The Lévy Libor model with default risk

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Abstract

In this paper we present a model for the dynamic evolution of the term structure of default-free and defaultable interest rates. The model is set in the Libor market model framework but in contrast to the classical diffusion-driven setup, its dynamics are driven by a time-inhomogeneous Lévy process which allows us to better capture the real-world dynamics of credit spreads. We present necessary and sufficient conditions for absence of arbitrage in the dynamics of the spreads, and provide pricing formulae for defaultable bonds, credit default swaps and options on credit default swaps in this setup.

1 Introduction

The market for credit-related financial instruments and credit derivatives has grown significantly in recent years. In particular, for a large number of reference obligors there exist liquid markets for credit default swaps (CDS) of different maturities which allow the construction of a *term structure of credit spreads*.

In this paper we present a framework for the modelling of the dynamic evolution of such a full term structure of credit spreads, jointly with the dynamics of a full term structure of forward interest rates. The framework is inspired by the famous *Libor market models* by Miltersen, Sandmann and Sondermann (1997), Brace, Gatarek, and Musiela (1997), and Jamshidian (1997) for the default-free case and in particular by its defaultable extension presented in Schönbucher (1999).

For various reasons we believe that an accurate representation of the term structure of credit spreads and their dynamics will become increasingly more important for credit risk models in the near future. First, the market for credit default swaps is moving in this direction: Besides the 5-year point which is still the most liquid reference maturity, for many reference entities there is liquid trading on the standard maturities of 1,3,5,7 and 10 years, and many brokers also quote CDS spreads for all maturities between 1 and 10 years. Secondly, this increase in liquidity is driven by

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the fact that the credit risk of a reference entity may indeed vary significantly when viewed over different time horizons, also under "objective" probabilities: There exists a variety of typical shapes for the term structure of credit spreads for obligors of different risk classes e.g. upward-sloping spreads for high quality credits and downward sloping ones for low credit quality. Additional variations in the shape of the term structure of credit risk may also be caused by obligor-specific circumstances e.g. the maturity of a large fraction of outstanding debt around a particular date. Furthermore, the information about the term structure of credit risk of an obligor is also of importance for the pricing of certain credit-related instruments, for example credit commitments, lines of credit or options on credit protection. And finally, a new class of credit derivatives has recently gained popularity, the so-called *constant-maturity credit default swaps* which are directly related to the shape of the term structure of credit spreads.

Similar points can be made for the ability to capture *joint* dynamics of interest rates and credit spreads, in particular given the increasing liquidity of CDS markets and the emergence of interest rate/credit hybrid derivatives.

While the default mechanism of the model presented in this paper is still an intensity-based model, it allows a direct specification of a full initial term structure of interest rates and credit spreads and of their dynamics, based upon CDS quotes and the pricing formulae given in section 6. This has obvious advantages in terms of calibration and specification of the model, as in purely intensity-based models, the calibration is usually much more involved. This advantage becomes even more compelling in a Lévy-driven framework.

The main contribution of this paper is the combination of the Libor market model based discrete-tenor setup with dynamics of the default-free and defaultable term structures that are driven by a much more general stochastic process than the standard Brownian motion which is used in Schönbucher (1999) (and almost all other models of spread-dynamics): a process with independent increments and absolutely continuous characteristics (PIIAC), also known as a *time-inhomogeneous Lévy process*.

As in the case of default-free interest rates, the main advantage of a discrete-tenor setup is a major increase in flexibility in terms of the specification of the volatility functions; in particular, a specification of *lognormal* dynamics is possible without endangering the existence of solutions of the stochastic differential equations governing the evolution of the term structure of interest rates. This advantage translates directly into the credit risk domain and distinguishes our approach from Schönbucher (1998), Bielecki and Rutkowski (2000), and Eberlein and Özkan (2003).

Diffusion-based models may be roughly adequate for models of default-free interest rates, although even in this case there is much evidence in favor of using Lévy processes. Yet in the credit risk area the arguments for models with jumps are even stronger: First, empirical evidence suggests that credit spreads (and by extension default hazard rates) have features which may be better captured by a process which exhibits jumps in its paths. For example the volatility of spreads is very high (Schönbucher (2004) finds values between 50 and 80% p.a.), and the dependence of volatilities on the level of spreads seems to be very high, too (Schönbucher (2004) finds an "volatility exponent" of around 1.5). Both phenomena can be explained well by introducing *jumps* in spreads, rather than bending a diffusion-based model to the data. Furthermore, there are good fundamental reasons for including jumps in the dynamics of credit spreads: Credit risk-related information often arrives in big “lumps”, e.g. rating adjustments, statements on the financial health of the firm, or surprising credit events in related firms may all cause large, discontinuous changes in the probabilities of survival of the affected obligors. Finally, there is evidence (e.g. Mortensen (2005)) that it is not possible to build a *multivariate* intensity model with realistic levels of default dependence unless one introduces jumps in the intensity processes of the obligors: Purely diffusion-driven intensity models usually do not exhibit levels of dependence that are high enough to reproduce current market prices of portfolio credit derivatives.

All these reasons lead us to believe that the introduction of a time-inhomogeneous Lévy process as driving process is a significant step towards making the dynamics of the model more realistic. The class of time-inhomogeneous Lévy processes is a very rich class of processes with a wide variety of possible patterns of behavior, including (amongst others) Brownian motions, Poisson jump processes (with very general jump size distributions), “classical” Lévy processes with infinite activity, e.g. generalized hyperbolic Lévy processes, and linear combinations of these. Most of the key results of this paper do not even depend on the defining properties of a PIIAC, the proofs given are directly transferable to a model driven by a general semimartingale (subject to certain regularity conditions). However, with a view towards implementation, we stick to the class of time-inhomogeneous Lévy processes.

