Generalized hyperbolic models

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Generalized hyperbolic Lévy motions constitute a broad subclass of Lévy processes which are generated by *generalized hyperbolic (GH) distributions*. GH distributions were introduced in [1] in connection with a project with geologists. The Lebesgue density of this 5-parameter class can be given in the following form

$$d_{GH(\lambda,\alpha,\beta,\delta,\mu)}(x) = a(\lambda,\alpha,\beta,\delta,\mu)(\delta^2 + (x-\mu)^2)^{(\lambda-\frac{1}{2})/2} \times K_{\lambda-\frac{1}{2}}(\alpha\sqrt{\delta^2+(x-\mu)^2})\exp(\beta(x-\mu)) \tag{1}$$

with the norming constant

$$a(\lambda,\alpha,\beta,\delta,\mu) = \frac{(\alpha^2-\beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}\delta^\lambda K_\lambda(\delta\sqrt{\alpha^2-\beta^2})}.$$

K_ν denotes the modified Bessel function of the third kind with index ν. The parameters can be interpreted as follows: $\alpha > 0$ determines the shape, β with $0 \leq |\beta| < \alpha$ the skewness and $\mu \in \mathbb{R}$ the location. $\delta > 0$ serves for scaling and $\lambda \in \mathbb{R}$ characterizes subclasses. It is essentially the weight in the tails which changes with λ. There are two alternative parametrizations which are scale- and location-invariant, i.e. they do not change under affine transformations $Y = aX + b$ for $a \neq 0$, namely $\zeta = \delta\sqrt{\alpha^2-\beta^2}$, $\rho = \beta/\alpha$ and $\xi = (1+\zeta)^{-1/2}$, $\chi = \xi\rho$. Since $0 \leq |\chi| < \xi < 1$, for a fixed λ the distributions parametrized by χ and ξ can be represented by the points of a triangle, the so-called shape triangle.

GH distributions arise in a natural way as variance-mean mixtures of normal distributions. Let d_{GIG} denote the density of a generalized inverse Gaussian distribution with parameters $\delta > 0, \gamma > 0$ and $\lambda \in \mathbb{R}$, i.e.

$$d_{GIG(\lambda,\delta,\gamma)}(x) = \left(\frac{\gamma}{\delta}\right)^{\lambda} \frac{1}{2K_{\lambda}(\delta\gamma)} x^{\lambda-1} \exp\left(-\frac{1}{2}(\frac{\delta^2}{x} + \gamma^2 x)\right) \mathbb{1}_{\{x>0\}} \quad (2)$$

Then if $N(\mu + \beta y, y)$ denotes a normal distribution with mean $\mu + \beta y$ and variance y one can easily verify that

$$d_{GH(\lambda,\alpha,\beta,\delta,\mu)}(x) = \int_0^\infty d_{N(\mu+\beta y,y)}(x) d_{GIG(\lambda,\delta,\sqrt{\alpha^2-\beta^2})}(y) dy. \quad (3)$$

Via maximum likelihood estimation one can fit GH distributions to empirical return distributions from financal time series such as daily stock or index prices. Fig. 1 shows a fit to daily closing prices of Telekom over a period of seven years. Fig. 2 shows the same densities on a log scale in order to make the

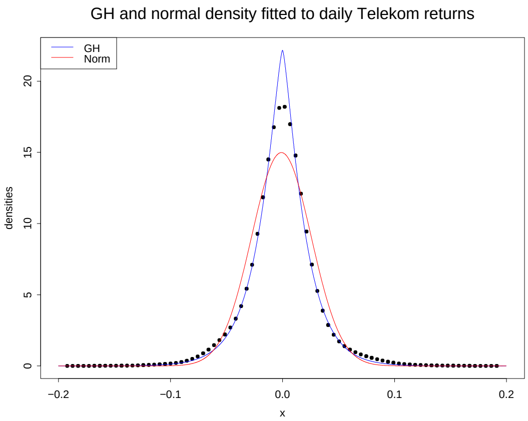

Fig. 1.

fit in the tails visible. One recognizes the hyperbolic shape of the GH density in comparison to the parabolic shape of the normal density. The characteristic function of the GH distribution is

$$\varphi_{GH}(u) = e^{iu\mu} \left(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + iu)^2}\right)^{\frac{\lambda}{2}} \frac{K_{\lambda}\left(\delta\sqrt{\alpha^2 - (\beta + iu)^2}\right)}{K_{\lambda}(\delta\sqrt{\alpha^2 - \beta^2})} \quad (4)$$

and expectation and variance are

$$E[GH] = \mu + \frac{\beta\delta^2}{\zeta}\frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)},$$
$$Var(GH) = \frac{\delta^2}{\zeta}\frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)} + \frac{\beta^2\delta^4}{\zeta^2}\left(\frac{K_{\lambda+2}(\zeta)}{K_\lambda(\zeta)} - \frac{K^2_{\lambda+1}(\zeta)}{K^2_\lambda(\zeta)}\right) \tag{5}$$

The moment generating function exists for all u such that $-\alpha-\beta < u < \alpha-\beta$. Therefore moments of all orders are finite.

