Jump-type Lévy processes

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1 Probabilistic structure of Lévy processes

The assumption that observations are normally distributed is predominant in many areas of statistics. So is the situation with time series of financial data where from the very beginning of continuous-time modeling Brownian motion itself or geometric Brownian motion became the favorites. This is largely due to the fact that the normal distribution as well as the continuous-time process it generates have nice analytic properties. The standard techniques to handle these objects are known to a large community which at the same time is less familiar with more sophisticated distributions and processes. On the other hand a thorough look at data from various areas of finance such as equity, fixed income, foreign exchange or credit, clearly reveals that assuming normality one gets a model which is only a poor approximation of reality. If $(S_t)_{t\geq 0}$ denotes a price process in continuous or discrete time, the quantity to be considered are the *log returns*

$$\log S_{t+\delta} - \log S_t = \log(S_{t+\delta}/S_t). \tag{1}$$

Usually log returns are preferred to *relative price changes* $(S_{t+\delta} - S_t)/S_t$ because by adding up log returns over n periods one gets the log return for the period $n\delta$. This is not the case for relative price changes. Numerically the difference between log returns and relative price changes is negligible because $x - 1$ is well approximated by $\log x$ at $x = 1$.

Whereas log returns taken from monthly stock prices (S_t) are reasonably represented by a normal distribution, the deviation becomes significant if one considers prices on a daily or even an intraday time grid (see e.g. Eberlein and Keller [12], Eberlein and Özkan [17]). As a consequence of the high volumes traded nowadays on electronic platforms, daily price changes of several percent are rather frequent also for big companies, i.e. companies with a high market capitalization. A model based on the Gaussian distribution however would allow this order of price change only as a very rare event. Let us underline that the deviation of probabilities is not restricted to tails only but can be observed on a lower scale for small price movements as well. Empirical return distributions have substantially more mass around the origin than the normal distribution. In order to improve the statistical accuracy of the models and thus to improve derivative pricing, risk management and portfolio optimization to name just some key areas of application, many extensions of the basic models have been introduced. Let us refer to adding stochastic volatility, stochastic interest rates, correlation terms and so on. Without any doubt these extensions typically reduce the deviation between model and reality. On the other side in most cases the simple analytic properties are sacrificed and in particular the distributions which the extended models produce are no longer known explicitly.

A more fundamental change in the modeling approach is to consider from the very beginning more realistic distributions and to keep the analytic form of the model itself simple. This leads naturally to a broader class of driving processes namely Lévy processes. A *Lévy process* $X = (X_t)_{t\geq 0}$ is a process with *stationary* and *independent increments*. Underlying is a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ to which the process $(X_t)_{t\geq 0}$ is adapted. It is well-known (Protter [29, Theorem 30.]) that a Lévy process has a version with *càdlàg paths*, i.e. paths which are right-continuous and have limits from the left. In the following we shall always consider processes with càdlàg paths. A (one-dimensional) Lévy process can be represented in the following way where we assume $X_0 = 0$ for convenience

$$X_t = bt + \sqrt{c}W_t + Z_t + \sum_{s\leq t} \Delta X_s 1\!\!1_{\{|\Delta X_s|>1\}}. \tag{2}$$

Here b and $c \geq 0$ are real numbers, $(W_t)_{t\geq 0}$ is a standard Brownian motion and $(Z_t)_{t\geq 0}$ is a purely discontinuous martingale which is independent of $(W_t)_{t\geq 0}$. $\Delta X_s := X_s - X_{s-}$ denotes the jump at time s if there is any and thus the last sum represents the jumps of the process with absolute jump size bigger than 1.

In case $c = 0$, i.e. if the continuous Gaussian part disappears, the process is a *purely discontinuous Lévy process*. As we will see later many examples which are important for modeling in finance are of this type. Let us also mention that in the general case where both martingales, (W_t) and (Z_t), are present, because of their independence they are orthogonal in a Hilbert space sense. This fact simplifies the analysis considerably because the two components of the process do not interact. As a consequence the classical formulae known for diffusion processes – for example Itô's formula – are complemented by a term or terms which come from the jump part of X, but no mixed terms have to be considered.

The decomposition of a Lévy process as given in (2) is known as the *Lévy–Itô decomposition*. At the same time every Lévy process is a *semimartingale* and (2) is the so-called *canonical representation* for semimartingales. For a semimartingale $Y = (Y_t)_{t\geq 0}$ the latter is obtained by the following procedure. One first substracts from Y the sum of the big jumps, e.g. the jumps with absolute jump size bigger than 1. The remaining process

$$Y_t - \sum_{s\le t} \Delta Y_s \mathbb{1}_{\{|\Delta Y_s|>1\}} \tag{3}$$

has bounded jumps and therefore is a *special semimartingale* (see Jacod and Shiryaev [24, I.4.21 and I.4.24]). Any special semimartingale admits a unique decomposition into a local martingale $M = (M_t)_{t\ge 0}$ and a predictable process with finite variation $V = (V_t)_{t\ge 0}$, i.e. the paths of V have finite variation over each finite interval $[0,t]$. For Lévy processes the finite variation component turns out to be the (deterministic) linear function bt. Any local martingale M (with $M_0 = 0$) admits a unique decomposition (see Jacod and Shiryaev [24, I.4.18]) $M = M^c + M^d$ where M^c is a local martingale with continuous paths and M^d is a purely discontinuous local martingale which we denoted Z in (2). For Lévy processes the continuous component M^c is a standard Brownian motion $W = (W_t)_{t\ge 0}$ scaled with a constant factor $\sqrt{c}$.