In credit risk modelling, Lévy processes have so far found successful applications mostly in the area of firm’s value based models. Cariboni and Schoutens (2004), and Hilberink and Rogers (2002) consider single-name firm’s value based models and by introducing Lévy processes are able to solve the “short-term spread problem” that diffusion-based firm’s value models usually suffer from. By construction, the dynamics of the term structure of credit spreads in these models exhibits Lévy jumps. The approach taken in our paper is very different from these papers as we do not try to explain the *reason* of the default (using the firm’s value) but only want to describe the term structure of default risk and its evolution. This gives us more flexibility in the specification of the term structure of default risk and its dynamics but at the cost of losing the intuitive appeal and the link to equity prices that firm’s value models have. Furthermore, we would like to mention Joshi and Stacey (2005) who present an interesting way to use the Gamma process in intensity-based portfolio default risk modelling, leading to a multivariate intensity model with joint default events.

By now, the literature on the default-free Libor market models has grown too large to be surveyed here. Besides the original papers mentioned above, we therefore only refer to Eberlein and Özkan (2005) who first introduced a Libor market model driven by a Lévy process. The literature on the defaultable version of the Libor market model has also grown. Brigo (2004) presents a defaultable version of the Libor market model which is based upon a slightly different representation of the term structure of defaultable assets using forward CDS contracts. This model has the advantage of allowing for cleaner pricing of CDS, but the disadvantage that the prices of defaultable zero coupon bonds are not obtainable in closed-form any more. Schönbucher (2004) presents and extends the survival-measure pricing technique introduced in Schönbucher (1999) and discusses its application to the pricing of options on CDS, and also presents empirical results on the dynamics of CDS spreads. Jumps in the dynamics of default intensities are usually modelled using exponentially affine models, the idea goes back to Duffie and Garleanu (2001) and was used frequently afterwards. Yet these models are models of the spot intensity, so they do not attempt to model a full term structure of defaultable bond prices as we do here. Furthermore, in our specification we allow for much more general jump processes than the class of affine processes.

The rest of the paper is structured as follows: We begin with a brief introduction of the variables defining the defaultable and default-free term structures of interest rates and a short description of the driving time-inhomogeneous Lévy process. Then, in section 2, the main assumptions regarding the dynamics of forward Libor rates and default-risk factors are presented. In particular, we quickly recall the setup and results of the default-free Lévy Libor model according to Eberlein and Özkan (2005) and then state the setup of the dynamic model for a "candidate system" of discrete default hazard rates $\widehat{H}(\cdot, T_k)$.

In the next step, we must ensure that the dynamics of the modelled discrete default hazard rates $\widehat{H}(\cdot, T_k)$ coincide with the evolution of the actual $H(\cdot, T_k)$ derived from the real probabilities of default. In section 3 we show by construction the existence of a default arrival process which is consistent with these dynamics, provided that the discrete default hazard rates satisfy a martingale/drift restriction under the corresponding forward measures. By explicitly constructing a default time via an extension of the probability space we are furthermore able to cleanly identify a "background"-filtration which includes no information about default itself but which will yield the full market-filtration when it is combined with the filtration generated by the default arrival process.

Section 4 treats the most important tool in the analysis of defaultable Libor market models: the survival measures or defaultable forward measures: We introduce two versions of these measures: an unrestricted version (the survival measure of Schönbucher (1999)) and the restriction of this measure to the background filtration (as it is used in Bielecki and Rutkowski (2002)). We derive the Radon-Nikodym densities of these measures with respect to their default-free counterparts and with respect to measures at different time horizons and show how these measures can be used to price survival-contingent payoffs.

The pricing of default-contingent payoffs, i.e. payoffs that are paid at the time of default, is treated in section 5. Such default-contingent payoffs are necessary in order to model recovery payoffs of real-world defaultable securities like coupon-bearing bonds or credit default swaps. In this paper, we chose the *recovery of par* parametrization of recovery which seemed to us to be the most realistic parametrization of recovery payoffs. This setup leads to an expression for the price of a recovery unit payoff in terms of an expectation of $H(\cdot, T_k)$ under the T_{k+1}-survival measure. As closed-form solutions for these expressions do not exist, we provide approximate solutions for them.

In order to demonstrate the flexibility and applicability of the modelling approach given here, we turn to the pricing of credit derivatives in the following sections. Section 6 treats the pricing of credit default swaps (CDS) in this model and section 7 the pricing of options on CDS. Both of these instruments are also important for the practical implementation of this model as an initial term structure of CDS prices would be the obvious calibration instrument for the initial term structure of default hazard rates, and the options on CDS would provide valuable volatility information.

1.1 Notation

We consider a fixed time horizon T^* and a discrete tenor structure $0 = T_0 < T_1 < \ldots < T_n = T^*$ with $\delta_k := T_{k+1} - T_k$ for $k = 0, \ldots, n-1$. We assume that default-free as well as defaultable zero coupon bonds with maturities $T_1, \ldots, T_n$ are traded on the market. By $B(t, T_k)$ (resp. $B^0(t, T_k)$) we denote the time-t price of a default-free zero coupon bond (resp. a defaultable zero coupon bond with zero recovery) with maturity T_k. Indicate the time of default by τ and the pre-default values of the defaultable bonds by $\overline{B}(\cdot, \cdot)$, then we have

$$B^0(t, T_i) = \mathbf{1}_{\{\tau > t\}} \overline{B}(t, T_i) \qquad \text{and} \qquad \overline{B}(T_i, T_i) = 1 \quad \text{for } i \in \{1, \ldots, n\}.$$

In what follows we are not going to model bond prices directly (it is only assumed that the processes describing the evolution of the bond prices $B(\cdot, T_i)$ and of the pre-default prices $\overline{B}(\cdot, T_i)$ are special semimartingales whose values as well as all left hand limits are strictly positive). Instead, we are going to specify the dynamics of forward Libor rates. The following notation will be used:

- The *default-free forward Libor rates* are given by

$$L(t, T_k) := \frac{1}{\delta_k} \left(\frac{B(t, T_k)}{B(t, T_{k+1})} - 1 \right) \qquad (k \in \{1, \ldots, n-1\}).$$

- The *defaultable forward Libor rates* are given by

$$\overline{L}(t, T_k) := \frac{1}{\delta_k} \left(\frac{\overline{B}(t, T_k)}{\overline{B}(t, T_{k+1})} - 1 \right) \qquad (k \in \{1, \ldots, n-1\}).$$

- The *forward Libor spreads* are given by

$$S(t, T_k) := \overline{L}(t, T_k) - L(t, T_k) \qquad (k \in \{1, \ldots, n-1\}).$$

- The *default risk factors* or *forward survival processes* are given by

$$D(t, T_k) := \frac{\overline{B}(t, T_k)}{B(t, T_k)} \qquad (k \in \{1, \ldots, n\}).$$

- The discrete-tenor *forward default intensities* are given by

$$H(t, T_k) := \frac{1}{\delta_k} \left(\frac{D(t, T_k)}{D(t, T_{k+1})} - 1 \right) \qquad (k \in \{1, \ldots, n-1\}).$$

1.2 The driving process

The model is driven by a d-dimensional stochastic process $L = (L_t)_{0 \le t \le T^*}$ with *independent increments* and *absolutely continuous characteristics*, henceforth abbreviated by *PIIAC*. These processes are also called *time-inhomogeneous* or *non-homogeneous Lévy processes*. More precisely, $L = (L^1, \ldots, L^d)$ has independent increments, and for every t the law of L_t is characterized by the characteristic function

$$\mathbb{E}\left[e^{\mathrm{i}\langle u, L_t \rangle} \right] = \exp \int_0^t \left(\mathrm{i}\langle u, b_s \rangle - \frac{1}{2} \langle u, c_s u \rangle + \int_{\mathbb{R}^d} (e^{\mathrm{i}\langle u, x \rangle} - 1 - \mathrm{i}\langle u, x \rangle) F_s(\mathrm{d}x) \right) \mathrm{d}s.$$

Here, $b_s \in \mathbb{R}^d$, c_s is a symmetric nonnegative-definite $d \times d$ matrix, and F_s is a measure on $\mathbb{R}^d$ that integrates $(|x|^2 \wedge |x|)$ and satisfies $F_s(\{0\}) = 0$. The Euclidian scalar product on $\mathbb{R}^d$ is denoted by $\langle \cdot, \cdot \rangle$, the respective norm by $|\cdot|$. It is assumed that

$$\sup_{0 \le s \le T^*} \left(|b_s| + ||c_s|| + \int_{\mathbb{R}^d} (|x|^2 \wedge |x|) F_s(\mathrm{d}x) \right) < \infty \tag{1}$$

(where $||\cdot||$ denotes any norm on the set of $d \times d$ matrices) and that there are constants M, $\varepsilon > 0$ such that for every $u \in [-(1+\varepsilon)M, (1+\varepsilon)M]^d$

$$\sup_{0 \le s \le T^*} \left(\int_{\{|x|>1\}} \exp\langle u, x \rangle F_s(\mathrm{d}x) \right) < \infty. \tag{2}$$

We call $(b, c, F) := (b_s, c_s, F_s)_{0 \le s \le T^*}$ the *characteristics* of L.

2 Presentation of the model

Let us begin by building up the default-free part of the model. The dynamics of default-free forward Libor rates are specified in the same way as in the Lévy Libor model introduced in Eberlein and Özkan (2005), to whom we refer for a detailed construction. Here, we only give a very brief description of the Lévy Libor model.

The model is constructed via backward induction and driven by a non-homogeneous Lévy process L^{T^*} on a complete stochastic basis $(\widetilde{\Omega}, \widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}_{T^*}, \widetilde{\mathbf{F}} = (\widetilde{\mathcal{F}}_s)_{0 \le s \le T^*}, \mathbf{P}_{T^*})$. The measure $\mathbf{P}_{T^*}$ should be regarded as the forward measure associated with the settlement day T^*. Since L^{T^*} is required to satisfy assumption (1), it can be written in its canonical decomposition as

$$L_t^{T^*} = \int_0^t \sqrt{c_s}\, \mathrm{d}W_s^{T^*} + \int_0^t \int_{\mathbb{R}^d} x(\mu - \nu^{T^*})(\mathrm{d}s, \mathrm{d}x).$$

Here, W^{T^*} denotes a standard Brownian motion, μ is the random measure associated with the jumps of L^{T^*}, and $\nu^{T^*}(\mathrm{d}t, \mathrm{d}x) = F_t^{T^*}(\mathrm{d}x)\,\mathrm{d}t$ is the compensator of μ. The characteristics of L^{T^*} are given by $(0, c, F^{T^*})$. Note that without loss of generality L^{T^*} is assumed to be driftless. The following assumptions are made:

($\mathbb{LR}$.1): For any maturity T_i there is a deterministic and continuous function $\lambda(\cdot, T_i) : [0, T_i] \to \mathbb{R}_+^d$, which represents the volatility of the forward Libor rate process $L(\cdot, T_i)$. In addition,

$$\sum_{i=1}^{n-1} \lambda^j(s, T_i) \le M \qquad \text{for all } s \in [0, T^*] \text{ and } j \in \{1, \dots, d\}, \tag{3}$$

where M is the constant from assumption (2) and we set $\lambda(s, T_i) = 0$ for $s > T_i$.

($\mathbb{LR}$.2): The initial term structure $B(0, T_i)$ $(i \in \{1, \dots, n\})$ is strictly positive and strictly decreasing (in i).

