Sato Processes and the Valuation of Structured Products*

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Abstract

We report on the adequacy of using Sato processes to value equity structured products. In models used to price options on realized variance, the latter must be a random variable with a positive variance. An analysis of this variance of realized variance for Sato processes shows that these processes may be suited to option contracts on realized volatility. Nonlinear pricing principles based on hedging to acceptability are outlined for the purpose of pricing structured transactions. It is shown that typically different products should be priced using different models. Pricing comparisons of Sato process prices with other standard models like Heston stochastic volatility, with and without jumps, VGSA, local volatility and local CGMY are also provided. Sato processes tend to overprice cliquets relative to other models. They also maintain the value of long dated out-of-the-money realized variance options.

1 Introduction

Equity structured products have cash flows defined by functions of the stock price from contract initiation to either maturity or early termination. There are now a wide variety of structures issued by the financial industry including as typical examples locally and globally, capped and floored, arithmetic or product cliquets, options on realized variance, and swing or reverse swing cliquets. The industry has undergone considerable growth with the US notional in 2004 standing at 12 billion dollars, rising to 50 billion dollars in 2005 and a further projected growth of up to 25% in 2006. It is the fastest growing investment class in the United States. Structured Retail Products reports that there are over 140,000 individual product offerings from over 900 companies representing a total world sales of over 600 billion Euros.

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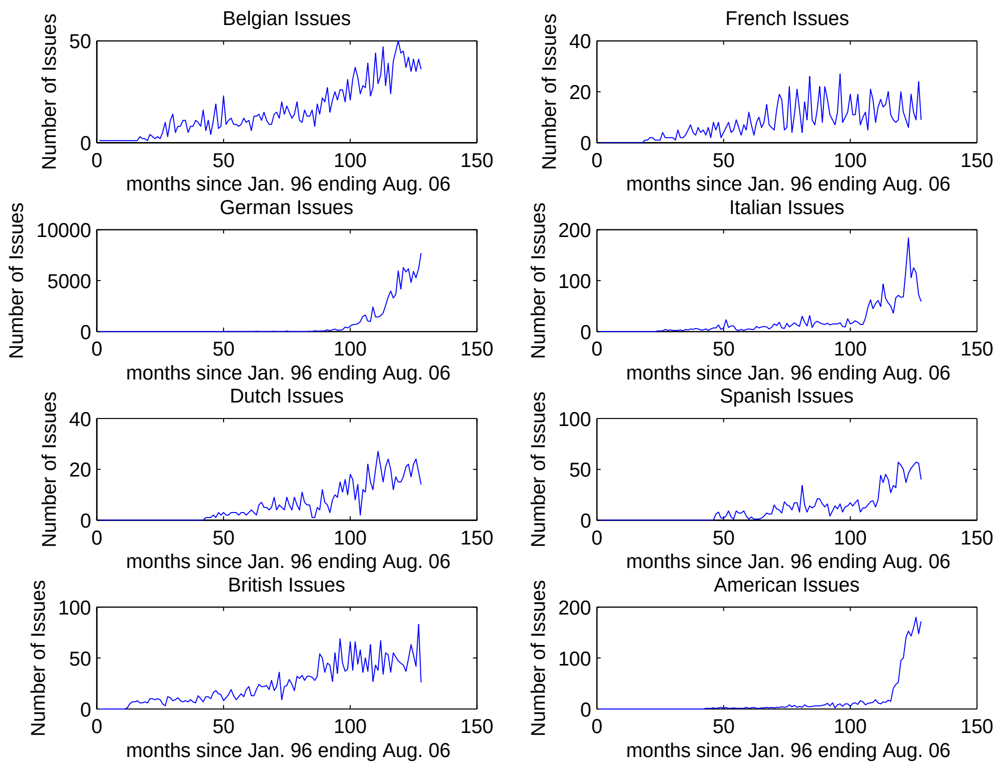

Figure 1: Total Number of Monthly Issues ending in August 2006

For a better appreciation of the development of this market we present graphs of the total number of monthly issues, and the financial value of these issues in eight major financial markets. These are Belgium, France, Germany, Italy, The Netherlands, Spain, the UK and USA. Figure (1) presents the total number of issues, while the financial values are presented in Figure (2). The longest series is for Belgium with 128 monthly observations, beginning in January 1996. We have kept the same length for all series ending in August 2006 and have filled in the earlier periods with zeros for the other series. The series lengths were for Belgium, 128, France 110, Germany 78, Italy 105, The Netherlands 86, Spain 82, the UK 117, and the USA 86. We observe a steady growth in issues in Belgium, The Netherlands, Spain and the UK, a stabilisation in France and rapid growth in the German, Italian and US markets. In terms of volume, there is a steady rise in most markets and rapid growth in the German and US markets. An appendix provides a list of product names and their common abbreviations.

The pricing, marking and risk management of these products is typically done by the simultaneous calibration, across strike and maturity, of a stochastic process model to the prices of vanilla options at market close. The calibrated

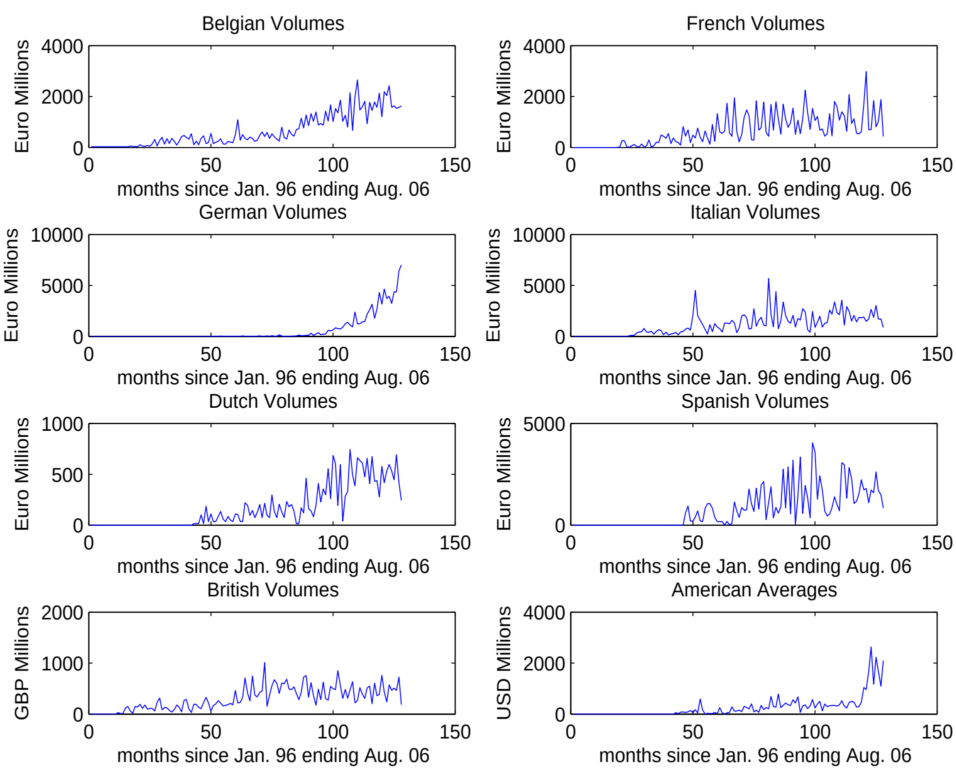

Figure 2: Total Financial Value of monthly Issues ending in August 2006

model is then employed to generate stock price paths and the product price is determined on computing a discounted expected cash flow of the specified payoff. The product is then regularly marked to market using a model selected for this purpose for the period the product is held by the issuer. Different products are generally marked using different models that address the product risks adequately. The validity of such a pricing methodology is grounded as we show below (See section 6) in the principles of pricing to acceptability (Carr, Geman, Madan (2001)).

Four important lessons emerge from this analysis (See section 6 for further details). First and foremost is the observation that risk neutral pricing though useful is not relevant for the pricing of structured products. We interpret risk neutral pricing in its classical form of pricing all products under a single change of measure. This linear pricing rule applies to liquid markets quoting bidirectional prices but the ask and bid prices for structured products are respectively convex and concave functionals. Second, price variation across models is incorrectly seen as model risk as the highest are close to ask prices while the lowest are close to bid prices and the other model valuations are not associated with any transaction and do not have the status of being prices. Third risk neutral models are useful as one seeks the maxima and minima of admissible risk neutral valuations to determine ask and bid prices respectively. It is expected, for example, that the maximum risk neutral valuation will occur for different products at different models. Hence, it is natural, proper and accurate for different products to be marked using different models. Fourth and finally, we show that if the same cone of acceptability is applied across the industry then even though ask and bid prices are not linear, arbitrage across market partcipants is nonetheless excluded.

Apart from the underlying asset, the relatively most liquid traded assets with market information are vanilla options on the underlying. The activity of issuing, marking to market, and risk managing structured products has naturally led to demands for creating stochastic process models capable of synthesizing the surface of vanilla option prices. These prices are typically represented by the matrix of implied volatilities indexed by the strike and maturity dimensions.

It is well recognized that the Black and Scholes (1973) and Merton (1973) geometric Brownian motion model is not capable of such a surface synthesis. Hence this model is never used as a candidate model for structured product valuation. Early improvements on this model offered by Lévy processes like the variance gamma model (Madan and Seneta (1990), Madan, Carr and Chang (1998)), or the hyperbolic model (Eberlein and Keller (1995), Eberlein (2001)) were found to be successful in synthesizing the slice across strikes for a given maturity, and on occasion the slice across maturity for a given strike. These models are on occasion employed to price structures possessing a dominating cash flow at a point of time, like a structured note or a credit default swap. However, these models have a theoretical term structure of skewness and kurtosis inconsistent with observed market surfaces. For Lévy processes (see Konikov and Madan 2002), the skewness decreases like the reciprocal of the square root of maturity while excess kurtosis decreases like the reciprocal of maturity, but these entities are often constant or increasing in market implied surfaces. As a consequence, these models are not employed to value structures with a balanced cash flow at a variety of dates.

Models that can synthesize the surface include the Heston stochastic volatility model (Heston (1993)), preferably including jumps (Bakshi, Cao and Chen (1997)), Lévy stochastic volatility models like VGSA proposed in Carr, Geman, Madan and Yor (2003), and nonparametric structures like local volatility (Dupire (1994), Derman and Kani (1994)) and the local Lévy model (Carr, Geman, Madan and Yor (2004)). The stochastic volatility models are two dimensional Markov models and it is known that the vanilla option surface is not really capable of identifying entities like the rate of mean reversion in variance or the volatility of the instantaneous variance. The nonparametric structures like local volatility and local Lévy processes are one dimensional Markov processes with local volatility accessing all its skewness from the leverage effect engineered by the dependence of volatility on the asset price. Local Lévy processes have an additional source of skewness embedded in a skewed local motion.

Recently, Carr, Geman, Madan and Yor (2007) showed that a wide class of additive processes (with independent but inhomogeneous increments) can also synthesize the surface of option prices, remarkably with as few as four parameters. These processes are associated with the law at unit time of a subclass of Lévy processes defined by the condition that the law at unit time be self decomposable or a limit law. Sato (1991) showed that when such a self decomposable law is scaled to define the marginal law at time t as that of t^{γ} times the unit time law, then there always exists an additive process possessing these marginal laws. We term this additive process the Sato process.

The fact that Sato processes fit the surface well on many underliers for most days does not automatically qualify them as good models for stuctured product valuation. In this regard we answer the title of Schoutens, Simons and Tistaert (2004), "A Pefect Calibration! Now what?". This is done by first investigating how forward implied volatility curves in the model contrast with those observed in reality across the spectrum of assets and time periods.