What we see so far is that a Lévy process has two simple components, a linear function and a Brownian motion. Now let us look more carefully into the *jump part*. Because we assumed càdlàg paths, over finite intervals $[0,t]$ any path has only a finite number of jumps with absolute jump size larger than ε for any $\varepsilon > 0$. As a consequence the sum of jumps along $[0,t]$ with absolute jump size bigger than 1 is a finite sum for each path.

Of course instead of the threshold 1 one could use any number $\varepsilon > 0$ here. Contrary to the sum of the big jumps the sum of the small jumps

$$\sum_{s\le t} \Delta X_s \mathbb{1}_{\{|\Delta X_s|\le 1\}} \tag{4}$$

does not converge in general. There are too many small jumps to get convergence. One can force this sum to converge by *compensating* it, i.e. by subtracting the corresponding average increase of the process along $[0,t]$. The average can be expressed by the intensity $F(dx)$ with which the jumps arrive. More precisely the following limit exists in the sense of convergence in probability

$$\lim_{\varepsilon\to 0} \Big(\sum_{s\le t} \Delta X_s \mathbb{1}_{\{\varepsilon\le|\Delta X_s|\le 1\}} - t\int x \mathbb{1}_{\{\varepsilon\le|x|\le 1\}} F(dx) \Big). \tag{5}$$

Note that the first sum represents the (finitely many) jumps of absolute jump size between ε and 1. The integral is the average increase of the process in a unit interval when jumps with absolute size smaller than ε or larger than 1 are eliminated. One cannot separate this difference, because in general non of the two expressions has a finite limit as $\varepsilon \to 0$.

There is a more elegant way to express (5). For this one introduces the *random measure of jumps* of the process X denoted by μ^X,

$$\mu^X(\omega; dt, dx) = \sum_{s>0} \mathbb{1}_{\{\Delta X_s \ne 0\}} \varepsilon_{(s,\Delta X_s(\omega))}(dt, dx). \tag{6}$$

If a path of the process given by ω has a jump of size $\Delta X_s(\omega) = x$ at time point s, then the random measure $\mu^X(\omega; \cdot, \cdot)$ places a unit mass $\varepsilon_{(s,x)}$ at the point (s,x) in $\mathbb{R}_+ \times \mathbb{R}$. Consequently for a time interval $[0,t]$ and a set $A \subset \mathbb{R}$, $\mu^X(\omega; [0,t] \times A)$ counts how many jumps of jump size within A occur for this particular path ω from time 0 to t,

$$\mu^X(\omega; [0,t] \times A) = |\{(s,x) \in [0,t] \times A \mid \Delta X_s(\omega) = x\}|. \tag{7}$$

This number is compared to the average number of jumps with size within A. The latter can be expressed by an intensity measure $F(A)$,

$$E\big[\mu^X(\,\cdot\,; [0,t] \times A)\big] = tF(A). \tag{8}$$

With this notation one can write the sum of the big jumps at the end of (2) in the form

$$\int_0^t \int_{\mathbb{R}} x 1\!\!1_{\{|x|>1\}} \mu^X(ds, dx) \tag{9}$$

and one can express (Z_t), the martingale of compensated jumps of absolute size less than 1 in the form

$$\int_0^t \int_{\mathbb{R}} x 1\!\!1_{\{|x|\leq 1\}} (\mu^X(ds, dx) - dsF(dx)). \tag{10}$$

Note that $\mu^X(\omega, ds, dx)$ is a random measure, i.e. it depends on ω, whereas $dsF(dx)$ is a product measure on $\mathbb{R}_+ \times \mathbb{R}$ not depending on ω. Again μ^X and $dsF(dx)$ cannot be separated in general.