The dynamics of the forward Libor rates are specified as

$$L(t, T_k) = L(0, T_k) \exp\left(\int_0^t b^L(s, T_k, T_{k+1})\, \mathrm{d}s + \int_0^t \lambda(s, T_k)\, \mathrm{d}L_s^{T_{k+1}} \right) \tag{4}$$

with initial condition

$$L(0,T_k) = \frac{1}{\delta_k}\left(\frac{B(0,T_k)}{B(0,T_{k+1})} - 1\right).$$

$L^{T_{k+1}}$ equals L^{T^*} plus some – in general non-deterministic – drift term which is chosen in such a way that $L^{T_{k+1}}$ is driftless under the forward measure associated with the settlement day T_{k+1}, henceforth denoted by $\mathbf{P}_{T_{k+1}}$. More precisely,

$$L_t^{T_{k+1}} = \int_0^t \sqrt{c_s}\,\mathrm{d}W_s^{T_{k+1}} + \int_0^t \int_{\mathbb{R}^d} x(\mu - \nu^{T_{k+1}})(\mathrm{d}s, \mathrm{d}x), \tag{5}$$

where $W^{T_{k+1}}$ is a standard Brownian motion with respect to $\mathbf{P}_{T_{k+1}}$ and $\nu^{T_{k+1}}$ is the $\mathbf{P}_{T_{k+1}}$-compensator of μ. The drift term $b^L(s,T_k,T_{k+1})$ is specified in such a way that $L(\cdot,T_k)$ becomes a $\mathbf{P}_{T_{k+1}}$-martingale, i.e.

$$\begin{aligned} b^L(s,T_k,T_{k+1}) &= -\frac{1}{2}\langle\lambda(s,T_k), c_s\lambda(s,T_k)\rangle \\ &\quad - \int_{\mathbb{R}^d}\left(e^{\langle\lambda(s,T_k),x\rangle} - 1 - \langle\lambda(s,T_k),x\rangle\right) F_s^{T_{k+1}}(\mathrm{d}x). \end{aligned} \tag{6}$$

The connection between different forward measures is given by

$$\frac{\mathrm{d}\mathbf{P}_{T_{k+1}}}{\mathrm{d}\mathbf{P}_{T^*}} = \prod_{l=k+1}^{n-1}\frac{1+\delta_l L(T_{k+1},T_l)}{1+\delta_l L(0,T_l)} = \frac{B(0,T^*)}{B(0,T_{k+1})}\prod_{l=k+1}^{n-1}(1+\delta_l L(T_{k+1},T_l)). \tag{7}$$

Once restricted to the σ-field $\widetilde{\mathcal{F}}_t$ this becomes

$$\left.\frac{\mathrm{d}\mathbf{P}_{T_{k+1}}}{\mathrm{d}\mathbf{P}_{T^*}}\right|_{\widetilde{\mathcal{F}}_t} = \frac{B(0,T^*)}{B(0,T_{k+1})}\prod_{l=k+1}^{n-1}(1+\delta_l L(t,T_l)) \qquad (t\in[0,T_{k+1}]). \tag{8}$$

The Brownian motions and compensators with respect to the different measures are connected via

$$W_t^{T_{k+1}} = W_t^{T^*} - \int_0^t \sqrt{c_s}\left(\sum_{l=k+1}^{n-1}\alpha(s,T_l,T_{l+1})\right)\mathrm{d}s \tag{9}$$

with

$$\alpha(s,T_l,T_{l+1}) := \frac{\delta_l L(s-,T_l)}{1+\delta_l L(s-,T_l)}\lambda(s,T_l) \tag{10}$$

and

$$\nu^{T_{k+1}}(\mathrm{d}t,\mathrm{d}x) = \left(\prod_{l=k+1}^{n-1}\beta(s,x,T_l,T_{l+1})\right)\nu^{T^*}(\mathrm{d}t,\mathrm{d}x) =: F_t^{T_{k+1}}(\mathrm{d}x)\,\mathrm{d}t, \tag{11}$$

where

$$\beta(s,x,T_l,T_{l+1}) := \frac{\delta_l L(s-,T_l)}{1+\delta_l L(s-,T_l)}\left(e^{\langle\lambda(s,T_l),x\rangle} - 1\right) + 1. \tag{12}$$

Note that $L^{T_{k+1}}$ is usually not a (non-homogeneous) Lévy process under any of the measures $\mathbf{P}_{T_i}$ (except for $k=n-1$, since L^{T^*} is by definition a PIIAC under $\mathbf{P}_{T^*}$).

The construction by backward induction guarantees that $\frac{B(\cdot,T_j)}{B(\cdot,T_k)}$ is a $\mathbf{P}_{T_k}$-martingale for all $j,k \in \{1,\dots,n\}$. Our goal in what follows is to include defaultable forward Libor rates in the Lévy Libor model.

At first sight, an evident way to build up the defaultable part of the model is to specify the dynamics of the defaultable forward Libor rates by an expression similar to (4). However, $\overline{L}(T_k,T_k) < L(T_k,T_k)$ implies $\overline{B}(T_k,T_{k+1}) > B(T_k,T_{k+1})$, in which case there is an arbitrage opportunity in the market, provided that $B^0(\cdot,T_{k+1})$ has not defaulted until T_k. It seems thus natural to specify the model in such a way that defaultable forward Libor rates are always higher than their default-free counterparts. This can be achieved by modelling forward Libor spreads or forward default intensities as positive processes, instead of specifying defaultable forward Libor rates directly. We can then get the defaultable forward Libor rates through

$$\overline{L}(t,T_k) = S(t,T_k) + L(t,T_k)$$

or

$$\overline{L}(t,T_k) = H(t,T_k)(1+\delta_k L(t,T_k)) + L(t,T_k). \tag{13}$$

Unfortunately, H or S cannot be specified directly since their dynamics depend on the specification of the default time τ (compare equation (18) and the discussion preceding it). In other words, as soon as τ is specified we cannot freely choose the dynamics of H (or S). What we can and will do in the sequel is the following: We give a pre-specification for H and then construct τ in such a way that the dynamics of H implied by τ will match this pre-specification. The following assumptions are made in addition to ($\mathbb{LR}$.1) and ($\mathbb{LR}$.2):