Additionally, our interest in Sato processes is further motivated by contracts trading options on realized variance or volatility, with a long maturity. The underlier for these contracts is an average realized variance that in many models converges to its expected value. Consequently the variance of the cash flow declines towards zero. As a result, out-of-the-money options on realized variance have negligible model values. In order that long maturity out-of-the-money options on realized variance be valuable in a model, the average realized variance must remain a random variable and not converge to a constant. We show that the Sato processes have this feature. An alternative approach, not pursued here, is to employ processes with long range dependence (Heyde and Yang, 1997).

The object of this paper is to report on the adequacy of Sato processes as valuation models for equity structured products. We report on the properties of the forward return distributions embedded in these processes and compare the valuations they provide for structured products with the stochastic volatility, local volatility and local Lévy models. In the process we develop a uniform methodology for simulating a large class of Sato processes based on the Ziggurat method of Marsaglia and Tsang (1984).

We find that the inhomogeneity embedded in Sato processes enhances forward return skews and reduces at the money volatilities. This leads to a possible overpricing of cliquets relative to other models. In contrast, Sato processes do maintain the value of long dated out of the money options on variance or volatility, while in contrast these values drop sharply towards zero for a number of parametric models.

The outline of the paper is as follows. Section 2 presents the Sato process, its spot and forward characteristic functions and Lévy systems. A preliminary evaluation of embedded forward returns is presented in section 3 based on an illustrative calibration to S&P 500 options on May 17, 2006. Section 4 presents a theoretical analysis of the variance of long maturity average realized variance for Sato processes. The uniform strategy for the simulation of a large class of Sato processes is developed in Section 5. Section 6 outlines the pricing principles relevant for the pricing of structured products. Section 7 summarizes a number of standard models used for structured product pricing. Section 8 presents the parameter values for the calibrated models. The products priced in this study are described in Section 9. Pricing results are reported in Section 10. Section 11 concludes.

2 The Sato Process

The Sato processes we consider, as models for the logarithm of the stock price, are constructed from the law at unit time of a subclass of pure jump Lévy processes. Let $L(t)$ denote this Lévy process and we suppose that the Lévy density is $k(x)$, $x \in \mathbb{R}-\{0\}$. The appropriate subclass is defined by the condition that $L(1) = X$ be a zero mean self decomposable (SD) random variable with no Gaussian component. It is shown in Sato (1999) that self decomposability is equivalent to $h(x) = |x|k(x)$ being a decreasing function of x for $x > 0$ and an increasing function of x for $x < 0$. Carr, Geman, Madan and Yor (2007) refer to the size scaled Lévy density h as the self decomposability characteristic.

For the Sato process the marginal laws of $(X(t), t \geq 0)$ are given by scaling and for a scaling constant $\gamma > 0$, we have

$$X(t) \overset{(d)}{=} t^{\gamma} X \tag{1}$$

It follows that the characteristic function of $X(t)$ is of the form

$$\begin{aligned}
\phi_{X(t)}(u) &= E[e^{iuX(t)}] \\
&= \phi_X(ut^{\gamma}). \\
&= \exp\left(\int_{-\infty}^{\infty} \left(e^{iut^{\gamma}x} - 1 - iut^{\gamma}x\mathbf{1}_{|x|<1} \right) k(x)dx \right)
\end{aligned}$$

The process $X(t)$ is employed in modelling the process for the logarithm of the stock price on incorporating in addition a drift correction. Since for a risk neutral stock price model the continuously compounded return equals $r - q$, where r is the continuously compounded interest rate and q is the dividend yield, we define

$$S(t) = S(0)e^{(r-q)t+X(t)}(E[\exp(X(t))])^{-1},$$

where we assume that $E[\exp(X(t))]$ is finite. We may then write

$$\ln\left(\frac{S(t)}{S(0)}\right) = (r-q)t - \ln\left(\phi_{X(t)}(-i)\right) + X(t). \tag{2}$$

It is shown in Carr, Geman, Madan and Yor (2007) that $X(t)$ is an additive process with the inhomogeneous Lévy density $g(x,t)$ given by

$$g(x,t) = \mathbf{1}_{x>0}\left(-\frac{h'\left(\frac{x}{t^\gamma}\right)\gamma}{t^{1+\gamma}}\right) + \mathbf{1}_{x<0}\left(\frac{h'\left(\frac{x}{t^\gamma}\right)\gamma}{t^{1+\gamma}}\right) \tag{3}$$

The self decomposability property of $h' < 0$ for $x > 0$ and $h' > 0$ for $x < 0$ is then critical to $g(x,t)$ being positive and thus a density. The subclass of Sato processes we focus on here is one in which the inhomogeneous Lévy density is itself also decreasing in x for positive x and increasing in x for negative x. This is the case for example when $h(x)$ and $h(-x)$ are both completely monotone functions for $x > 0$.

Since we know that we have an additive process consistent with the marginals the forward log returns have an easily defined characteristic function given by

$$\phi_{\ln\left(\frac{S(t+h)}{S(t)}\right)}(u) =$$
$$\exp\left(iu((r-q)h - \ln\phi_X(-i(t+h)^\gamma) + \ln\phi_X(-it^\gamma)\right)\frac{\phi_X(u(t+h)^\gamma)}{\phi_X(ut^\gamma)} \tag{4}$$

We may employ these characteristic functions to construct forward log return densities by Fourier inversion. We use these densities via the transform methods of Carr and Madan (1999) to price options on the forward gross return $\left(\frac{S(t+h)}{S(t)}\right)$. We report on the prices of options on the forward gross return and construct the Black-Merton-Scholes implied volatility curves for these forward starting options at a variety of forward dates t for a range of maturities h and associated strikes a. A forward starting call option with a 100 dollar notional on the gross return pays at $t + h$ for the strike a the cash flow

$$100\left(\frac{S(t+h)}{S(t)} - a\right)^+.$$

This option has a time t price $w_t(a,h)$ in the model obtained given by

$$w_t(a,h) = E_t\left[e^{-rh}\left(\frac{S(t+h)}{S(t)} - a\right)^+\right].$$

Note that the conditional expectation E_t by the independence of the increments depends only on the calendar time t.

The self decomposable characteristics we employ, in addition to the Variance Gamma (VG) (Madan, Carr and Chang (1998)), the Normal Inverse Gaussian (NIG) (Barndorff-Nielsen (1998)), and the Meixner ($MXNR$) process (Schoutens (2002), (2003)) reported in Carr, Geman, Madan and Yor (2007) are the $CGMY$ (Carr, Geman, Madan and Yor (2002)) and the GH, the generalized hyperbolic model (Eberlein (2001), Eberlein, Prause (2002)).

3 Properties of Forward Returns

We have observed that many implied volatility surfaces on many underliers and on many days are well fitted by the marginal distributions associated with the Sato process and the spot characteristic functions (4) taken at $t = 0$. This finding was reported in Carr, Geman, Madan and Yor (2007).

We now assess the impact of the time inhomogeneity built into the Sato process on the distribution of forward log returns. For this purpose we construct forward Black-Merton-Scholes implied volatility surfaces $\sigma_t(a, h)$ from the forward gross return call prices $w_t(a, h)$. These are the forward, at time $t > 0$, gross return implied volatility surfaces associated with any Sato process model calibrated to market implied volatilities at time 0. Every Sato process when calibrated at time 0 produces via the characteristic function (4) the forward call option surface $w_t(a, h)$ for any forward date t, across the range of strikes a and maturities h. These prices may then be transformed into Black-Merton-Scholes implied volatilities $\sigma_t(a, h)$. Our objective here is to ascertain the difference between these forward implied volatility surfaces and those produced by the Sato process at time 0. The latter are consistent with wide market observations as documented in Carr, Geman, Madan and Yor (2007), the former may not be.

With a view to detecting whether these forward return distributions are close to the spot distributions associated with the Sato process at time 0, we first ask if these forward Sato option prices are well fitted by the time 0 Sato process or the spot Sato model. In essence we ask if the family of forward Sato return distributions are similar to the type of distributions seen daily in the spot option surfaces.

For this exercise we consider 8 models. These are the scaled self decomposable forms of the $VG, NIG, MXNR, GH$ and the $CGMY$ model with Y fixed at the levels $.25, .5, .75$ and 1.25. These models are all fitted to European option prices on the $S\&P500$ index for May 17, 2006 with maturities between one month and two years. The data is obtained from OptionMetrics available at $WRDS$ the Wharton Research Data Service. Only out-of-the money options are utilized to minimize the impact of American features. Furthermore, European prices are constructed by first fitting a geometric Brownian motion model of constant volatility to the American price and using this volatility via the Black Scholes formula to generate a European price that is subsequently used in the calibrations. There were 230 options in all. The fit statistics reported are the root mean square error ($RMSE$), the average absolute error (AAE) and the average percentage error (APE) defined as the average absolute error divided by the average option price. Table 1 presents the fit statistics of the eight models on the spot surface.

TABLE 1
Spot Surface Fit Statistics

Model	RMSE	AAE	APE
VGSSD	0.6942	0.5502	0.0331
CGMYSSD1	0.6905	0.5468	0.0328
CGMYSSD2	0.6968	0.5525	0.0332
CGMYSSD3	0.7129	0.5662	0.0340
CGMYSSD4	0.7767	0.6073	0.0365
NIGSSD	0.7084	0.5636	0.0338
MXNRSSD	0.6889	0.5471	0.0328
GHSSD	0.6892	0.5466	0.0329

We observe that all the models fit the 230 options of the spot surface at a comparable level.

We next construct the forward option price surfaces $w_t(a,h)$ for $t = 1, 2, 5$ years. We construct three surfaces using 21 strikes a ranging from 80 to 120 in 2 dollar intervals, and four maturities h, ranging from .25 to one year in steps of .25. This gives us a total of 84 options for each of the three forward dates. These prices are obtained by Fourier inversion of the characteristic function for forward returns (4) modified as described in Carr and Madan (1999) for option pricing. The parameter values used are those estimated for the spot model and hence we investigate the nature of future implied volatilities of the estimated spot model in the model.

We then fit each Sato process model at time 0, to each of its three forward surfaces. The fit statistics of all eight models are presented in Tables 2, 3 and 4.

TABLE 2
One Year Forward Surface

Model	RMSE	AAE	APE
VGSSD	0.0907	0.0715	0.0537
CGMYSSD1	0.0991	0.0814	0.0395
CGMYSSD2	0.0859	0.0718	0.0348
CGMYSSD3	0.1459	0.1223	0.0594
CGMYSSD4	0.1426	0.0943	0.0457
NIGSSD	0.0693	0.0561	0.0272
MXNRSSD	0.0783	0.0638	0.0309
GHSSD	0.0626	0.0494	0.0240

TABLE 3
Two Year Forward Surface

Model	RMSE	AAE	APE
VGSSD	0.0898	0.0716	0.0596
CGMYSSD1	0.1051	0.0863	0.0447
CGMYSSD2	0.0869	0.0694	0.0359
CGMYSSD3	0.1683	0.1378	0.0709
CGMYSSD4	0.2056	0.1492	0.0764
NIGSSD	0.0758	0.0605	0.0312
MXNRSSD	0.0905	0.0752	0.0389
GHSSD	0.0848	0.0682	0.0353

TABLE 4
Five Year Forward Surface

Model	RMSE	AAE	APE
VGSSD	0.0783	0.0623	0.0640
CGMYSSD1	0.3362	0.2478	0.1453
CGMYSSD2	0.1319	0.1042	0.0611
CGMYSSD3	0.2046	0.1621	0.0929
CGMYSSD4	0.2882	0.2166	0.1217
NIGSSD	0.1018	0.0803	0.0461
MXNRSSD	0.1248	0.1034	0.0603
GHSSD	0.1290	0.1061	0.0622

We observe from the reported average percentage errors in these tables, that for each model as we go out in the forward date, the quality of the fit of the spot model to the forward surface deteriorates. This is indicative of a divergence of the internal forward return distributions of the model from what is seen in spot markets.