2 Distributional description of Lévy processes

The distribution of a Lévy process $X = (X_t)_{t>0}$ is completely determined by any of its marginal distributions $\mathcal{L}(X_t)$. Let us consider $\mathcal{L}(X_1)$ and write for any natural number n

$$X_1 = X_{1/n} + (X_{2/n} - X_{1/n}) + (X_{3/n} - X_{2/n}) + \cdots + X_{n/n} + X_{n-1/n}. \tag{11}$$

By stationarity and independence of the increments we see that $\mathcal{L}(X_1)$ is the n-fold convolution of the laws $\mathcal{L}(X_{1/n})$,

$$\mathcal{L}(X_1) = \mathcal{L}(X_{1/n}) * \cdots * \mathcal{L}(X_{1/n}). \tag{12}$$

Consequently $\mathcal{L}(X_1)$ and analogously any $\mathcal{L}(X_t)$ is an infinitely divisible distribution. Conversely any infinitely divisible distribution ν generates in a natural way a Lévy process $X = (X_t)_{t\geq 0}$ which is uniquely determined by setting $\mathcal{L}(X_1) = \nu$. If for $n > 0$, ν_n is the probability measure such that $\nu = \nu_n * \cdots * \nu_n$, then one gets immediately for rational time points $t = k/n$, $\mathcal{L}(X_t)$ as the k-fold convolution of ν_n. For irrational time points t, $\mathcal{L}(X_t)$ is determined by a continuity argument (see Breiman [8, chap. 14.4]). Because the process to be constructed has independent increments, it is sufficient to know the one-dimensional distributions.

As we have seen the class of infinitely divisible distributions and the class of Lévy processes are in a one-to-one relationship. Therefore if a specific infinitely divisible distribution is characterized by a few parameters the same holds for the corresponding Lévy process. This fact is crucial for the estimation of parameters in financial models which are driven by Lévy processes. The classical example is Brownian motion which is characterized by the parameters μ and σ^2 of the normal distribution $N(\mu, \sigma^2)$. A number of examples which allow more realistic modeling in finance will be considered in the last section.

For an infinitely divisible distribution ν which we can write as $\nu = \mathcal{L}(X_1)$ for a Lévy process $X = (X_t)_{t \geq 0}$, the *Fourier transform* in its *Lévy–Khintchine form* is given by

$$E[\exp(\mathrm{i}uX_1)] = \exp\left[\mathrm{i}ub - \frac{1}{2}u^2 c + \int_{\mathbb{R}} \left(e^{\mathrm{i}ux} - 1 - \mathrm{i}ux 1\!\!1_{\{|x| \leq 1\}}\right) F(dx)\right]. \quad (13)$$

The three quantities (b, c, F) are those which appeared in (2) and (8) resp. (10) already. They determine the law of X_1, $\mathcal{L}(X_1)$, and thus the process $X = (X_t)_{t \geq 0}$ itself completely. (b, c, F) is called *Lévy–Khintchine triplet* or in semimartingale terminology the *triplet of local characteristics*. The truncation function $h(x) = x 1\!\!1_{\{|x| \leq 1\}}$ used in (13) could be replaced by other versions of truncation functions, e.g. smooth functions which are identical to the identity in a neighbourhood of the origin and go to 0 outside of this neighbourhood. Changing h results in a different *drift parameter* b, whereas the *diffusion coefficient* $c \geq 0$ and the *Lévy measure* F remain unaffected. We note that F does not have mass on 0, $F(\{0\}) = 0$, and satisfies the following integrability condition

$$\int_{\mathbb{R}} \min(1, x^2) F(dx) < \infty. \quad (14)$$

Conversely any measure on the real line with these two properties together with parameters $b \in \mathbb{R}$ and $c \geq 0$ defines via (13) an infinitely divisible distribution and thus a Lévy process. Let us write (13) in the short form

$$E[\exp(\mathrm{i}uX_1)] = \exp(\psi(u)). \quad (15)$$

ψ is called the *characteristic exponent*. Again by independence and stationarity of the increments of the process (see (11) and (12)) one derives that the characteristic function of $\mathcal{L}(X_t)$ is the t-th power of the characteristic function of $\mathcal{L}(X_t)$,

$$E[\exp(\mathrm{i}uX_t)] = \exp(t\psi(u)). \quad (16)$$

This property is useful when one has to compute numerically values of derivatives which are represented as expectations of the form $E[f(X_T)]$ where X_T is the value of a Lévy process at maturity T, and the parameters of the Lévy process were estimated as the parameters of $\mathcal{L}(X_1)$.

A lot of information on the process can be derived from *integrability properties* of the Lévy measure F. The following Proposition shows that finiteness of moments of the process depends only on the frequency of large jumps since it is related to integration by F over $\{|x| > 1\}$.

Proposition 1. *Let $X = (X_t)_{t\geq 0}$ be a Lévy process with Lévy measure F.*

(a) *X_t has finite p-th moment for $p \in \mathbb{R}_+$, i.e. $E[|X_t|^p] < \infty$, if and only if $\int_{\{|x|>1\}} |x|^p F(dx) < \infty$.*

(b) *X_t has finite p-th exponential moment for $p \in \mathbb{R}$, i.e. $E[\exp(pX_t)] < \infty$, if and only if $\int_{\{|x|>1\}} \exp(px) F(dx) < \infty$.*