($\mathbb{DLR}$.1): For any maturity T_i there is a deterministic and continuous function $\gamma(\cdot,T_i) : [0,T_i] \to \mathbb{R}^d_+$, which represents the volatility of the forward default intensity $H(\cdot,T_i)$. We set $\gamma(s,T_i) = 0$ for $T_i < s \le T^*$ and tighten condition (3) by assuming that

$$\sum_{i=1}^{n-1} (\lambda^j(s,T_i) + \gamma^j(s,T_i)) \le M \qquad \text{for all } s \in [0,T^*] \text{ and } j \in \{1,\dots,d\}. \tag{14}$$

($\mathbb{DLR}$.2): The initial term structure $\overline{B}(0,T_i)$ ($i \in \{1,\dots,n\}$) of defaultable zero coupon bond prices satisfies $0 < \overline{B}(0,T_i) \le B(0,T_i)$ for all T_i as well as $\overline{L}(0,T_i) \ge L(0,T_i)$, i.e.

$$\frac{\overline{B}(0,T_i)}{\overline{B}(0,T_{i+1})} \ge \frac{B(0,T_i)}{B(0,T_{i+1})}.$$

To avoid confusion, let us denote by $\widehat{H}$ the pre-specified forward default intensities, which we postulate to be given by

$$\begin{aligned}\widehat{H}(t,T_k) \;&=\; H(0,T_k)\exp\Bigg(\int_0^t b^H(s,T_k,T_{k+1})\,\mathrm{d}s + \int_0^t \sqrt{c_s}\gamma(s,T_k)\,\mathrm{d}W_s^{T_{k+1}} \\ &\qquad + \int_0^t \int_{\mathbb{R}^d} \langle \gamma(s,T_k), x\rangle (\mu - \nu^{T_{k+1}})(\mathrm{d}s,\mathrm{d}x)\Bigg)\end{aligned} \tag{15}$$

subject to the initial condition

$$H(0,T_k) = \frac{1}{\delta_k}\left(\frac{\overline{B}(0,T_k)B(0,T_{k+1})}{B(0,T_k)\overline{B}(0,T_{k+1})} - 1\right).$$

$W^{T_{k+1}}$ and $\nu^{T_{k+1}}$ are defined in (9) and (11). The drift term $b^H(\cdot,T_k,T_{k+1})$ will be specified later. For the moment we only assume $b^H(s,T_k,T_{k+1}) = 0$ for $T_k < s \le T^*$, i.e. we require that $\widehat{H}(t,T_k) = \widehat{H}(T_k,T_k)$ for $t \in [T_k,T^*]$.

3 Construction of the time of default

The construction of the default time will be done in the canonical way, that is for a given $\widetilde{\mathbf{F}}$-hazard process Γ a stopping time τ on an enlarged probability space will be constructed. We will do the construction for a general Γ first. The key question then will be which particular hazard process to choose to make H match $\widehat{H}$. For more details on the canonical construction we refer to Bielecki and Rutkowski (2002), from whom the notation is adopted.

Let Γ be an $\widetilde{\mathbf{F}}$-adapted, right-continuous, increasing process on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \mathbf{P}_{T^*})$ satisfying $\Gamma_0 = 0$ and $\lim_{t\to\infty}\Gamma_t = \infty$. Furthermore, let η be a random variable on some probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbf{P}})$ that is uniformly distributed on $[0,1]$. Consider the product space $(\Omega, \mathcal{G}, \mathbf{Q}_{T^*})$ defined by

$$\Omega := \widetilde{\Omega}\times\hat{\Omega}, \qquad \mathcal{G} := \widetilde{\mathcal{F}}\otimes\hat{\mathcal{F}}, \qquad \mathbf{Q}_{T^*} := \mathbf{P}_{T^*}\otimes\hat{\mathbf{P}}$$

and denote by $\mathbf{F}$ the trivial extension of $\widetilde{\mathbf{F}}$ to the enlarged probability space $(\Omega, \mathcal{G}, \mathbf{Q}_{T^*})$, i.e. each $A \in \mathcal{F}_t$ is of the form $\widetilde{A}\times\hat{\Omega}$ for some $\widetilde{A} \in \widetilde{\mathcal{F}}_t$. We extend all stochastic processes from the default-free part of the model to the extended probability space (by setting $L^{T^*}(\widetilde{\omega},\hat{\omega}) := L^{T^*}(\widetilde{\omega})$ and similarly for all other processes).

Define a random variable $\tau : \Omega \to \mathbb{R}_+$ by

$$\tau := \inf\{t \in \mathbb{R}_+ : e^{-\Gamma_t} \le \eta\}.$$

and denote $\mathcal{H}_t := \sigma\left(\mathbf{1}_{\{\tau\le u\}} \,|\, 0 \le u \le t\right)$ and $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$ for $t \in [0,T^*]$. Then τ is a stopping time with respect to the filtration $\mathbf{G} := (\mathcal{G}_s)_{0\le s\le T^*}$ since $\{\tau \le t\} \in \mathcal{H}_t \subset \mathcal{G}_t$. Moreover, for $0 \le s \le t \le T^*$ we have (compare Bielecki and Rutkowski (2002, (8.14)))

$$\mathbf{Q}_{T^*}\{\tau > s|\mathcal{F}_{T^*}\} = \mathbf{Q}_{T^*}\{\tau > s|\mathcal{F}_t\} = \mathbf{Q}_{T^*}\{\tau > s|\mathcal{F}_s\} = e^{-\Gamma_s}, \tag{16}$$

i.e. Γ is the $\mathbf{F}$-hazard process of τ under $\mathbf{Q}_{T^*}$.