We may also measure the average difference in implied volatilities for all strikes and maturities between the forward implied volatility curves and the spot curve for each model. These are measured by the root mean square difference in implied volatilities and are reported in Table 5.

TABLE 5
RMSE of Spot and Forward Implied Volatility Curve

Model	One Year	Two Years	Five Years
VGSSD	0.0255	0.0323	0.0407
CGMYSSD1	0.0534	0.0551	0.0555
CGMYSSD2	0.0524	0.0534	0.0537
CGMYSSD3	0.0508	0.0517	0.0517
CGMYSSD4	0.0494	0.0504	0.0525
NIGSSD	0.0511	0.0519	0.0517
GHSSD	0.0536	0.0549	0.0549
MXNRSSD	0.0534	0.0544	0.0538

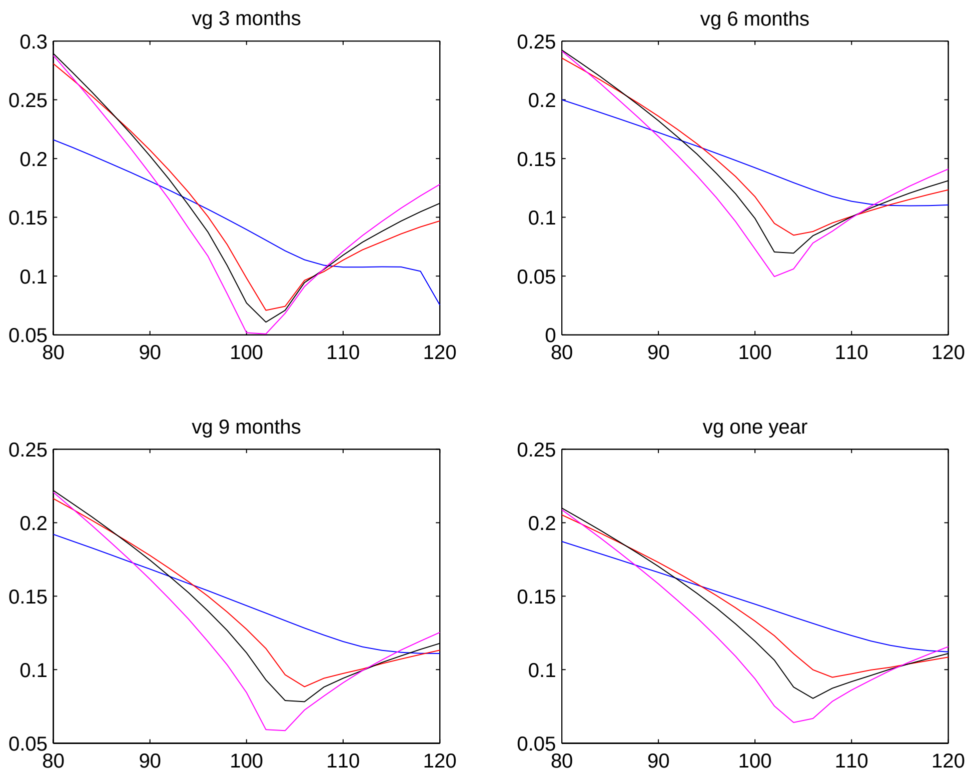

Figure 3: Forward Implied Volatilities for the VG Sato Process.

We observe that generally the inhomogeneity tends to take the implied volatility curves further from the initial spot curve as we move further into the future of the Sato process models.

To further investigate the nature of this divergence we graph the forward implied volatility curves for each model. The curves are for fixed maturities of 3 months, 6 months, 9 months and one year. For each maturity we show in blue the spot curve, while in red, black, and magenta we show the curve one year forward, two years forward, and five years forward. We present 5 graphs, one for $CGMY$ with $Y = .5$, and the others for Sato processes associated with the VG, NIG, $MXNR$, and GH distributions. Each graph contains four subgraphs, one for each of four maturities, with four curves on each subgraph, for the spot implied volatility curve, and the one, two and five year forward implied volatility curves.

We see from these graphs that the forward return distributions are closer to each other than they are to the spot distribution. They also have substantially lower at the money volatilities and sharper skews than those of the spot curve.