For the proof see Sato [34, Theorem 25.3]. From part (a) we see that if the generating distribution $\mathcal{L}(X_1)$ has *finite expectation* then $\int_{\{|x|>1\}} xF(dx) < \infty$. This means that we can add $-\int \mathrm{i}ux 1\!\!1_{\{|x|>1\}} F(dx)$ to the integral in (13) and get the simpler representation for the Fourier transform,

$$E[\exp(\mathrm{i}uX_1)] = \exp\left[\mathrm{i}ub - \frac{1}{2}u^2 c + \int_{\mathbb{R}} \left(e^{\mathrm{i}ux} - 1 - \mathrm{i}ux\right) F(dx)\right]. \tag{17}$$

In the same way in this case where the expectation of $\mathcal{L}(X_1)$ is finite and thus $\int_0^t \int_{\mathbb{R}} x 1\!\!1_{\{|x|>1\}} ds F(dx)$ exists, we can add

$$\int_0^t \int_{\mathbb{R}} x 1\!\!1_{\{|x|>1\}} \left(\mu^X(ds, dx) - dsF(dx)\right) \tag{18}$$

to (10). Note that the sum of the big jumps which is $\int_0^t \int_{\mathbb{R}} x 1\!\!1_{\{|x|>1\}} \mu^X(ds, dx)$ always exists for every path. As a result of this we get (2) in the simpler representation

$$X_t = bt + \sqrt{c}W_t + \int_0^t \int_{\mathbb{R}} x\left(\mu^X(ds, dx) - dsF(dx)\right). \tag{19}$$

Of course the drift coefficient b in (19) is different from the drift coefficient b in the general representation (2). Actually b in (19) is nothing but the expectation $E[X_1]$ because the Brownian motion $(W_t)_{t\geq 0}$ and the pure jump integral process are both martingales with expectation zero and $E[X_t] = tE[X_1]$. From representation (19) it is immediately clear that (X_t) is a martingale if $b = E[X_1] = 0$, it is a submartingale if $b > 0$ and a supermartingale if $b < 0$.

The case of finite expectation $E[X_1]$ under which we get (19) is of particular interest because all Lévy processes which we use in finance have finite first moments.

Whereas the frequency of the big jumps determines existence of moments of the process, the *fine structure* of the paths of the process can be read off the frequency of the small jumps. We say the process has *finite activity* if almost all paths have only a finite number of jumps along any time interval of finite length. In case almost all paths have infinitely many jumps along any time interval of finite length, we say the process has *infinite activity*.

Proposition 2. *Let $X = (X_t)_{t\geq 0}$ be a Lévy process with Lévy measure F.*

(a) *X has finite activity if $F(\mathbb{R}) < \infty$.*
(b) *X has infinite activity if $F(\mathbb{R}) = \infty$.*

Note that by definition a Lévy measure satisfies $\int_{\mathbb{R}} 1\!\!1_{\{|x|>1\}} F(dx) < \infty$. Therefore the assumption $F(\mathbb{R}) < \infty$ or $F(\mathbb{R}) = \infty$ is equivalent to assume finiteness or infiniteness of $\int_{\mathbb{R}} 1\!\!1_{\{|x|\leq 1\}} F(dx) < \infty$. It is well-known that the paths of Brownian motion have infinite variation. Consequently it follows from representation (2) or (19) that Lévy processes a priori have infinite variation paths as soon as $c > 0$. If the purely discontinuous component (Z_t) in (2) or the purely discontinuous integral process in (19) produce paths with finite or infinite variation again depends on the frequency of the small jumps.

Proposition 3. *Let $X = (X_t)_{t\geq 0}$ be a Lévy process with triplet (b, c, F).*

(a) *Almost all paths of X have finite variation if $c = 0$ and $\int_{\{|x|\leq 1\}} |x| F(dx) < \infty$.*
(b) *Almost all paths of X have infinite variation if $c \neq 0$ or $\int_{\{|x|\leq 1\}} |x| F(dx) = \infty$.*

For the proof see Sato [34, Theorem 21.9]. The integrability of F in the sense that $\int_{\{|x|\leq 1\}} |x| F(dx) < \infty$ guarantees also that the sum of the small jumps as given in (4) converges for (almost) every path. Therefore in this case one can separate the integral in (10) or (19) and write e.g. in (19)

$$\int_0^t \int_{\mathbb{R}} x\Big(\mu^X(ds,dx) - dsF(dx)\Big) = \int_0^t \int_{\mathbb{R}} x\mu^X(ds,dx) - t\int_{\mathbb{R}} xF(dx). \tag{20}$$

3 Financial modeling

The classical model in finance for *stock prices* or *indices* which goes back to Samuelson [33] and which became the basis for the Black–Scholes option pricing theory is the *geometric Brownian motion* given by the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \tag{21}$$