A question that arises naturally is whether or not L^{T^*} is a non-homogeneous Lévy process with respect to $\mathbf{Q}_{T^*}$ and the enlarged filtration $\mathbf{G}$.

Proposition 1 *L^{T^*} is a non-homogeneous Lévy process with characteristics $(0, c, F^{T^*})$ on the stochastic basis $(\Omega, \mathcal{G}_{T^*}, \mathbf{G}, \mathbf{Q}_{T^*})$.*

Proof: L^{T^*} is clearly an adapted, càdlàg process and satisfies $L_0^{T^*} = 0$. Its characteristic function is given by

$$\begin{aligned}\mathbb{E}_{\mathbf{Q}_{T^*}}[\exp(\mathrm{i}uL_t^{T^*})] &= \int_{\widetilde{\Omega}\times\hat{\Omega}} \exp(\mathrm{i}uL_t^{T^*}(\widetilde{\omega},\hat{\omega}))\,\mathrm{d}(\mathbf{P}_{T^*}\otimes\hat{\mathbf{P}})(\widetilde{\omega},\hat{\omega})\\ &= \int_{\widetilde{\Omega}} \exp(\mathrm{i}uL_t^{T^*}(\widetilde{\omega}))\,\mathrm{d}\mathbf{P}_{T^*}(\widetilde{\omega}) \;=\; \mathbb{E}_{\mathbf{P}_{T^*}}[\exp(\mathrm{i}uL_t^{T^*})].\end{aligned}$$

Hence, the characteristic function of $L_t^{T^*}$ and thus also the characteristics of L^{T^*} are preserved. It remains to show that $L_t^{T^*} - L_s^{T^*}$ is independent of $\mathcal{G}_s$ for $s < t$. Note that equation (16) is equivalent to the following statement (see Bielecki and Rutkowski (2002, (p. 166/167))): for any bounded, $\mathcal{F}_{T^*}$-measurable random variable X we have

$$E_{\mathbf{Q}_{T^*}}[X|\mathcal{G}_s] = E_{\mathbf{Q}_{T^*}}[X|\mathcal{F}_s] \qquad (0 \leq s \leq T^*). \tag{17}$$

Let $B \in \mathcal{B}^d$ and $A \in \mathcal{G}_s$, then using (17) with $X := \mathbf{1}_B(L_t^{T^*} - L_s^{T^*})$ and the fact that $L_t^{T^*} - L_s^{T^*}$ is independent of $\mathcal{F}_s$ we get

$$\begin{aligned} \mathbf{Q}_{T^*}(A \cap \{(L_t^{T^*} - L_s^{T^*}) \in B\}) &= \int_A \mathbf{1}_B(L_t^{T^*} - L_s^{T^*})\,\mathrm{d}\mathbf{Q}_{T^*} \\ &= \int_A \mathbb{E}_{\mathbf{Q}_{T^*}}[\mathbf{1}_B(L_t^{T^*} - L_s^{T^*})|\mathcal{F}_s]\,\mathrm{d}\mathbf{Q}_{T^*} \\ &= \int_A \mathbb{E}_{\mathbf{Q}_{T^*}}[\mathbf{1}_B(L_t^{T^*} - L_s^{T^*})]\,\mathrm{d}\mathbf{Q}_{T^*} \\ &= \mathbf{Q}_{T^*}(A)\mathbf{Q}_{T^*}(\{(L_t^{T^*} - L_s^{T^*}) \in B\}). \qquad \square \end{aligned}$$

In particular, each forward Libor rate $L(t, T_k)_{0 \leq t \leq T_k}$ is a martingale with respect to the filtration $(\mathcal{G}_s)_{0 \leq s \leq T_k}$ and the measure $\mathbf{Q}_{T_{k+1}}$, which is constructed from $\mathbf{Q}_{T^*}$ in the same way as $\mathbf{P}_{T_{k+1}}$ is constructed from $\mathbf{P}_{T^*}$.

Γ is not only the $\mathbf{F}$-hazard process of τ under $\mathbf{Q}_{T^*}$, but also the $\mathbf{F}$-hazard process of τ under all other forward measures, as the following lemma shows:

Lemma 2 *Γ is the $\mathbf{F}$-hazard process of τ under $\mathbf{Q}_{T_k}$ for all $k \in \{1, \ldots, n\}$.*

Proof: Fix a k and denote by ψ the ($\mathcal{F}_{T_k}$-measurable) Radon–Nikodym derivative of $\mathbf{Q}_{T_k}$ with respect to $\mathbf{Q}_{T^*}$. Note that (16) is equivalent to the conditional independence of $\mathcal{F}_{T^*}$ and $\mathcal{H}_s$ given $\mathcal{F}_s$ under $\mathbf{Q}_{T^*}$, that is for any bounded $\mathcal{F}_{T^*}$-measurable random variable X and any bounded $\mathcal{H}_s$-measurable random variable Y we have

$$E_{\mathbf{Q}_{T^*}}[XY|\mathcal{F}_s] = E_{\mathbf{Q}_{T^*}}[X|\mathcal{F}_s]\, E_{\mathbf{Q}_{T^*}}[Y|\mathcal{F}_s] \qquad (0 \leq s \leq T^*)$$