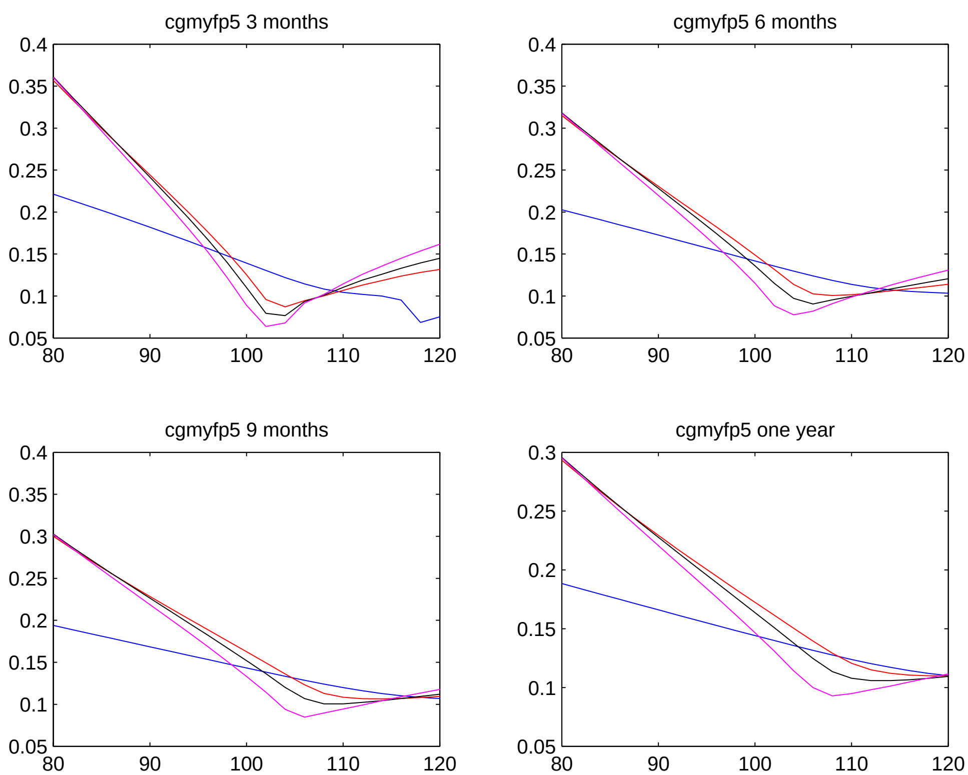

Figure 4: Forward Implied Volatilities for the CGMY Sato process with Y=0.5

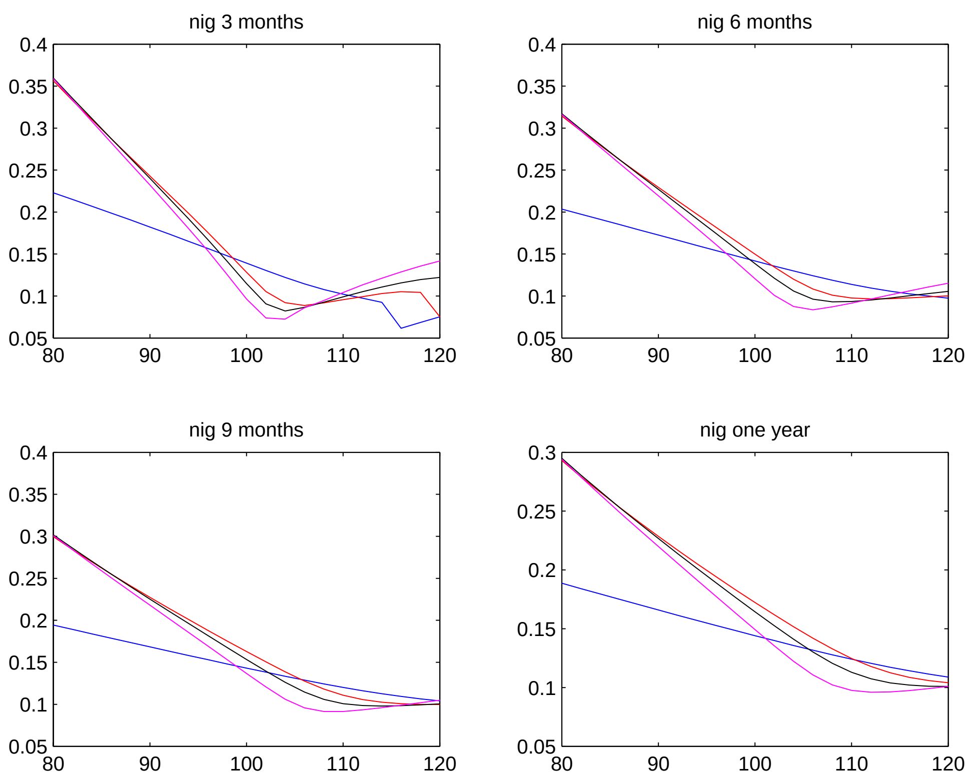

Figure 5: Forward Implied Volatilities for the NIG Sato process

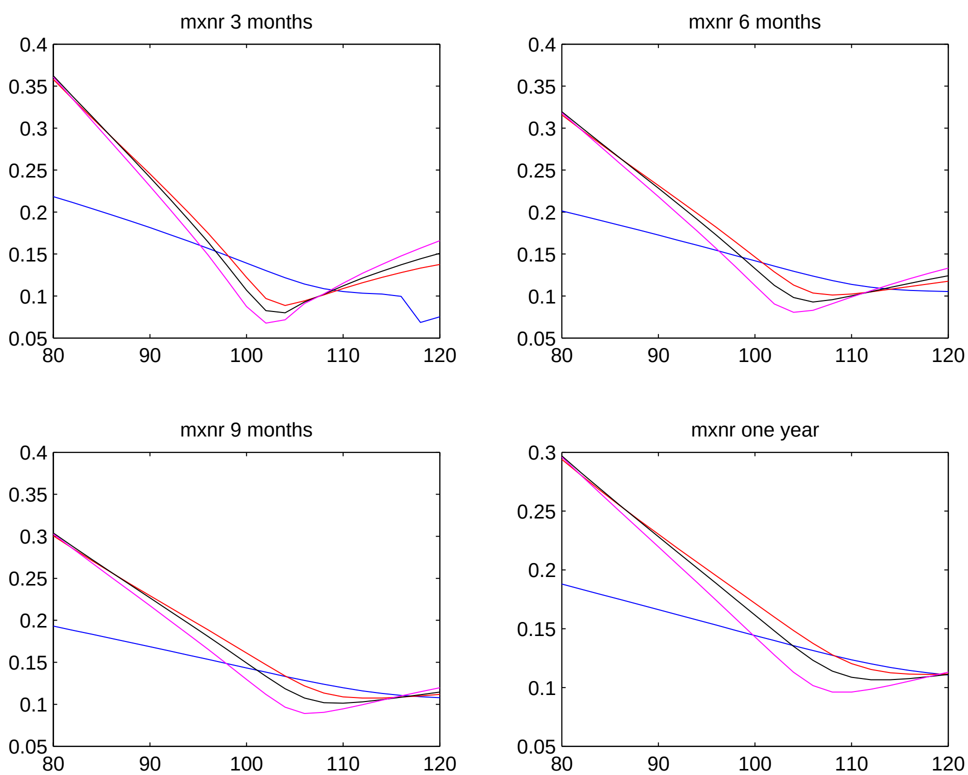

Figure 6: Forward Implied Volatilities for the MXNR Sato process

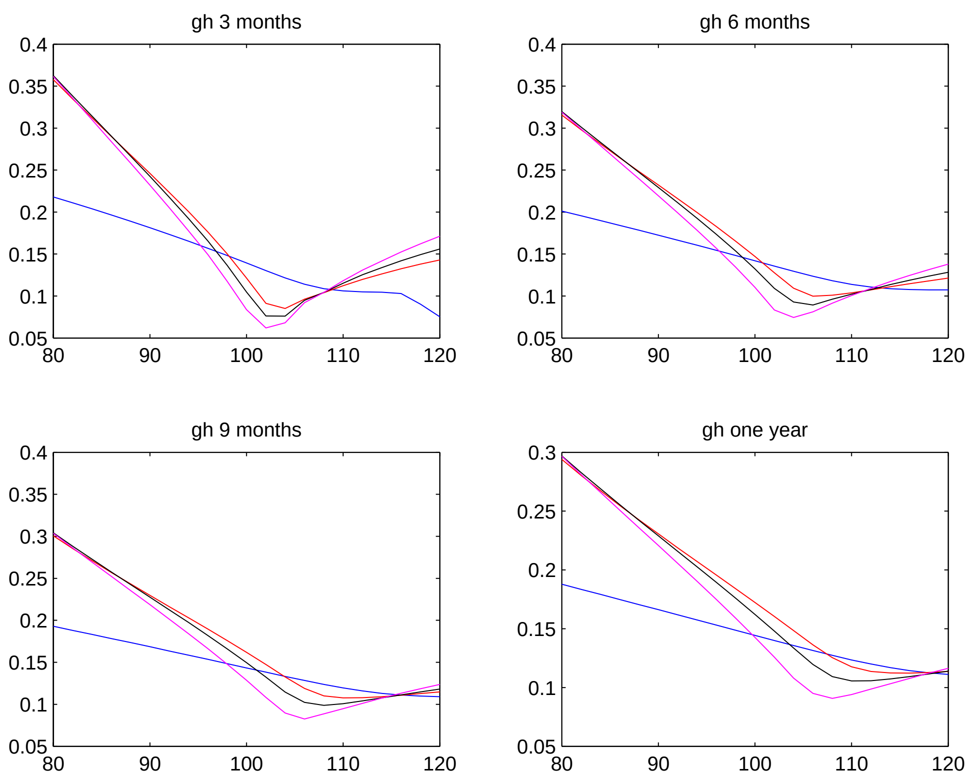

Figure 7: Forward Implied Volatilities for the GH Sato process

4 The Variance of Realized Variance for Sato Processes

We are interested here in analysing the long term behavior of average realized quadratic variation for the additive Sato processes associated with the scaled marginals of a self decomposable law at unit time. For this purpose it is sufficient to focus attention on the symmetric case. The inhomogeneous Lévy density for the jumps is given on the positive side by

$$g(x,t) = -\ h'\left(\frac{x}{t^{\gamma}}\right)\frac{\gamma}{t^{\gamma+1}}. \tag{5}$$

We define by $q(t)$ the realized quadratic variation to time t per unit time or

$$\begin{aligned} q(t) &= \frac{V(t)}{t} \\ V(t) &= \sum_{s\leq t}(\Delta X(s))^2 \end{aligned}$$

The expectation of $V(t)$ is easily evaluated on noting that

$$M(t) = V(t) - \int_0^t \int_{-\infty}^{\infty} x^2 g(x,u)dudx$$

is a martingale. Hence

$$E\left[V(t)\right] = -2\int_0^t du \int_0^{\infty} dx\ x^2 h'\left(\frac{x}{u^{\gamma}}\right)\frac{\gamma}{u^{\gamma+1}}$$

Making the change of variable $yu^{\gamma} = x$ we get

$$\begin{aligned} E\left[V(t)\right] &= -2\int_0^{\infty} dy\ y^2 h'(y)\left(\int_0^t du\ u^{2\gamma}u^{\gamma}\frac{\gamma}{u^{\gamma+1}}\right) \\ &= \left(-\int_0^{\infty} dy\ y^2 h'(y)\right)\left(\int_0^t du\ 2\gamma u^{2\gamma-1}\right) \\ &= \left(-\int_0^{\infty} dy\ y^2 h'(y)\right)t^{2\gamma} \end{aligned}$$

The expectation of $V(t)/t$ is then

$$\left(-\int_0^{\infty} dy\ y^2 h'(y)\right)t^{2\gamma-1}$$

and this is constant, increasing or decreasing according as γ equals, exceeds, or is below $1/2$. We note that this expectation is the price of the variance swap contract to time t. For Lévy processes $X(t)$ this price is a constant independent of maturity and here with scaling we have the possibility of a term structure in the variance swap rate.

From the perspective of options on $q(t)$, our interest shifts to the random variable $q(t)$. The Laplace transform of $q(t)$ is

$$E\left[\exp\left(-\lambda q(t)\right)\right] = E\left[\exp\left(-\frac{\lambda}{t}V(t)\right)\right]$$

and we may determine the variance of $q(t)$ from the Laplace transform of $V(t)$.

It is shown in Carr, Geman, Madan and Yor (2005) that when $X(t)$ is a Lévy process with Lévy density $g(x)$ then $V(t)$ is a Lévy process with Lévy density

$$v(y) = \frac{g(\sqrt{y})}{2\sqrt{y}} + \frac{g(-\sqrt{y})}{2\sqrt{y}},\ y > 0$$

It follows that the Laplace transform of $V(t)$ is

$$E\left[\exp\left(-\lambda V(t)\right)\right] = \exp\left(-t\int_0^\infty (1-e^{-\lambda y})v(y)dy\right)$$

and so for $q(t)$ we get

$$E\left[\exp\left(-\frac{\lambda}{t}V(t)\right)\right] = \exp\left(-t\int_0^\infty \left(1-e^{-\lambda y/t}\right)v(y)dy\right)$$

The negative of the partial derivative of this Laplace transform with respect to λ evaluated at $\lambda = 0$ is

$$E\left[\frac{V(t)}{t}\right] = \int_0^\infty yv(y)dy$$

and this is constant across t as expected. For the second partial derivative we have

$$\frac{\partial}{\partial\lambda}\left(\exp\left(-t\int_0^\infty \left(1-e^{-\lambda y/t}\right)v(y)dy\right)\left(-\int_0^\infty e^{-\lambda y/t}yv(y)dy\right)\right)$$

$$= \exp\left(-t\textstyle\int_0^\infty \left(1-e^{-\lambda y/t}\right)v(y)dy\right)\left(-\textstyle\int_0^\infty e^{-\lambda y/t}yv(y)dy\right)^2$$

$$+\ \exp\left(-t\textstyle\int_0^\infty \left(1-e^{-\lambda y/t}\right)v(y)dy\right)\textstyle\int_0^\infty e^{-\lambda y/t}y^2v(y)dy\frac{1}{t}$$

Evaluating at $\lambda = 0$ and subtracting the square of the expectation we get that the variance is

$$\frac{\int_0^\infty y^2v(y)dy}{t}$$

and thus the variance of $q(t)$ decreases like t, with the variable approaching a constant, as we expect.

In the case of a Sato process, however, we have that the inhomogeneous Lévy density for $V(t)$ is

$$v(y,t) = \frac{g(\sqrt{y},t)}{2\sqrt{y}} + \frac{g(-\sqrt{y},t)}{2\sqrt{y}}$$

The Laplace transform of $V(t)$ is now

$$E\left[\exp\left(-\lambda V(t)\right)\right] = \exp\left(-\int_0^t\int_0^\infty (1-e^{-\lambda y})v(y,u)dydu\right)$$

and for $q(t)$ we get

$$E\left[\exp\left(-\frac{\lambda}{t}V(t)\right)\right] = \exp\left(-\int_0^t\int_0^\infty (1-e^{-\lambda y/t})v(y,u)dydu\right)$$

The mean value is given by

$$\frac{1}{t}\int_0^t\int_0^\infty yv(y,u)dydu$$

and in the symmetric case we have

$$\frac{1}{t}\int_0^t\int_0^\infty y\frac{g(\sqrt{y},u)}{\sqrt{y}}dydu = -\frac{1}{t}\int_0^t du\int_0^\infty dy\sqrt{y}h'\left(\frac{\sqrt{y}}{u^\gamma}\right)\frac{\gamma}{u^{\gamma+1}}$$

We now make the substitution $\sqrt{y} = u^\gamma w$ to get

$$-\frac{1}{t}\int_0^t du\int_0^\infty dw\ h'(w)u^\gamma w^2\frac{\gamma}{u^{\gamma+1}}u^{2\gamma}2 = \left(-\int_0^\infty dw\ h'(w)w^2\right)t^{2\gamma-1}$$

as we computed before.

For the variance we wish to evaluate the second partial derivative with respect to λ and on subtracting the square of the mean we will have on evaluating at $\lambda = 0$ the term

$$\begin{aligned}
\frac{1}{t^2}\int_0^t\int_0^\infty y^2v(y,u)dydu &= \frac{1}{t^2}\int_0^t\int_0^\infty y^2\frac{g(\sqrt{y},u)}{\sqrt{y}}dydu \\
&= -\frac{1}{t^2}\int_0^t\int_0^\infty \sqrt{y^3}h'\left(\frac{\sqrt{y}}{u^\gamma}\right)\frac{\gamma}{u^{\gamma+1}}dydu \\
&= -\frac{2}{t^2}\int_0^t\int_0^\infty u^{3\gamma}w^3h'(w)\frac{\gamma}{u^{\gamma+1}}u^{2\gamma}wdwdu \\
&= \left(-\int_0^\infty w^4h'(w)dw\right)\frac{1}{t^2}\int_0^t 2\gamma u^{4\gamma-1}du \\
&= \left(-\int_0^\infty w^4h'(w)dw\right)\frac{t^{2(2\gamma-1)}}{2}
\end{aligned}$$

and we see that unlike the Lévy case the variance of $q(t)$ is constant, increasing or decreasing according as 2γ equals, exceeds or falls short of unity. Hence, for the Sato process $q(t)$ can remain a random variable with a positive variance at all t.

5 Uniformly Simulating Sato Processes

Suppose we wish to generate the path of the Sato process from time 0 to time T. We do this by generating the paths of the log price and then taking the exponential. The log price over an interval of length t has a drift as implicitly specified in the equation (2) plus a sum of jumps with the compensator given through the density in equation (3). Making the change of variable $u = x/t^\gamma$ we may write the density for the compensator of the corresponding process $U(t)$ with jumps u as

$$l(u,t) = \mathbf{1}_{u>0}\left(-\frac{h'(u)\gamma}{t}\right) + \mathbf{1}_{u<0}\left(\frac{h'(u)\gamma}{t}\right). \tag{6}$$

We may then simulate the process $\ln S(t)$ as

$$\ln S(t) = \ln S(0) + a(t) + \sum_{s\le t} s^\gamma \Delta U(s)$$

where $U(t)$ is a jump process with compensator specified via the density (6) and $a(t)$ is the drift component.

The process $U(t)$ has an arrival rate of jumps decreasing in time as is evidenced by the division by t in equation (6). The distribution of jump sizes is however independent of t and has a decreasing density in the absolute value of the jump size. Of course these jumps in the process U have to be scaled by t^γ at time t to get the jumps of the process for the logarithm of the stock price.

The arrival rate of jumps in the process $U(t)$ on the positive side, for jumps exceeding ε, is given by

$$\frac{A_p}{t} = \frac{\gamma}{t}\int_\varepsilon^\infty -h'(u)du$$

while on the negative side we get

$$\frac{A_n}{t} = \frac{\gamma}{t}\int_{-\infty}^{-\varepsilon} h'(u)du.$$

The distribution of jump sizes $\varepsilon + v$ on the positive side is

$$f_p(v) = \frac{h'(\varepsilon+v)}{\int_\varepsilon^\infty h'(u)du},\ v>0$$

while on the negative side it is

$$f_n(v) = \frac{h'(-\varepsilon-v)}{\int_{-\infty}^{-\varepsilon} h'(u)du},\ v>0$$

Both $f_p(v), f_n(v)$ are decreasing densities, for all Sato processes that we consider. This makes them ideally suited to simulation using a Ziggurat cover as described in Marsaglia and Tsang (1984). We draw from these densities by the Ziggurat method where for the distribution of the tail we approximate by an exponential density, except for the $CGMY$. In the case of the $CGMY$ we have an exact expression in terms of the incomplete gamma function. The shape parameter can be organized to be $2-Y>0$, on integrating by parts two times.