This equation is solved by

$$S_t = S_0 \exp\big(\sigma W_t + (\mu - \sigma^2/2)t\big). \tag{22}$$

The exponent of this price process is a Lévy process as given in (2) with $b = \mu - \sigma^2/2$, $\sqrt{c} = \sigma$, $Z_t \equiv 0$ and no big jumps either. Log returns $\log S_{t+1} - \log S_t$ produced by this process along a time grid with span 1 are normally distributed variables $N(\mu - \sigma^2/2, \sigma^2)$ which are far from being realistic for most time series of financial data. Once one has identified a more realistic parametric distribution ν by fitting an empirical return distribution – several classes of candidate distributions will be presented in section 4 – the right starting point for a statistically more accurate model for prices is equation (22). Considering the Lévy process $X = (X_t)_{t\geq 0}$ such that $\mathcal{L}(X_1) = \nu$, the model

$$S_t = S_0 \exp(X_t) \tag{23}$$

which we call *exponential Lévy model*, produces along a time grid with span 1 log returns which are exactly equal to ν. This way one can implement an empirically derived (infinitely divisible) distribution into a model with exact returns. If one would start instead with a stochastic differential equation, i.e. with the equivalent of (21), which is the equation

$$dS_t = S_{t-} dX_t, \tag{24}$$

one gets as solution the stochastic exponential

$$S_t = S_0 \exp(X_t - ct/2) \prod_{s\leq t} (1 + \Delta X_s) \exp(-\Delta X_s). \tag{25}$$

The distribution of the log returns of this process is not known in general. As one can see directly from the term $(1 + \Delta X_s)$ in (25), another drawback of this model is that it can produce negative stock prices as soon as the driving Lévy process X has negative jumps with absolute jump size larger than 1. The Lévy measures of all interesting classes of distributions which we shall consider in the next section have strictly positive densities on the whole negative half line and therefore the Lévy processes generated by these distributions have negative jumps of arbitrary size. For completeness we mention that the stochastic differential equation which describes the process (23) is

$$dS_t = S_{t-}\big(dX_t + (c/2)dt + e^{\Delta X_t} - 1 - \Delta X_t\big). \tag{26}$$

For the pricing of derivatives it is interesting to characterize when the price process given by (23) is a *martingale* because pricing is done by taking expectations under a risk neutral or *martingale measure*. For $(S_t)_{t\geq 0}$ to be a martingale, first of all the expectation $E[S_t]$ has to be finite. Therefore candidates for the role of the driving process are Lévy processes X which have a finite first exponential moment

$$E[\exp(X_t)] < \infty. \tag{27}$$

Proposition 1 characterizes these processes in terms of their Lévy measure. At this point one should mention that the necessary assumption (27) a priori excludes stable processes as suitable driving processes for models in finance. Second let X be given in the representation (19) (still assuming (27)) then $S_t = S_0 \exp(X_t)$ is a martingale if

$$b = -\frac{c}{2} - \int_{\mathbb{R}} \big(e^x - 1 - x\big) F(dx). \tag{28}$$

This can be seen by applying Itô's formula to $S_t = S_0 \exp(X_t)$, where (28) guarantees that the drift component is 0. An alternative way to derive (28) is to verify that the process $(M_t)_{t\geq 0}$ given by

$$M_t = \frac{\exp(X_t)}{E[\exp(X_t)]} \tag{29}$$

is a martingale. Stationarity and independence of increments have to be used here. (28) follows once one verifies (see (15)–(17)) that

$$E[\exp(X_t)] = \exp\left[t\left(b + \frac{c}{2} + \int_{\mathbb{R}} (e^x - 1 - x)F(dx)\right)\right]. \tag{30}$$

The simplest models for fixed income markets take the short rate r_t as the basic quantity and derive all other rates from r_t. More sophisticated approaches model simultaneously the whole term structure of rates for a continuum of maturities $[0, T^*]$ or as in the case of the LIBOR model the rates corresponding to the maturities of a tenor structure $T_0 < T_1 < \cdots < T_N = T^*$. As a consequence these models are mathematically more demanding. A survey of interest rate modeling in the classic setting of diffusions is given by Björk [6] in this volume. The interest rate theory for models driven by Lévy processes has been developed in a series of papers by the author with varying coauthors (Eberlein and Kluge [13, 14], Eberlein and Özkan [16, 18], Eberlein and Raible [20], Eberlein, Jacod, and Raible [22]).

Two basic approaches are the forward rate approach and the LIBOR approach. In the first case one assumes the dynamics of the *instantaneous forward rate* with maturity T, contracted at time t, $f(t, T)$ in the form

$$f(t, T) = f(0, T) + \int_0^t \alpha(s, T)ds - \int_0^t \sigma(s, T)dX_s \tag{31}$$

for any $T \in [0, T^*]$. The coefficients $\alpha(s, T)$ and $\sigma(s, T)$ can be deterministic or random. Starting with (31) one gets zero-coupon bond prices $B(t, T)$ in a form comparable to the stock price model (23), namely

$$B(t, T) = B(0, T) \exp\left(\int_0^t (r(s) - A(s, T))ds + \int_0^t \varSigma(s, T)dX_s\right). \tag{32}$$

Here $r_s = r(s) = f(s, s)$ is the short rate and $A(s, T)$ resp. $\varSigma(s, T)$ are derived from $\alpha(s, T)$ resp. $\sigma(s, T)$ by integration.

