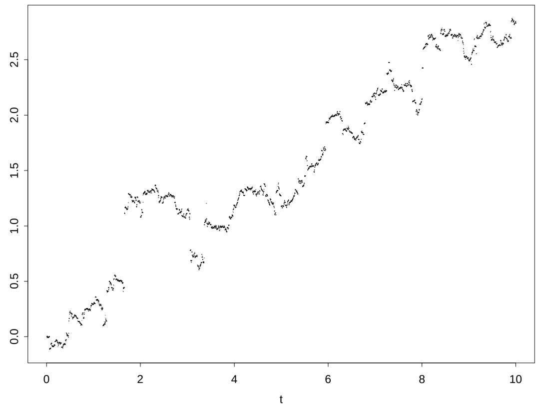

Fig. 2. Simulation of a purely discontinuous Lévy process with infinite activity and infinite variation

$$X_t = bt + \sqrt{c}W_t + \sum_{k=1}^{N_t} Y_k. \tag{14}$$

This is the model introduced by Merton [14] to describe asset returns. Merton chose normally distributed variables Y_k. In a later paper Kou [12] used double-exponentially distributed jump sizes Y_k. If one replaces in (14) the Brownian motion with drift by a general diffusion process one gets a *jump diffusion*. The key property of jump diffusions is that one adds only a finite number of jumps in any finite time interval to a process with continuous paths. In other words, the jump times can be given by successive stopping times $T_1 < T_2 < T_3 < \cdots$. We also note that the distribution of X_t is not known for diffusions in general. The same holds for jump diffusions. This reduces their applicability in mathematical finance. A key advantage of the pure jump Lévy processes we discuss now is that they are distributionally very flexible and the distributions are known explicitly.

Generalized hyperbolic Lévy motions $(X_t)_{t\geq 0}$ (see *Generalized hyperbolic models* in this book or [6], [9]) represent a very large class of Lévy processes which are generated by generalized hyperbolic (GH) distributions (Barndorff-Nielsen (1978) [1]), i.e. the distribution of X_1, $\mathcal{L}(X_1)$, is GH. Via (12) this determines all other distributions $\mathcal{L}(X_t)$. GH distributions have an explicit

Lebesgue density as has the corresponding Lévy measure. GH distributions can be represented as normal mean-variance mixtures where the mixing distribution is a generalized inverse Gaussian (GIG) distribution. Moments of any order exist. Since $c = 0$ in (13) generalized hyperbolic Lévy motions have purely discontinuous paths. They are infinite activity processes. Important subclasses are hyperbolic Lévy motions (Eberlein and Keller (1995) [7]) and normal inverse Gaussian (NIG) Lévy motions (Barndorff-Nielsen (1998) [2]). Many well-known distributions can be obtained as limiting cases of GH distributions, which generate the corresponding processes (see Eberlein and von Hammerstein (2004) [10]). Among those are the Variance Gamma distribution (see [13]), scaled and shifted Cauchy distributions, shifted Student-t distributions, GIG distributions, the Gamma as well as the normal distribution.

The CGMY process introduced in Carr, Geman, Madan, and Yor (2002) [4] is another purely discontinuous Lévy process which can be defined via the Lévy density of $\mathcal{L}(X_1)$

$$g_{\mathrm{CGMY}}(x) = \begin{cases} C\frac{\exp(-G|x|)}{|x|^{1+Y}} & x < 0 \\ C\frac{\exp(-Mx)}{x^{1+Y}} & x > 0 \end{cases} \tag{15}$$

where $Y \in (-\infty, 2)$. The process has infinite activity iff $Y \in [0, 2)$ and it has infinite variation iff $Y \in [1, 2)$. For $Y = 0$ it reduces to the Variance Gamma process.

A very classical class are α-stable Lévy processes where $0 < \alpha \leq 2$. For $\alpha = 2$ one gets Brownian motion whereas for $\alpha < 2$ one gets purely discontinuous processes. Only for three special cases explicit densities are known: the Gaussian, the Cauchy, and the Lévy distribution.

An easy to handle extension of Lévy processes are time-inhomogeneous Lévy processes, i.e. processes with independent increments and absolutely continuous characteristics, called PIIAC in [11]. For any fixed t, the triplet of $\mathcal{L}(X_t)$ for these processes is given in the form $b = \int_0^t b_s ds$, $c = \int_0^t c_s ds$, and

$K(dx) = \int_0^t K_s(dx)ds$. This class of processes has been used extensively in the context of interest rate models (see e.g. Eberlein and Kluge (2006) [8]).

Jump processes with paths which are rather different from those discussed so far were introduced by Barndorff-Nielsen and Shephard [3] in the context of stochastic volatility models. Let $(Z_t)_{t\geq 0}$ be a *subordinator*, i.e. a Lévy process, starting at 0 with increasing paths and consequently without a Gaussian component. The volatility process $(\sigma_t^2)_{t\geq 0}$ is modeled via an Ornstein–Uhlenbeck type stochastic differential equation

$$d\sigma_t^2 = -\lambda\sigma_t^2 dt + dZ_{\lambda t}$$

for some $\lambda > 0$. The solution $(\sigma_t^2)_{t\geq 0}$ moves up entirely by jumps and then tails of exponentially. σ_t is fed into a Brownian semimartingale which then represents the price process.

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