We use the Ziggurat to simulate the paths on a daily basis. For the first day however the division of arrival rates by t amounts to a division by zero and we cannot do the simulation of the first day using the Ziggurat. Here we use the time 0 characteristic function of the log price for one day and construct the cumulative distribution function by Fourier inversion. An application of the inverse uniform method then yields a draw from the log price process at the first day. Subsequently we employ the Ziggurat method, first drawing a Poisson number of positive and negative jumps, and then using the Ziggurat to sample the actual jump sizes of the process for $U(t)$. The move in the logarithm of the stock price is then formed by adding to the required drift, the positive jumps in $U(t)$, scaled at time t by t^{γ}, and subtracting the similarly scaled jumps on the negative side.

6 Structured Product Pricing Principles

The standard methodology of risk neutral pricing is not relevant for the pricing of structured products, though we shall see that risk neutral models play a critical role. Under the principles of risk neutral pricing all products are priced under a single change of measure and this linearity of the pricing operator is a consequence of the absence of arbitrage (Harrison and Kreps (1979), Harrison and Pliska (1981)). These are the pricing principles relevant to liquid markets where one may transact in both directions at the same price. For specially designed structured products there are two transaction prices and one buys at the ask price and sells at the bid price and these prices are not generally given by a linear operator.

The properties of the bid and ask prices may be seen by considering the consequences of hedging to acceptability. A structured product may be represented as a state contingent cash flow whereby we payout in present value terms the sum $x=(x_s, s\in\mathcal{S})$ for a finite set of states s. We may hedge this liability by trading in zero cost or self financed liquid assets with asset j yielding the cash flow $(Y_{js}, s\in\mathcal{S})$. With the position α_j in liquid asset j and the sale of x at the ask price a we have the net cash flow

$$c=a+\alpha' Y-x.$$

The cash flow c is certainly acceptable if $c_s\geq 0$ for all states $\geq s$. More generally we adopt the definition of acceptable cash flows of Artzner, Eber, Delbaen and Heath (1998), also studied in the form presented here in Carr, Geman and Madan (2001). In this formulation the cash flow c is acceptable if it belongs to a convex cone containing the positive orthant. More specifically we suppose the existence of a finite set of valuation or test measures given by the columns of the positive matrix B such that c is acceptable just if cB or

$$a + (\alpha' Y - x)B \geq 0.$$

The ask price for the cash flow is the smallest such price and is then the solution to the linear programming problem

$$\begin{aligned} a(x) &= Min_{(a,\alpha)}\ a \\ &\quad s.t.\ a + \alpha' Y B \geq xB \end{aligned}$$

By virtue of being a solution to such a minimization problem we see that the ask price is a convex functional of the liability x and linear pricing does not hold.

A similar argument shows that the bid price $b(x)$ is the solution to

$$\begin{aligned} b(x) &= Max_{(b,\beta)}\ b \\ &\quad s.t.\ b + \beta' Y B \leq xB \end{aligned}$$

The solution to a maximization problem is a concave functional and again we do not have a linear pricing rule.

These considerations establish our remark in the introduction that directional ask prices are convex functionals while bid prices are concave functionals.

We may learn more about the ask and bid prices by considering the dual to these linear programming problems. The dual to the ask price problem yields

$$\begin{aligned} a(x) &= Max_{q \geq 0}\ q' x \\ &\quad s.t.\ Y B q = 0 \\ &\quad \mathbf{1}' q = 1 \\ &\quad q \geq 0 \end{aligned}$$

Similarly we have for the bid price problem that

$$\begin{aligned} b(x) &= Min_{q \geq 0}\ q' x \\ &\quad s.t.\ Y B q = 0 \\ &\quad \mathbf{1}' q = 1 \\ &\quad q \geq 0 \end{aligned}$$

The nonnegativity of q along with the condition that the entries sum to unity yields that Bq is a special measure on the set of scenarios, one that is in the convex hull of the test measures B. Furthermore, the measure Bq is a risk neutral measure as it reprices the zero cost liquid assets Y at their zero market price. The measures Bq are then a subcollection of risk neutral models with the bid price the smallest of these model valuations while the ask price is the largest. Hence in general we may compute a wide range of admissible model valuations with the ask price being the largest valuation and the bid price, the smallest valuation. All other valuations do not represent prices but are just candidate valuations. The differences in model valuations are therefore not evidence of model risk but just intermediate calculations in the procedure to locate the bid and ask prices.

We also learn from the dual that the ask price will be attained for different products at different models and the use of different models to mark different products is no inconsistency but just the logical consequence of hedging to acceptability.

Though risk neutral valuation is not relevant for structured product valuation as linear pricing fails, risk neutral models and valuations by market tested equivalent martingale measures remain useful. This is because the models defined by Bq are risk neutral models by virtue of the repricing condition $YBq = 0$. However the search is constrained to be in the convex hull of B and not over all risk neutral models or equivalent martingale measures. Were we to take the latter, it is well known that the ask price would be a superreplication price while the bid price would be a subreplication price and the spread too wide. Our subsequent analysis restricts attention to market tested equivalent martingale measures as embedded in calibrated pricing models commonly used in the financial industry.

Finally we observe that these nonlinear, convex and concave ask and bid prices are free of arbitrage provided market participants use the same cone of acceptability. In this case, for $0 < \lambda < 1$, the purchase of the package $\lambda x_1 + (1-\lambda)x_2$ at the ask price $a(\lambda x_1 + (1-\lambda)x_2)$ coupled by the sale of the components at the bid prices $\lambda b(x_1), (1-\lambda)b(x_2)$ results in the non-positive cash flow

$$\begin{aligned} & -a(\lambda x_1 + (1-\lambda)x_2) + \lambda b(x_1) + (1-\lambda)b(x_2) \\ \leq \; & -a(\lambda x_1 + (1-\lambda)x_2) + b(\lambda x_1 + (1-\lambda)x_2) \leq 0. \end{aligned}$$

The first inequality is a consequence of the concavity of the bid price and the second follows from the domination of bid prices by ask prices.

There are then four important consequences of hedging to acceptabilty, noted in the introduction. The ask and bid prices are not given by a linear or risk neutral pricing principle. The former is a convex functional obtained as the maximum of admissible risk neutral valuations while the latter is a concave functional obtained as the minimum of a similar set of risk neutral valuations. The model valuations supporting products vary with the product being priced. Finally, agreement on the cone of acceptability leaves these nonlinear pricing rules free of exposure to arbitrage.

7 The Reference Models

Apart from the Sato processes used to calibrate the surface there are a number of standard models used in the financial industry for calibrating a surface of option prices and then valuing a variety of structured products. These include the Heston stochastic volatility (HSV) model (Heston (1993)), a stochastic volatility extension of the Merton jump diffusion model that we term (SVJ), (Bakshi, Cao and Chen (1997)). A stochastic volatility extension of a Lévy process termed ($VGSA$) (Carr, Geman, Madan and Yor (2003)). In addition we consider two nonparametric structures, local volatility (LV) (Dupire (1994), Derman and Kani (1994)) and a local Lévy (LL) (Carr, Geman, Madan and Yor (2004)) process that in particular is a localized form of $CGMY$. We briefly summarize these processes in separate subsections of this section.

7.1 HSV

The Heston stochatic volatility model has the risk neutral stock price $S(t)$ evolving in accordance with the stochastic differential equations

$$\begin{aligned} dS(t) &= (r-q)S(t)dt + \sqrt{v(t)}S(t)dW_S(t) \\ dv(t) &= \kappa\left(\theta^2 - v\right)dt + \lambda\sqrt{v(t)}dW_v(t) \\ dW_S(t)dW_v(t) &= \rho dt \end{aligned}$$

where r is the risk free rate, q the dividend yield, κ is the mean reversion rate in the variance, θ is the long term volatility, λ is the volatility of volatility, $(W_S(t), W_v(t), t \geq 0)$ are standard Brownian motions with instantaneous correlation ρ. This is a five parameter process where in addition to $\kappa, \theta, \lambda,$ and ρ one also estimates the level of the initial variance $v(0)$. The model was simulated with a Milstein first order correction to the diffusion evolution and as zero is a reflecting boundary with positive θ, we replaced the simulated value of y on each path by 10^{-7} whenever it went negative.

7.2 SVJ

The Merton jump diffusion model is extended to incorporate stochastic volatility by the model specification in differential form

$$\begin{aligned} dS(t) &= (r-q)S(t)dt + \sqrt{v(t)}S(t)dW_S(t) \\ &\quad + S(t\_)\int_{-\infty}^{\infty}\left(e^x - 1\right)\left(\mu(dx,dt) - \lambda_J k(x)dxdt\right) \\ dv(t) &= \kappa\left(\eta - v\right)dt + \lambda\sqrt{v(t)}dW_v(t) \\ dW_S(t)dW_v(t) &= \rho dt \\ k(x) &= \frac{1}{\sigma_J\sqrt{2\pi}}\exp\left(-\frac{(x-\mu_J)^2}{2\sigma_J^2}\right) \end{aligned}$$

This model extends the Heston model by allowing for jumps of size x at time t as counted by the random measure $\mu(dx, dt)$ with arrival rate λ_J and Gaussian jump sizes of mean μ_J and jump variance σ_J^2. There are in all 8 parameters, given by $v(0), \lambda_J, \mu_J, \sigma_J, \kappa, \eta, \lambda,$ and ρ.

7.3 VGSA

This is a VG Lévy process running at a speed given by a CIR or square root process. The specific dynamics are

$$\begin{aligned} S(t) &= S(0)\exp((r-q)t)\,\frac{\exp(X(Y(t))}{E[\exp(X(Y(t))]} \\ Y(t) &= \int_0^t y(u)du \\ dy(t) &= \kappa(\theta - y(t))dt + \lambda\sqrt{y(t)}dW_y(t) \end{aligned}$$

where $(X(t), t \geq 0)$ is the two parameter VG process with *Lévy* measure

$$\begin{aligned} k(x) &= \exp(Ax - B|x|) \\ A &= (G-M)/2 \\ B &= (G+M)/2 \end{aligned}$$

There are in all six parameters, $y(0), G, M, \kappa, \theta$, and λ.

7.4 LV

The local volatility model is a nonparametric model with continuous sample paths and stock price dynamics given by

$$dS(t) = (r-q)S(t)dt + \sigma(S(t),t)S(t)dW(t)$$

with a nonparametric local volatility surface $\sigma(S,t)$ and $(W(t), t \geq 0)$ a driving Brownian motion. The function $\sigma(S,t)$ may be inferred from the call option price surface $C(K,T)$ for the call price of strike K and maturity T via the Dupire equation

$$\sigma^2(K,T) = 2\frac{C_T + qC + (r-q)KC_K}{K^2 C_{KK}}$$

For smooth derivatives with respect to both arguments we employ model prices from a calibration of $VGSSD$ to infer the local volatility function.