In the *Lévy LIBOR market model* (Eberlein and Özkan [18]) the forward LIBOR rates $L(t, T_j)$ for the time points $T_j (0 \leq j \leq N)$ of a tenor structure are chosen as the basic rates. As a result of a backward induction one gets for each j the rate in the following uniform form

$$L(t, T_j) = L(0, T_j) \exp\left(\int_0^t \lambda(s, T_j)dX_s^{T_{j+1}}\right) \tag{33}$$

where $\lambda(s, T_j)$ is a volatility structure, $X^{T_{j+1}} = (X_t^{T_{j+1}})_{t \geq 0}$ is a process derived from an initial (time-homogeneous or time-inhomogeneous) Lévy process $X^{T_N} = (X_t^{T_N})_{t \geq 0}$ and equation (33) is considered under $\mathbb{P}^{T_{j+1}}$, the forward martingale measure which is derived during the backward induction. Closely related to the LIBOR model is the *forward process model* where forward processes $F(t, T_j, T_{j+1}) = B(t, T_j)/B(t, T_{j+1})$ are chosen as the basic quantities and modeled in a form analogous to (33). An extension of the Lévy LIBOR approach to a multicurrency setting taking exchange rates into account has been developed in Eberlein and Koval [15]. In all implementations of these models pure jump processes have been chosen as driving processes.

4 Examples of Lévy processes with jumps

4.1 Poisson and compound Poisson processes

The simplest Lévy measure one can consider is ε_1, a point mass in 1. Adding an intensity parameter $\lambda > 0$ one gets $F = \lambda \varepsilon_1$. Assuming $c = 0$ this Lévy measure generates a process $X = (X_t)_{t \geq 0}$ with jumps of size 1 which occur with an average rate of λ in a unit time interval. Otherwise the paths are constant. X is called a *Poisson process* with intensity λ. The drift parameter b in (17) is $E[X_1]$ which is λ. Therefore the Fourier transform takes the form

$$E[\exp(\mathrm{i}uX_t)] = \exp\left[\lambda t(e^{\mathrm{i}u} - 1)\right]. \tag{34}$$

Any variable X_t of the process has a Poisson distribution with parameter λt, i.e.

$$P[X_t = k] = \exp(-\lambda t)\frac{(\lambda t)^k}{k!}.$$

One can show that the successive waiting times from one jump to the next are independent exponentially distributed random variables with parameter λ. Conversely, starting with a sequence $(\tau_i)_{i \geq 1}$ of independent exponentially distributed random variables with parameter λ and setting $T_n = \sum_{i=1}^n \tau_i$, the associated *counting process*

$$N_t = \sum_{n \geq 1} 1\!\!1_{\{T_n \leq t\}} \tag{35}$$

is a Poisson process with intensity λ.

A natural extension of the Poisson process with jump height 1 is a process where the jump size is random. Let $(Y_i)_{i \leq 1}$ be a sequence of independent, identically distributed random variables with $\mathcal{L}(Y_1) = \nu$.

$$X_t = \sum_{i=1}^{N_t} Y_i, \tag{36}$$

where $(N_t)_{t\geq 0}$ is a Poisson process with intensity $\lambda > 0$ which is independent of $(Y_i)_{i\geq 1}$, defines a *compound Poisson process* $X = (X_t)_{t\geq 0}$ with intensity λ and jump size distribution ν. Its Fourier transform is given by

$$E[\exp(\mathrm{i}uX_t)] = \exp\left[\lambda t \int_{\mathbb{R}} \big(e^{\mathrm{i}ux} - 1\big)\nu(dx)\right]. \tag{37}$$

Consequently the Lévy measure is given by $F(A) = \lambda\nu(A)$ for measurable sets A in $\mathbb{R}$.

4.2 Lévy jump diffusion

A Lévy jump diffusion is a Lévy process where the jump component is given by a compound Poisson process. It can be represented in the form

$$X_t = bt + \sqrt{c}W_t + \sum_{i=1}^{N_t} Y_i \tag{38}$$

where $b \in \mathbb{R}$, $c > 0$, $(W_t)_{t\geq 0}$ is a standard Brownian motion, $(N_t)_{t\geq 0}$ is a Poisson process with intensity $\lambda > 0$ and $(Y_i)_{i\geq 1})$ is a sequence of independent, identically distributed random variables which are independent of $(N_t)_{t\geq 0}$. For normally distributed random variables Y_i, Merton [28] introduced the process (38) as a model for asset returns. Kou [25] used double-exponentially distributed jump size variables Y_i. In principle any other distribution could be considered as well, but of course the question is if one can control explicitly the quantities one is interested in as for example $\mathcal{L}(X_t)$.