There are two important subclasses. For $\lambda = 1$ one gets the class of *hyperbolic* distributions with density

$$d_{H(\alpha,\beta,\delta,\mu)}(x) = \frac{\sqrt{\alpha^2-\beta^2}}{2\alpha\delta K_1(\delta\sqrt{\alpha^2-\beta^2})}\exp(-\alpha\sqrt{\delta^2+(x-\mu)^2}+\beta(x-\mu)) \tag{6}$$

whereas for $\lambda = -\frac{1}{2}$ one gets the class of normal inverse Gaussian (NIG) distributions with density

$$d_{NIG(\alpha,\beta,\delta,\mu)}(x) = \frac{\alpha}{\pi}\exp(\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu))\frac{K_1\left(\alpha\delta\sqrt{1+(\frac{x-\mu}{\delta})^2}\right)}{\sqrt{1+(\frac{x-\mu}{\delta})^2}}. \tag{7}$$

The latter one has a particularly simple characteristic function

$$\varphi_{NIG}(u) = e^{iu\mu}\frac{\exp(\delta\sqrt{\alpha^2-\beta^2})}{\exp(\delta\sqrt{\alpha^2-(\beta+iu)^2})}. \tag{8}$$

Many well-known distributions are limit cases of the class of GH distributions. For $\lambda > 0$ and $\delta \to 0$ one gets a Variance-Gamma distribution, in the special case of $\lambda = 1$ the result is a skewed and shifted Laplace distribution. Other limit cases are the Cauchy and the Student-t distribution as well as the Gamma, the reciprocal Gamma and the normal distribution. For details see [8].

GH distributions are infinitely divisible and therefore generate a Lévy process $L = (L_t)_{t\geq 0}$ such that the distribution of L_1, $\mathcal{L}(L_1)$, is the given GH distribution. Analysing the characteristic function in their Lévy–Khintchine form one sees that the Lévy measure has an explicit density. There is no Gaussian component. Consequently the generated Lévy process is a process

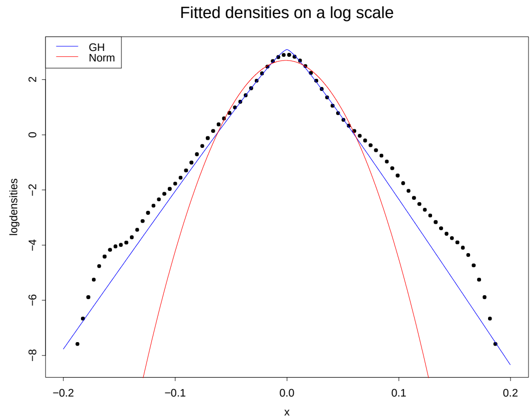

Fig. 2.

with purely discontinuous paths. The paths have infinite activity, which means that there are infinitely many jumps in any finite time interval (see *Jump Processes* in this book).

As a model for asset prices such as stock prices, indices or foreign exchange rates, we take the exponential of the Lévy process L

$$S_t = S_0 \exp L_t \tag{9}$$

For hyperbolic Lévy motions this model was introduced in [5], NIG Lévy processes were considered in [2] and the extension to GH Lévy motions appeared in [3] and [7]. The log returns from this model taken along time intervals of length 1 are $L_t - L_{t-1}$ and therefore they have exactly the GH distribution which generates the Lévy process. It was shown in [6] that the model (9) succeeds to produce empirically correct distributions on other time horizons as well. This time consistency property can e.g. be used to derive correct VAR-estimates on a two-week horizon according to the Basel II rules. (9) can be expressed by the following stochastic differential equation

$$dS_t = S_{t-}\left(dL_t + e^{\Delta L_t} - 1 - \Delta L_t\right) \tag{10}$$

The price of an European option with payoff $f(S_T)$ is

$$V = e^{-rT}\mathbb{E}\left[f(S_T)\right] \tag{11}$$

where r is the interest rate and expectation is taken with respect to a risk-neutral (martingale) measure. As is shown in [4] there are many equivalent martingale measures due to the rich structure of the driving process L. The simplest choice is the so-called Esscher transform which was used in [3]. For the process L to be again a GH Lévy motion under an equivalent martingale measure, the parameters δ and μ have to be kept fixed (see [9]). Since the density of the distribution of S_T can be derived via inversion of the characteristic function, the expectation in (11) can be computed directly. A numerically much more efficient method based on two-sided Laplace transforms which is applicable to a wide variety of options has be developed in [9]. Assume that $e^{-Rx}f(e^{-x})$ is bounded and integrable for some R such that the moment generating function of L_T is finite at $-R$. Write $g(x) = f(e^{-x})$ and $\psi_g(z) = \int_{\mathbb{R}} e^{-zx}g(x)dx$ for the bilateral Laplace transform of g. If $\zeta := -\log S_0$ then the option price V can be expressed in the form

$$V(\zeta) = \frac{e^{\zeta R - rT}}{2\pi}\int_{\mathbb{R}} e^{iu\zeta}\psi_g(R+iu)\varphi_{L_T}(iR-u)\,du \tag{12}$$

whenever the integral exists. φ_{L_T} denotes the characteristic function of the distribution of L_T.

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