7.5 LL

The local Lévy model is a generalization of local volatility that runs a pure jump Lévy process with Lévy measure $k(x)$ at a nonparametric space time dependent speed $a(S,t)$. The dynamics are given in compensated jump form by

$$dS(t) = (r-q)S(t)dt + S(t\_)\int_{-\infty}^{\infty}(e^x - 1)\left(\mu(dx,dt) - a(S(t\_),t)k(x)dxdt\right)$$

The specific Lévy measure we localize is $CGMY$ where the C parameter is absorbed by the speed function and

$$k(x) = \mathbf{1}_{x<0}\frac{\exp(-5|x|)}{|x|^{1.5}} + \mathbf{1}_{x>0}\frac{\exp(-10x)}{x^{1.5}}$$

This specification is the result of a number of studies on a variety of index options. The nonparametric speed function may be recovered on solving the convolution equation

$$\begin{aligned}
\int_{-\infty}^{\infty} b(y,t)\psi_e(k-y)dy &= c_T + qc + (r-q)c_k \\
c(k,T) &= C(e^k, T) \\
\psi_e(z) &= \mathbf{1}_{z<0}\int_{-\infty}^{z}\left(e^z - e^x\right)k(x)dx + \\
&\quad \mathbf{1}_{z>0}\int_{z}^{\infty}\left(e^x - e^z\right)k(x)dx \\
a(K,T) &= \frac{b(\ln(K),T)}{K^2 C_{KK}}.
\end{aligned}$$

Again for smooth derivatives we employed call option prices from the calibration of $VGSSD$.

7.6 The Sato processes

We employ the Ziggurat simulator described in section 4 to simulate the Sato processes. This method is readily applied to processes with an elementary closed form for the Lévy measure. Hence we restricted attention to the $MXNR$ and $CGMY$ processes. The VG is also amenable to analysis but as forward returns are in this case from a compound Poisson process with finite activity and a charge at zero we considered just the $CGMY$ and $MXNR$ processes. For $CGMY$ we considered three values of Y and these are $.25, .5$ and $.75$.

8 The Calibrated Models

We present here the parameters and fit statistics for the calibrated processes. The fit statistics are in order the root mean square error ($RMSE$), the average absolute error (AAE), and the average percentage error (APE). The number of options used in the calibration was 230 and they ranged in maturity from a month to two years.

The HSV parameters were initial volatility 0.1424, long term volatility 0.1564, mean reversion 3.1140, volatility of volatility 0.4764 and correlation -0.7124. The fit statistics are 0.6453, $0,4798$, 0.0288.

The SVJ parameters were initial variance 0.02064, jump arrival rate 2.52, mean jump size 0.000574, jump standard deviation 0.0155, mean reversion 2.701, volatility of volatility 0.5147, and correlation -0.6987. The fit statistics were 0.6251, 0.4705, 0.0283.

The $VGSA$ parameters were initial speed 10.8277, negative decay rate 24.9927, positive decay rate 46.0252, mean reversion of speed 2.7896, long term speed 5.9009 and volatility of speed 7.1422. The fit statistics were 0.7445, 0.5735, 0.0345.

Local volatility and Local Lévy used $VGSSD$ for generation with the parameters $\sigma = .1218$, $\nu = 0.6566$, $\theta = -0.1209$, $\gamma = 0.5326$. The fit statistics were 0.6942, 0.5502, 0.0331.

The $CGMYSSD$ parameters for $Y = .25$ were $C = 0.7461$, $G = 7.0953$, $M = 25.46$, $\gamma = 0.5329$. The fit statistics were 0.6905, 0.5468, 0.0328. The respective values for $Y = .5$ were $C = 0.3673$, $G = 5.9112$, $M = 28.3871$, and $\gamma = 0.5332$. The fit statistics were 0.6968, 0.5525, 0.0332. For $Y = .75$ we have $C = 0.1829, G = 4.8074$, $M = 38.7755$, $\gamma = 0.5337$. The fit statistics were 0.7129, 0.5662, 0.0340.

Finally for the Meixner process the parameters were $a = 0.1755$, $b = -1.7765$, $d = 0.6349$ and $\gamma = 0.5329$. The fit statistics were 0.6889, 0.5481, 0.0328.

9 The Structured Products Priced

On these path spaces we priced four cliquets, options on realized variance and options on volatility. The first two cliquets priced are arithmetical with local floors and global caps, or local caps and global floors. We denote by R_n the return over month n. For each product we consider five maturities given by the end of the first to the fifth year.

For the locally floored and globally capped cliquet with local floor LF and global cap GC the payoff at maturity is

$$LFGC(\omega) = Min\left[\sum_n (R_n \vee LF), GC \right]$$

while for the locally capped with cap LC and globally floored with floor GF it is

$$LCGF(\omega) = Max\left[\sum_n (R_n \wedge LC), GF \right]$$

We denote by ω a particular realization of the path space of prices and returns. The local floors and caps were $\pm 5, 10, 15$ percent while the global caps and floors were ± 25 and 50 percent. There were therefore 6 products in each of the two classes $LFGC$ and $LCGF$.

In addition to these basic cliquets we priced swing and reverse swing cliquets. Swing cliquets payoff on large local moves independent of direction and incorporate a global cap. The reverse swing cliquet pays on small moves locally and has the payoff dropping to zero for a large move. The exact payoff on the swing cliquet is

$$SC(\omega) = Min\left[\sum_n \left(|R_n| - k\right)^+, GC \right]$$

while the payoff on the reverse swing cliquet is

$$RSC(\omega) = Min\left[\sum_n \left(k - |R_n|\right)^+, GC \right].$$

The prices of these claims are spot prices and are discounted expected values of the associated cash flows. We denote these by $w_{LFGC}, w_{LCGF}, w_{SC}$, and w_{RSC}. For the swing cliquet we used the strikes of $5, 10, 15\%$ with global caps at $25, 50\%$ and this gave 6 products. For the reverse swing cliquet the values for k were still $5, 10, 15\%$ but the global caps were set at 500, and 1000 to provide us with again 6 products.

In addition to cliquets we considered options on variance and volatility. Denote by R_t the return on day t, measured as a log price relative. The realized variance to day T is defined by

$$\frac{1}{T}\sum_{t=1}^{T} R_t^2$$

For an option on variance with strike k quoted as a volatility in annualized terms the payoff to the option is

$$VarOpt(\omega) = 10000\left(\frac{252}{T}\sum_{t=1}^{T} R_t^2 - k^2\right)^+$$

For an option on volatility the payoff is

$$VolOpt(\omega) = 100\left(\sqrt{\frac{252}{T}\sum_{t=1}^{T} R_t^2} - k\right)^+$$

Our quote on the variance and volatility options are on a forward basis and are not discounted. For the variance option we have

$$w_{varopt} = \sqrt{E[VarOpt(\omega)]}$$

while for the vol option we quote

$$w_{volopt} = E\left[VolOpt(\omega)\right]$$

By the concavity of the square root function we expect that $w_{volopt} < w_{varopt.}$ For the strikes on the variance and volatility options we used *six* strikes starting at 10% and increasing by 5 percentage points to a maximum of 35%. This gave six products for both the variance and volatility options.

10 The Prices

The structured prices are presented in six tables for the six product classes, $LFGC, LCGF, SC, RSC, VarOpt$, and $VolOpt$. Each table has 9 columns for the nine models $HSV, SVJ, VGSA, LV, LL, CGMYSATO(Y = .25, .5$ and $.75)$, and $MXNRSATO$. There are thirty rows in each table to cover the five annual maturities for each of six products. For the cliquets the first fifteen rows employ the lower absolute global cap for three local caps or floors while the last fifteen rows use the larger absolute global cap. For the variance and volatility options we have six strikes and five maturities for each strike. We comment on each of the six tables in the following subsections, in the order of the products listed above, combining the comments on the variance and volatility options into one subsection.

10.1 LFGC

The Sato processes give a higher value for these cliquets with SVJ and LL giving values closer to these from among the other processes. Local volatility gives the lowest values excepting the large local floor of -15%. In each case the values rise with maturity and the global cap and fall as we lower the local floor. The sharper skews for forward returns in Sato processes observed in Section 3 probably account for the higher cliquet prices.

10.2 LCGF

For the local cap of 5% the Sato processes give a substantially higher value, excepting $VGSA$. For higher caps the values are closer with SVJ and LV giving among the highest values for the 15% local cap, though the $CGMYSATO(Y = .75)$ is close for this cap. As we raise the local cap the optionality disappears and we have an in the money situation with the calibrated models in basic agreement. For an effective cap the Sato processes have higher prices possibly due to lower at the money volatilities as seen in section 3.

The values rise as we raise the local cap and the maturity and fall as the global floor is dropped. The increase is quite pronounced for most models as we raise the local cap, but it is particularly so for HSV and LV for the 5% local cap and the global floor of -25%.

10.3 SC

For the 5% strike the Sato processes give intermediate values when compared with the values of the other models. At higher strikes the Sato processes give a higher value, excepting LL that remains comparable with the Sato processes. The Sato processes reach the tails more easily than the models dominated by diffusion components. This is also a feature of the sharply rising implied volatility curves for forward returns seen in Section 3.

The values fall with the strike and rise with maturity. The drop with strike is quite marked for the parametric models ($HSV, SVJ, VGSA$) and LV. It is less so for LL and the Sato processes.

10.4 RSC

The reverse swing cliquet with a lower global cap was more like a bond and we raised the cap to see the differences between the models. Focusing on the

Locally Floored and Globally Capped Cliquets

LF		Global Cap 25				Sato processes			
	HSV	SVJ	VGSA	LV	LL	Y=.25	Y=.5	Y=.75	MXNR
	6.51	7.18	6.33	5.97	6.95	7.83	7.81	7.84	8.07
	11.17	11.91	10.19	10.28	11.87	13.24	13.33	13.35	13.39
-5	13.79	14.35	12.26	13.06	14.87	16.24	16.25	16.35	16.28
	15.04	15.56	13.32	14.61	16.37	17.62	17.52	17.61	17.48
	15.60	16.05	13.94	15.27	16.85	17.93	17.86	17.92	17.76
	3.49	4.06	3.83	3.14	4.77	5.16	5.21	5.13	5.29
	5.86	6.40	5.96	5.29	8.22	9.41	9.50	9.27	9.47
-10	7.18	7.44	6.91	6.86	10.43	12.17	12.14	11.96	12.17
	7.83	8.13	7.36	7.87	11.71	13.89	13.70	13.59	13.74
	8.30	8.63	7.73	8.28	12.31	14.82	14.60	14.52	14.59
	2.77	3.24	3.09	2.72	3.76	3.93	4.02	3.94	3.97
	4.49	4.77	4.70	4.51	6.44	7.33	7.46	7.23	7.32
-15	5.29	5.14	5.24	5.75	8.11	9.65	9.68	9.43	9.60
	5.55	5.40	5.39	6.53	9.05	11.24	11.12	10.96	11.10
	5.79	5.60	5.58	6.74	9.45	12.27	12.08	11.95	12.07

		Global Cap 50							
	6.66	7.43	6.42	6.13	7.26	7.95	7.92	7.92	8.20
	12.99	14.21	11.26	11.59	13.29	14.38	14.48	14.53	14.73
-5	18.43	19.80	15.44	16.63	18.45	19.44	19.59	20.00	20.00
	22.53	23.97	18.83	20.70	22.75	23.30	23.55	24.30	23.96
	25.29	26.70	21.37	23.59	25.86	26.09	26.40	27.31	26.66
	3.59	4.23	3.91	3.29	5.06	5.27	5.31	5.21	5.41
	7.10	7.92	6.85	6.29	9.36	10.37	10.43	10.21	10.55
-10	10.11	10.88	9.38	9.24	13.07	14.65	14.63	14.57	14.95
	12.38	13.32	11.39	11.79	16.21	18.07	18.06	18.21	18.45
	14.17	15.12	12.94	13.61	18.54	20.73	20.72	20.95	21.03
	2.87	3.40	3.17	2.87	4.05	4.04	4.11	4.01	4.09
	5.67	6.20	5.58	5.50	7.51	8.25	8.33	8.14	8.33
-15	8.03	8.33	7.63	8.05	10.52	11.92	11.94	11.81	12.12
	9.75	10.11	9.22	10.30	13.03	14.96	14.96	14.97	15.23
	11.14	11.42	10.45	11.85	14.90	17.41	17.36	17.40	17.59