4.3 Hyperbolic Lévy processes

Hyperbolic distributions which generate *hyperbolic Lévy processes* $X = (X_t)_{t\geq 0}$ – also called *hyperbolic Lévy motions* – constitute a four parameter class of distributions. Their Lebesgue density is given by

$$d_H(x) = \frac{\sqrt{\alpha^2 - \beta^2}}{2\alpha\delta K_1(\delta\sqrt{\alpha^2 - \beta^2})} \exp\big(-\alpha\sqrt{\delta^2 + (x-\mu)^2} + \beta(x-\mu)\big). \tag{39}$$

Here K_ν denotes the modified Bessel function of the third kind with index ν. The four parameters of this distribution have the following meaning: $\alpha > 0$ determines the shape, β with $0 \leq |\beta| < \alpha$ the skewness, $\mu \in \mathbb{R}$ the location and $\delta > 0$ is a scaling parameter comparable to σ in the normal distribution. Taking the logarithm of d_H one gets a hyperbola. This explains the name hyperbolic distribution. Based on an extensive empirical study of stock prices hyperbolic Lévy processes were first used in finance in [12].

The Fourier transform ϕ_H of a hyperbolic distribution can be easily derived because of the exponential form of d_H in (39).

$$\phi_H(u) = \exp(\mathrm{i}u\mu)\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+\mathrm{i}u)^2}\right)^{1/2}\frac{K_1(\delta\sqrt{\alpha^2-(\beta+\mathrm{i}u)^2})}{K_1(\delta\sqrt{\alpha^2-\beta^2})}. \tag{40}$$

Moments of all orders exist. In particular the expectation is given by

$$E[X_1] = \mu + \frac{\beta\delta}{\sqrt{\alpha^2-\beta^2}}\frac{K_2(\delta\sqrt{\alpha^2-\beta^2})}{K_1(\delta\sqrt{\alpha^2-\beta^2})}. \tag{41}$$

Analyzing ϕ_H in the form (17) one sees that $c = 0$. This means that hyperbolic Lévy motions are *purely discontinuous* processes. The Lévy measure F has an explicit *Lebesgue density* (see (46)).

4.4 Generalized hyperbolic Lévy processes

Hyperbolic distributions are a subclass of a more powerful five parameter class, the *generalized hyperbolic distributions* (Barndorff-Nielsen [2]). The additional class parameter $\lambda \in \mathbb{R}$ has the value 1 for hyperbolic distributions. The Lebesgue density for these distributions with parameters $\lambda, \alpha, \beta, \delta, \mu$ is

$$\begin{aligned} d_{GH}(x) = a(\lambda,\alpha,\beta,\delta)\big(\delta^2+(x-\mu)^2\big)^{(\lambda-\frac{1}{2})/2} \\ + K_{\lambda-\frac{1}{2}}\Big(\alpha\sqrt{\delta^2+(x-\mu)^2}\Big)\exp(\beta(x-\mu)) \end{aligned} \tag{42}$$

where the normalizing constant is given by

$$a(\lambda,\alpha,\beta,\delta) = \frac{(\alpha^2-\beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}\delta^\lambda K_\lambda(\delta\sqrt{\alpha^2-\beta^2})}.$$

Other parametrizations are in use as well. Generalized hyperbolic distributions can be represented as normal mean-variance mixtures

$$d_{GH}(x;\lambda,\alpha,\beta,\delta,\mu) = \int_0^\infty d_{N(\mu+\beta y,y)}(x)d_{GIG}(y;\lambda,\delta,\sqrt{\alpha^2-\beta^2})dy \tag{43}$$

where the mixing distribution is *generalized inverse Gaussian* with density

$$d_{GIG}(x;\lambda,\delta,\gamma) = \Big(\frac{\gamma}{\delta}\Big)^\lambda\frac{1}{2K_\lambda(\delta\gamma)}x^{\lambda-1}\exp\Big(-\frac{1}{2}\Big(\frac{\delta^2}{x}+\gamma^2x\Big)\Big) \quad (x>0). \tag{44}$$

The exponential Lévy model with generalized hyperbolic Lévy motions as driving processes was introduced in [11] and [19].

The *moment generating function* $M_{GH}(u)$ exists for u with $|\beta+u| < \alpha$ and is given by

$$M_{GH}(u) = \exp(\mu u)\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+u)^2}\right)^{\lambda/2}\frac{K_\lambda(\delta\sqrt{\alpha^2-(\beta+u)^2})}{K_\lambda(\delta\sqrt{\alpha^2-\beta^2})}. \tag{45}$$