(compare Bielecki and Rutkowski (2002, (p. 166))). Using the abstract Bayes rule and this conditional independence (plus a dominated convergence argument) we get

$$\begin{aligned} \mathbf{Q}_{T_k}\{\tau > s|\mathcal{F}_s\} &= \frac{\mathbb{E}_{\mathbf{Q}_{T^*}}[\psi \mathbf{1}_{\{\tau > s\}}|\mathcal{F}_s]}{\mathbb{E}_{\mathbf{Q}_{T^*}}[\psi|\mathcal{F}_s]} \\ &= \frac{\mathbb{E}_{\mathbf{Q}_{T^*}}[\psi|\mathcal{F}_s]\,\mathbb{E}_{\mathbf{Q}_{T^*}}[\mathbf{1}_{\{\tau > s\}}|\mathcal{F}_s]}{\mathbb{E}_{\mathbf{Q}_{T^*}}[\psi|\mathcal{F}_s]} \\ &= e^{-\Gamma_s}. \qquad \square \end{aligned}$$

To clarify the relationship between default time and default intensities remember that the time-t value of a defaultable bond is given by

$$B^0(t, T_k) = \mathbf{1}_{\{\tau > t\}}\overline{B}(t, T_k).$$

In the model for the default-free Libor rates, the time-t price of a contingent claim X paying $\mathbf{1}_{\{\tau > T_k\}}$ at T_k is given by

$$X_t := B(t, T_k)\mathbb{E}_{\mathbf{Q}_{T_k}}[\mathbf{1}_{\{\tau > T_k\}}|\mathcal{G}_t] = \mathbf{1}_{\{\tau > t\}}B(t, T_k)\mathbb{E}_{\mathbf{Q}_{T_k}}[\mathbf{1}_{\{\tau > T_k\}}|\mathcal{G}_t].$$

To have a consistent model, we thus have to have

$$B^0(t,T_k) = \mathbf{1}_{\{\tau>t\}} B(t,T_k) \mathbb{E}_{\mathbf{Q}_{T_k}}[\mathbf{1}_{\{\tau>T_k\}} | \mathcal{G}_t]$$

and consequently (at least on $\{\tau > t\}$)

$$\overline{B}(t,T_k) = B(t,T_k) \mathbb{E}_{\mathbf{Q}_{T_k}}[\mathbf{1}_{\{\tau>T_k\}} | \mathcal{G}_t]$$

or equivalently

$$D(t,T_k) = \mathbb{E}_{\mathbf{Q}_{T_k}}[\mathbf{1}_{\{\tau>T_k\}} | \mathcal{G}_t], \tag{18}$$

which immediately provides a formula for H (and also for S).

Let us now turn to the question which hazard process Γ to choose to make H match its pre-specification. As pointed out, to have a consistent model we have to have

$$\begin{aligned} B^0(t,T_k) &= B(t,T_k) \mathbf{Q}_{T_k}\{\tau > T_k | \mathcal{G}_t\} &(19)\\ &= B(t,T_k) \mathbf{1}_{\{\tau>t\}} \frac{\mathbb{E}_{\mathbf{Q}_{T_k}}[\mathbf{1}_{\{\tau>T_k\}} | \mathcal{F}_t]}{\mathbb{E}_{\mathbf{Q}_{T_k}}[\mathbf{1}_{\{\tau>t\}} | \mathcal{F}_t]}, \end{aligned}$$

where the last equality follows from Bielecki and Rutkowski (2002, (5.2)). Let

$$\overline{B}(t,T_k) := B(t,T_k) \frac{\mathbb{E}_{\mathbf{Q}_{T_k}}[\mathbf{1}_{\{\tau>T_k\}} | \mathcal{F}_t]}{\mathbb{E}_{\mathbf{Q}_{T_k}}[\mathbf{1}_{\{\tau>t\}} | \mathcal{F}_t]} = B(t,T_k) \frac{\mathbb{E}_{\mathbf{Q}_{T_k}}[\mathbf{1}_{\{\tau>T_k\}} | \mathcal{F}_t]}{e^{-\Gamma_t}}, \tag{20}$$

then

$$D(t,T_k) = \mathbb{E}_{\mathbf{Q}_{T_k}}\left[e^{\Gamma_t - \Gamma_{T_k}} \middle| \mathcal{F}_t \right].$$

In particular,

$$H(t,T_k) = \frac{1}{\delta_k}\left(\frac{D(t,T_k)}{D(t,T_{k+1})} - 1 \right) = \frac{1}{\delta_k}\left(\frac{\mathbb{E}_{\mathbf{Q}_{T_k}}\left[e^{-\Gamma_{T_k}} \middle| \mathcal{F}_t \right]}{\mathbb{E}_{\mathbf{Q}_{T_{k+1}}}\left[e^{-\Gamma_{T_{k+1}}} \middle| \mathcal{F}_t \right]} - 1 \right). \tag{21}$$

It is clear from the previous equation that, in order to make H match its pre-specification $\widehat{H}$, we only need to specify the hazard process Γ at the points T_k for $k \in \{1,\ldots,n\}$ in a suitable way. The values of Γ in between these points do not have an influence on the value of H. Moreover, we know from equation (21) that

$$\mathbb{E}_{\mathbf{Q}_{T_k}}\left[e^{-\Gamma_{T_k}} \middle| \mathcal{F}_{T_{k-1}} \right] = e^{-\Gamma_{T_{k-1}}} (1 + \delta_{k-1} H(T_{k-1}, T_{k-1}))^{-1}.$$

We now choose the hazard process and define Γ recursively by setting $\Gamma_0 := 0$,

$$\begin{aligned} \Gamma_{T_k} &:= \Gamma_{T_k-1} + \log(1 + \delta_{k-1}\widehat{H}(T_{k-1}, T_{k-1})) \qquad (k \in \{1,\ldots,n\})\\ &= \sum_{l=0}^{k-1} \log(1 + \delta_l \widehat{H}(T_l, T_l)), &(22) \end{aligned}$$

and for $t \in (T_{k-1}, T_k)$

$$\Gamma_t := (1 - \alpha_k(t))\Gamma_{T_{k-1}} + \alpha_k(t)\Gamma_{T_k},$$

where $\alpha_k : [T_{k-1}, T_k] \to [0,1]$ is a continuous, strictly increasing function satisfying $\alpha_k(T_{k-1}) = 0$ and $\alpha_k(T_k) = 1$. Obviously Γ is a continuous, strictly increasing (since $\widehat{H}(\cdot,\cdot) > 0$ by construction), and $\widetilde{\mathbf{F}}$-adapted process (since Γ_{T_k} is $\widetilde{\mathcal{F}}_{T_{k-1}}$-measurable) and can be used for the canonical construction.