Locally Capped and Globally Floored Cliquets

Global Floor -25

						Sato Processes			
LC	HSV	SVJ	VGSA	LV	LL	Y=.25	Y=.5	Y=.75	MXNR
	0.48	0.53	2.44	0.43	0.74	2.35	2.50	2.81	2.51
	1.33	1.01	4.90	1.43	2.03	4.63	4.83	5.44	4.85
5	1.94	1.44	6.90	2.16	3.28	6.42	6.73	7.70	6.84
	2.45	1.81	8.53	2.95	4.31	7.99	8.41	9.57	8.47
	3.03	2.22	10.01	3.50	5.24	9.40	9.82	11.02	9.81
	2.82	3.10	3.35	3.12	2.29	3.11	3.18	3.29	3.24
	5.63	5.68	6.25	6.19	4.69	5.96	6.06	6.34	6.13
10	7.84	7.80	8.64	8.62	6.84	8.19	8.42	8.99	8.56
	9.79	9.70	10.64	10.84	8.60	10.08	10.46	11.22	10.52
	11.62	11.36	12.45	12.62	10.13	11.72	12.15	12.97	12.16
	3.22	3.62	3.46	3.38	2.85	3.25	3.28	3.32	3.34
	6.43	6.73	6.43	6.69	5.68	6.31	6.32	6.44	6.40
15	9.01	9.29	8.86	9.36	8.17	8.75	8.85	9.19	9.01
	11.27	11.60	10.92	11.83	10.24	10.81	11.07	11.51	11.13
	13.39	13.63	12.77	13.86	12.02	12.60	12.89	13.36	12.92

Global Floor -50

	-0.31	-0.36	1.87	-0.35	0.17	1.92	2.06	2.37	2.05
	-0.38	-1.00	3.75	-0.50	0.76	3.68	3.96	4.55	3.90
5	-0.46	-1.55	5.41	-0.59	1.56	5.21	5.58	6.46	5.58
	-0.44	-1.72	6.82	-0.29	2.28	6.62	7.06	8.19	7.05
	-0.20	-1.76	8.17	-0.14	2.91	7.95	8.40	9.60	8.29
	2.21	2.44	2.82	2.54	1.76	2.70	2.76	2.87	2.80
	4.44	4.33	5.22	4.95	3.60	5.06	5.24	5.50	5.24
10	6.33	5.91	7.32	7.02	5.44	7.07	7.36	7.84	7.39
	8.04	7.56	9.14	9.06	7.01	8.83	9.22	9.93	9.23
	9.75	9.09	10.87	10.72	8.36	10.41	10.87	11.68	10.80
	2.65	3.02	2.94	2.83	2.32	2.84	2.86	2.89	2.89
	5.35	5.53	5.40	5.58	4.62	5.42	5.51	5.60	5.52
15	7.66	7.66	7.56	7.99	6.85	7.66	7.81	8.04	7.86
	9.73	9.77	9.44	10.30	8.76	9.60	9.86	10.23	9.87
	11.77	11.74	11.22	12.27	10.39	11.33	11.65	12.10	11.59

Swing Cliquet

Strike	HSV	SVJ	VGSA	LV	LL	Sato Processes Y=.25	Y=.5	Y=.75	MXNR
	6.56	7.06	4.47	6.28	6.53	5.66	5.47	5.21	5.80
	11.43	12.18	7.23	10.64	10.47	9.12	9.02	8.62	9.26
5	14.63	15.30	9.40	13.32	12.89	11.24	11.12	10.79	11.33
	16.38	16.91	11.12	14.77	14.37	12.41	12.39	12.16	12.53
	17.16	17.51	12.35	15.65	15.17	12.94	13.02	13.01	13.17
	1.46	1.82	1.15	0.82	3.12	2.49	2.46	2.26	2.55
	2.93	3.73	1.92	1.68	5.33	4.88	4.75	4.32	4.88
10	4.26	5.42	2.62	2.48	6.94	6.64	6.38	5.90	6.62
	5.38	6.71	3.25	3.09	8.27	7.88	7.54	7.08	7.86
	6.26	7.78	3.76	3.74	9.25	8.65	8.32	7.95	8.66
	0.32	0.48	0.29	0.11	1.68	1.19	1.25	1.12	1.21
	0.69	1.07	0.48	0.30	2.93	2.78	2.74	2.45	2.76
15	1.05	1.61	0.68	0.60	3.87	4.07	3.93	3.58	4.09
	1.36	2.08	0.85	0.81	4.77	5.12	4.85	4.49	5.14
	1.60	2.49	0.99	1.07	5.47	5.86	5.52	5.21	5.89

	7.06	7.84	4.62	6.57	7.13	6.04	5.87	5.57	6.18
	13.57	15.15	7.81	12.36	12.56	10.69	10.50	9.96	10.83
5	18.97	20.98	10.60	16.97	16.65	14.05	13.86	13.35	14.25
	23.16	25.13	13.05	20.20	19.88	16.41	16.25	15.86	16.69
	26.06	27.91	15.10	22.68	22.23	17.91	17.87	17.73	18.35
	1.49	1.90	1.15	0.83	3.37	2.61	2.58	2.37	2.66
	3.07	4.07	1.95	1.80	6.07	5.48	5.32	4.85	5.48
10	4.56	6.07	2.69	2.86	8.24	7.84	7.53	6.98	7.87
	5.87	7.76	3.35	3.70	10.23	9.73	9.29	8.71	9.78
	6.95	9.21	3.90	4.54	11.80	11.11	10.64	10.11	11.21
	0.33	0.49	0.29	0.11	1.81	1.24	1.29	1.16	1.25
	0.70	1.10	0.48	0.33	3.25	3.03	3.01	2.71	3.03
15	1.07	1.67	0.68	0.77	4.40	4.66	4.50	4.13	4.70
	1.40	2.19	0.85	1.10	5.55	6.10	5.77	5.38	6.16
	1.65	2.63	0.99	1.44	6.48	7.22	6.80	6.41	7.30

Reverse Swing Cliquets

Global Cap 500

Local Cap	HSV	SVJ	VGSA	LV	LL	Sato Processes Y=.25	Y=.5	Y=.75	MXNR
	27.93	27.84	33.94	24.92	39.18	37.35	35.90	33.86	35.33
	53.45	53.17	67.44	47.57	75.88	78.76	75.44	70.78	74.15
5	76.39	75.76	97.69	67.79	109.06	118.27	113.52	106.21	111.08
	96.89	96.17	124.83	86.07	138.75	154.79	148.89	139.36	145.46
	115.41	114.40	149.07	102.45	165.49	188.23	181.34	169.94	177.00
	79.38	78.86	87.55	76.24	92.47	90.97	89.67	87.72	88.87
	151.07	149.72	170.10	145.08	177.61	181.93	178.68	174.09	177.19
10	215.52	213.17	244.55	206.62	254.47	266.46	261.65	254.29	259.08
	273.21	270.34	311.30	261.93	323.33	343.63	337.51	327.81	334.01
	325.12	321.44	369.74	311.19	377.57	389.27	388.96	386.75	388.43
	135.29	134.51	143.76	132.59	147.96	146.68	145.45	143.58	144.53
	257.27	255.29	277.22	252.15	283.30	288.02	284.90	280.49	283.28
15	366.93	363.59	397.47	359.12	405.13	417.31	413.19	406.23	410.59
	408.62	407.72	409.36	405.47	409.36	409.37	409.37	409.37	409.37
	389.40	389.39	389.40	388.56	389.40	389.40	389.40	389.40	389.40

Global Cap 1000

Local Cap	HSV	SVJ	VGSA	LV	LL	Y=.25	Y=.5	Y=.75	MXNR
	27.93	27.84	33.94	24.92	39.18	37.35	35.90	33.86	35.33
	53.45	53.17	67.44	47.57	75.88	78.76	75.44	70.78	74.15
5	76.39	75.76	97.69	67.79	109.06	118.27	113.52	106.21	111.08
	96.89	96.17	124.83	86.07	138.75	154.79	148.89	139.36	145.46
	115.41	114.40	149.07	102.45	165.49	188.23	181.34	169.94	177.00
	79.38	78.86	87.55	76.24	92.47	90.97	89.67	87.72	88.87
	151.07	149.72	170.10	145.08	177.61	181.93	178.68	174.09	177.19
10	215.52	213.17	244.55	206.62	254.47	266.46	261.65	254.29	259.08
	273.21	270.34	311.30	261.93	323.33	343.63	337.51	327.81	334.01
	325.12	321.45	370.96	311.19	385.05	413.62	406.35	394.58	401.93
	135.29	134.51	143.76	132.59	147.96	146.68	145.45	143.58	144.53
	257.27	255.29	277.22	252.15	283.30	288.02	284.90	280.49	283.28
15	366.93	363.59	397.47	359.12	405.37	418.06	413.42	406.25	410.69
	465.15	461.03	505.29	455.09	514.75	536.19	530.23	520.72	526.58
	553.35	548.14	601.68	540.58	612.71	642.89	635.72	624.13	631.18

Options on Variance

	HSV	SVJ	VGSA	LV	LL	Sato Processes Y=.25	Y=.5	Y=.75	MXNR
	11.8791	12.7620	8.9487	11.5133	14.1940	13.1588	13.3587	12.9085	13.3285
	12.0206	13.0562	7.7240	11.7427	13.5320	13.5212	13.8933	13.5326	13.9245
10	12.1142	13.2807	7.1442	12.2678	13.0491	13.7560	13.8767	13.9932	14.3499
	12.1654	13.3500	6.7817	12.6359	12.9323	13.9556	14.1271	14.2751	14.6620
	12.1563	13.3747	6.4981	12.8979	12.7962	14.0972	14.0347	14.4664	14.5286
	8.1392	9.3108	6.0495	6.8958	12.2783	11.1961	11.4310	10.8681	11.2451
	7.7291	9.1799	4.4159	7.2976	11.2930	11.5657	11.9597	11.5111	11.8985
15	7.4704	9.0933	3.5502	8.0429	10.5452	11.7767	11.8967	12.0204	12.3530
	7.2324	8.9292	2.9470	8.5832	10.2718	11.9774	12.1616	12.3059	12.6815
	6.9869	8.7466	2.4973	8.8597	9.9986	12.1395	12.0530	12.4923	12.5132
	4.8132	6.1330	3.6499	3.3338	10.5534	9.3995	9.7202	9.0711	9.3441
	4.0476	5.5808	2.0494	3.9071	9.2655	9.7601	10.2006	9.7872	10.0478
20	3.4764	5.1572	1.3040	5.0538	8.3185	9.9415	10.0947	10.3148	10.5328
	2.9698	4.6919	0.8544	5.8544	7.8956	10.1316	10.3980	10.5869	10.8478
	2.5095	4.2866	0.5553	6.2488	7.4968	10.3077	10.2756	10.7604	10.6199
	2.5277	3.6707	1.9767	1.2372	9.0655	7.8797	8.2572	7.5724	7.6912
	1.7801	3.0391	0.8328	2.0268	7.5714	8.1646	8.6804	8.3398	8.4910
25	1.2416	2.4830	0.3096	3.6436	6.4692	8.3364	8.5130	8.8676	8.9477
	0.8855	2.0342	0.1855	4.6138	5.9115	8.5224	8.8855	9.1145	9.2502
	0.6122	1.6399	0.0687	5.0554	5.4720	8.6939	8.7138	9.2896	8.9534
	1.1798	1.9485	1.0322	0.3217	7.8162	6.6335	6.9531	6.3033	6.3069
	0.6276	1.4898	0.2446	1.1083	6.1761	6.7945	7.3828	7.1109	7.1961
30	0.3646	0.9440	0.0000	2.9246	4.9938	6.9532	7.1784	7.6353	7.6253
	0.2147	0.5885	0.0000	3.9018	4.3778	7.1192	7.5799	7.8702	7.9161
	0.0982	0.5040	0.0000	4.3325	3.9207	7.3230	7.3614	8.0550	7.5051
	0.4894	0.9056	0.5195	0.0000	6.7858	5.5801	5.8307	5.2559	5.2037
	0.2666	0.5917	0.0000	0.5220	5.0295	5.6411	6.2995	6.0708	6.1135
35	0.0000	0.1407	0.0000	2.3457	3.7904	5.7883	6.0046	6.6057	6.5177
	0.0000	0.1235	0.0000	3.3295	3.1796	5.9442	6.4886	6.8007	6.7560
	0.0000	0.1093	0.0000	3.7683	2.7163	6.2026	6.2109	7.0003	6.2191