As a consequence exponential moments (27) are finite. This fact is crucial for pricing of derivatives under martingale measures. The Fourier transform ϕ_{GH} is obtained from the relation $\phi_{GH}(u) = M_{GH}(\mathrm{i}u)$. Analyzing ϕ_{GH} in its form (17) we see again that $c = 0$, i.e. generalized hyperbolic Lévy motions are purely discontinuous processes. The Lévy measure F has a density given by

$$g_{GH}(x) = \frac{e^{\beta x}}{|x|}\left(\int_0^\infty \frac{\exp\big(-\sqrt{2y+\alpha^2}|x|\big)}{\pi^2 y\big(J^2_{|\lambda|}(\delta\sqrt{2y}) + Y^2_{|\lambda|}(\delta\sqrt{2y})\big)}dy + 1\!\!1_{\{\lambda\geq 0\}}\lambda e^{-\alpha|x|}\right). \tag{46}$$

Setting $\lambda = -\frac{1}{2}$ in (42) we get another interesting subclass, the *normal inverse Gaussian (NIG)* distributions, which were first used in finance in [3]. Their Fourier transform is particularly simple since the Bessel function satisfies $K_{-1/2}(z) = K_{1/2}(z) = \sqrt{\pi/(2z)}e^{-z}$. Therefore

$$\phi_{NIG}(u) = \exp(\mathrm{i}u\mu)\exp\big(\delta\sqrt{\alpha^2-\beta^2}\big)\exp\big(-\delta\sqrt{\alpha^2-(\beta+\mathrm{i}u)^2}\big). \tag{47}$$

From the form (47) one sees immediately that NIG-distributions are closed under convolution in the two parameters δ and μ, because taking a power t in (47) one gets the same form with parameters $t\delta$ and $t\mu$.

4.5 CGMY and Variance Gamma Lévy processes

Carr, Geman, Madan, and Yor [9] introduced a class of infinitely divisible distributions – called CGMY – which extends the Variance Gamma (V.G.) model due to Madan and Seneta [27] and Madan and Milne [26]. *CGMY Lévy processes* have purely discontinuous paths and the Lévy density is given by

$$g_{CGMY}(x) = \begin{cases} C\dfrac{\exp(-G|x|)}{|x|^{1+Y}} & x<0,\\[2ex] C\dfrac{\exp(-Mx)}{x^{1+Y}} & x>0.\end{cases} \tag{48}$$

The parameter space is C, G, $M > 0$ and $Y \in (-\infty, 2)$. The process has infinite activity iff $Y \in [0, 2)$ and the paths have infinite variation iff $Y \in [1, 2)$. For $Y = 0$ one gets the three parameter *Variance Gamma distributions*. The latter are also a subclass of the generalized hyperbolic distributions (see Eberlein and von Hammerstein [21] or Raible [31]). For $Y < 0$ the Fourier transform of CGMY is given by

$$\phi_{CGMY}(u) = \exp\Big(C\Gamma(-Y)\big[(M-\mathrm{i}u)^Y - M^Y + (G+\mathrm{i}u)^Y - G^Y\big]\Big). \tag{49}$$

4.6 α-stable Lévy processes

Stable distributions are a classical subject in probability. They constitute a four parameter class of distributions with Fourier transform given by

$$\phi_{stab}(x) = \exp\left[\sigma^\alpha(-|\theta|^\alpha) + \mathrm{i}\theta\omega(\theta,\alpha,\beta)) + \mathrm{i}\mu\theta\right]$$

where

$$\omega(\theta,\alpha,\beta) = \begin{cases} \beta|\theta|^{\alpha-1}\tan\frac{\pi\alpha}{2} & \text{if } \alpha \neq 1, \\ -\beta\frac{2}{\pi}\ln|\theta| & \text{if } \alpha = 1. \end{cases} \tag{50}$$

The parameter space is $0 < \alpha \leq 2$, $\sigma \geq 0$, $-1 \leq \beta \leq 1$ and $\mu \in \mathbb{R}$. For $\alpha = 2$ one gets the Gaussian distribution with mean μ and variance $2\sigma^2$. For $\alpha < 2$ there is no Gaussian part which means the paths of an α-stable Lévy motion are purely discontinuous in this case.

Explicit densities are known in three cases only: the Gaussian distribution ($\alpha = 2$, $\beta = 0$), the Cauchy distribution ($\alpha = 1$, $\beta = 0$), and the Lévy distribution ($\alpha = 1/2$, $\beta = 1$). Stable distributions have been used in risk management (see [30]) where the heavy tails are exploited. As pointed out earlier, their usefulness for modern financial theory in particular as a pricing model is limited for $\alpha \neq 2$ by the fact that the basic requirement (27) is not satisfied.

4.7 Meixner Lévy processes

The Fourier transform of Meixner distributions is given by

$$\phi_{Meix}(u) = \left(\frac{\cos(\beta/2)}{\cosh((\alpha u - \mathrm{i}\beta)/2)}\right)^{2\delta}$$

for parameters $\alpha > 0$, $|\beta| < \pi$, $\delta > 0$. The corresponding Lévy processes are purely discontinuous with Lévy density

$$g_{Meix}(x) = \delta\frac{\exp(\beta x/\alpha)}{x\sinh(\pi x/\alpha)}.$$

The process has paths of infinite variation. This process has been introduced by Schoutens in the context of financial time series (see [35]).

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