Options on Volatility

Strike	HSV	SVJ	VGSA	LV	LL	Sato Processes Y=.25	Y=.5	Y=.75	MXNR
	4.8266	5.3324	2.7537	4.8385	4.9675	4.4794	4.5474	4.4019	4.7096
	5.1094	5.7390	2.2129	4.9472	4.9559	4.7065	4.8432	4.6752	4.9447
10	5.2857	6.0455	1.9711	5.1765	4.8958	4.8567	4.9039	4.8789	5.1692
	5.4012	6.1938	1.8251	5.3062	4.9502	4.9719	4.9881	5.0528	5.3471
	5.4520	6.2870	1.7101	5.4434	4.9571	5.0258	4.9794	5.1583	5.3590
	1.7579	2.2164	0.9655	1.3239	2.8967	2.5301	2.5949	2.4169	2.6201
	1.6378	2.2174	0.5465	1.4473	2.6859	2.7050	2.8023	2.6046	2.8101
15	1.5643	2.2242	0.3646	1.6319	2.4817	2.7961	2.8206	2.7795	2.9930
	1.4906	2.1829	0.2563	1.7545	2.4266	2.8812	2.8782	2.9121	3.1386
	1.4121	2.1237	0.1871	1.8189	2.3504	2.9324	2.8683	2.9822	3.1371
	0.4964	0.7834	0.2830	0.2486	1.7557	1.4575	1.5462	1.3776	1.4859
	0.3605	0.6626	0.0935	0.3274	1.4770	1.5900	1.6733	1.5396	1.6331
20	0.2713	0.5785	0.0391	0.4789	1.2614	1.6416	1.6727	1.6751	1.7862
	0.2007	0.4855	0.0169	0.5946	1.1717	1.6992	1.7256	1.7717	1.8892
	0.1450	0.4102	0.0073	0.6566	1.0766	1.7419	1.7187	1.8157	1.8692
	0.1144	0.2368	0.0689	0.0285	1.1002	0.8672	0.9562	0.8172	0.8535
	0.0580	0.1645	0.0128	0.0720	0.8359	0.9482	1.0291	0.9504	0.9847
25	0.0287	0.1125	0.0019	0.1990	0.6458	0.9831	1.0119	1.0521	1.0929
	0.0147	0.0765	0.0007	0.2991	0.5545	1.0254	1.0709	1.1207	1.1687
	0.0071	0.0498	0.0001	0.3500	0.4845	1.0537	1.0536	1.1537	1.1367
	0.0213	0.0575	0.0161	0.0017	0.7138	0.5356	0.5929	0.4943	0.4979
	0.0061	0.0341	0.0010	0.0186	0.4842	0.5724	0.6472	0.6026	0.6124
30	0.0021	0.0140	0.0000	0.1122	0.3352	0.5956	0.6282	0.6796	0.6910
	0.0007	0.0055	0.0000	0.1877	0.2645	0.6232	0.6772	0.7335	0.7492
	0.0002	0.0040	0.0000	0.2254	0.2163	0.6514	0.6552	0.7615	0.7001
	0.0032	0.0109	0.0036	0.0000	0.4809	0.3365	0.3708	0.3069	0.3012
	0.0010	0.0048	0.0000	0.0037	0.2851	0.3504	0.4178	0.3904	0.3893
35	0.0000	0.0003	0.0000	0.0650	0.1712	0.3656	0.3886	0.4526	0.4486
	0.0000	0.0002	0.0000	0.1236	0.1239	0.3855	0.4395	0.4907	0.4860
	0.0000	0.0002	0.0000	0.1547	0.0922	0.4166	0.4135	0.5158	0.4268

larger global cap of 1000, we see that the Sato processes are close to LL with the parametric models ($HSV, SVJ, VGSA$) and LV giving substantially lower values. The values rise with the global cap, the local payout strike and the maturity.

10.5 VarOpt and VolOpt

Options on variance are priced above options on volatility as expected. The values drop with the strike quite substantially for the parametric models HSV, SVJ, and $VGSA$. This drop is less marked for LV but is still quite substantial at the lower maturities when compared with LL. For the longer maturities LV and LL are comparable. The values rise with maturity for LV and fall for LL. The Sato processes maintain value with both strike and maturity with the values not falling that fast with strike or maturity. This is presumably a consequence of the effects of scaling as outlined in the theoretical analysis of the variance of realized quadratic variation for Sato processes presented in Section 4.

Finally we present graphs of prices for the six products using six models. The products are the locally floored and globally capped cliquet $LFGC$ with the floor at -10 and the cap at 25, the locally capped at 10 and globally floored at -25 $LCGF$ cliquet, the swing cliquet SC with a strike at 10 and cap 25, the reverse swing cliquet RSC with strike 10 and cap 500 and the variance VAR $OPTION$ and volatility options VOL $OPTIONS$ struck at 25. The six models are HSV, SVJ, $VGSA$, LV, LL, and $CGMY$ Sato with $Y = .5$. The other Sato prices were similar and we graph just one Sato price.

11 Conclusion

This paper presents the properties of Sato processes, introduced in Carr, Geman, Madan and Yor (2007), as models for structured product pricing. These are additive processes with inhomogeneous but independent increments. We observe that the forward return distributions depart from spot return distributions with falling at the money volatilities and sharper skews. To their advantage they are possibly suitable as models for options on realized variance, as the inhomogeneity induces realized variance to remain a random variable over the longer horizon, unlike the situation for Lévy processes or more generally short memory processes. In addition we develop a uniform simulation method for a wide class of Sato processes based on the construction of a Ziggurat from the base Lévy density.

With respect to pricing structured products it is shown that linear or risk neutral pricing principles are not relevant. Instead the ask and bid prices are shown to be concave and convex functionals respectively, consistent with the principles of hedging to acceptability. It is also observed that ask prices are the maximum of a range of risk neutral valuations while bid prices are the minimum of a similar range. The risk neutral valuations correspond to selected equivalent martingale measures. A natural consequence is that different products should

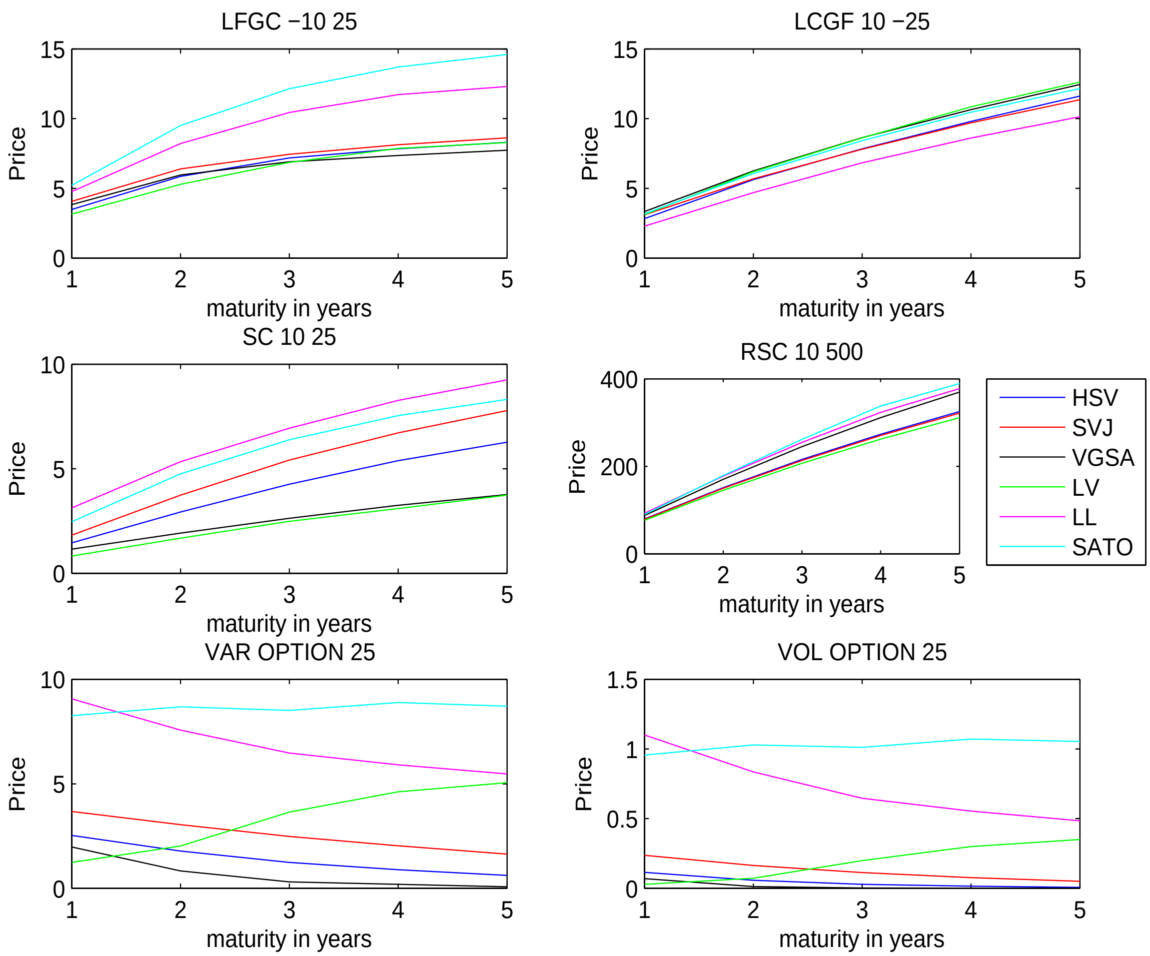
LFGC −10 25
LCGF 10 −25
SC 10 25
RSC 10 500
VAR OPTION 25
VOL OPTION 25
Price
maturity in years
HSV
SVJ
VGSA
LV
LL
SATO

Figure 8:

be priced by different risk neutral models. The resulting nonlinear pricing is also free of arbitrage provided there is agreement across market participants on the cone of acceptable risks.

Path spaces are generated for a variety of Sato processes and a number of reference models commonly used in the industry. Comparative prices are provided for a range of structured products including options on realized variance. It is observed that the Sato processes over price cliquets relative to other models. They also maintain the value of long dated options on realized variance.

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Appendix on Some Product Names and Abbreviations
AL Altiplano
AN Annapurna
BB Bull Bear
BO Best of Option
BR Bear
CA Callable
CC Capped Call
CF Click Fund
CL Credit Linked
CO Call Overwriting
CQ Cliquet
DC Downside Cliquet
DD Dividend
DI Digital
DP Dispersion
DRC Digital Reverse Convertible
ET Enhanced Tracker
EXO Exotic
FL Floater
FU Fix Upside
GRC Geared Reverse Convertible
HM Himalaya
IC Investment Certificate
KJ Kilimanjaro
KO Knock Out
LA Ladder
LB Lookback
Lev. Long w. SL Leverage Long with stop loss
Lev. Short w. SL Leverage Short with stop loss
NA Napoleon
PB Precipice Bond
PF Profiled
PI Portfolio Insurance
PO Podium
PT Protected Tracker
PU Putable
PW Power Option
RA Range
RB Rainbow
RC Reverse Convertible
RF Reverse Floater
SO Spread Option
ST Steepener
TR Target Return
UC Uncapped Call

VS Volatility Swap
WH Whale
WO Worst Of Option