It still has to be checked whether the implied dynamics of H match those of $\widehat{H}$. Using (21) and (22) we get

$$\begin{aligned} H(t,T_1) &= \frac{1}{\delta_1}\left(\frac{1}{\mathbb{E}_{\mathbf{Q}_{T_2}}[e^{-\Gamma_{T_2}+\Gamma_{T_1}} \,|\, \mathcal{F}_t]} - 1\right) \\ &= \frac{1}{\delta_1}\left(\frac{1}{\mathbb{E}_{\mathbf{Q}_{T_2}}\left[\frac{1}{1+\delta_1\widehat{H}(T_1,T_1)} \Big| \, \mathcal{F}_t\right]} - 1\right) \end{aligned}$$

or, written differently,

$$\mathbb{E}_{\mathbf{Q}_{T_2}}\left[\frac{1}{1+\delta_1\widehat{H}(T_1,T_1)}\Big| \, \mathcal{F}_t\right] = \frac{1}{1+\delta_1 H(t,T_1)}.$$

Consequently, $H(\cdot,T_1)$ meets its pre-specification if $\left(\frac{1}{1+\delta_1\widehat{H}(t,T_1)}\right)_{0\le t\le T_1}$ is a $\mathbf{Q}_{T_2}$-martingale. More generally we have the following result. Recall that $\widehat{H}(t,T_i) = \widehat{H}(T_i,T_i)$ for $t \in [T_i, T^*]$.

Lemma 3 *$H(\cdot,T_k)$ meets its pre-specification if $\left(\prod_{i=1}^{l}\frac{1}{1+\delta_i\widehat{H}(t,T_i)}\right)_{0\le t\le T_l}$ is a $\mathbf{Q}_{T_{l+1}}$-martingale for all $l \in \{1,\ldots,k\}$.*

Proof: The result for $k = 1$ has been proven above. Using (21), (22) and the prerequisite we get for $k > 1$

$$\begin{aligned} H(t,T_k) &= \frac{1}{\delta_k}\left(\frac{\mathbb{E}_{\mathbf{Q}_{T_k}}\left[\prod_{i=0}^{k-1}\frac{1}{1+\delta_i\widehat{H}(T_i,T_i)}\Big|\mathcal{F}_t\right]}{\mathbb{E}_{\mathbf{Q}_{T_{k+1}}}\left[\prod_{i=0}^{k}\frac{1}{1+\delta_i\widehat{H}(T_i,T_i)}\Big|\mathcal{F}_t\right]} - 1\right) \\ &= \frac{1}{\delta_k}((1+\delta_k\widehat{H}(t,T_k)) - 1) = \widehat{H}(t,T_k). \end{aligned}$$ □

Remember that we can still choose the drift coefficients $b^H(\cdot,T_k,T_{k+1})$ in (15) in order to satisfy the prerequisite of the previous lemma. This choice is done in appendix A. In the subsequent sections, we assume that the drift terms $b^H(\cdot,T_k,T_{k+1})$ are chosen as described in proposition 13 and do not distinguish between H and $\widehat{H}$ anymore.

4 Defaultable forward measures

It is well known that pricing of derivatives in default-free interest rate models can often be facilitated considerably by changing numeraires, i.e. changing measures, in a suitable way. In particular, forward measures prove to be useful in many situations. Similarly, valuation of contingent claims in our model can be simplified by using two counterparts to default-free forward measures. The first definition traces back to Schönbucher (1999):

Definition 4 *The* defaultable forward (martingale) measure *or* survival measure $\overline{\mathbf{Q}}_{T_i}$ *for the settlement day* T_i *is defined on* $(\Omega, \mathcal{G}_{T_i})$ *by*

$$\frac{\mathrm{d}\overline{\mathbf{Q}}_{T_i}}{\mathrm{d}\mathbf{Q}_{T_i}} := \frac{B(0,T_i)}{B^0(0,T_i)} B^0(T_i,T_i) = \frac{B(0,T_i)}{\overline{B}(0,T_i)} \mathbf{1}_{\{\tau > T_i\}}.$$

Equation (19) ensures that the preceding expression is indeed a density. $\overline{\mathbf{Q}}_{T_i}$ corresponds to the choice of $B^0(\cdot, T_i)$ as a "numeraire". We use quotation marks since $B^0(\cdot, T_i)$ is not a strictly positive process with probability one. Consequently, $\overline{\mathbf{Q}}_{T_i}$ is absolutely continuous with respect to $\mathbf{Q}_{T_i}$, but the two measures are not mutually equivalent. In particular, the set $A = \{\tau \le t\}$ for $t \in (0, T_i]$ has a strictly positive probability under $\mathbf{Q}_{T_i}$ but zero probability under $\overline{\mathbf{Q}}_{T_i}$. The term "survival measure" is justified by the fact that

$$\overline{\mathbf{Q}}_{T_i}(A) = \frac{\mathbf{Q}_{T_i}(A \cap \{\tau > T_i\})}{\mathbf{Q}_{T_i}(\{\tau > T_i\})} = \mathbf{Q}_{T_i}(A | \{\tau > T_i\}) \qquad (A \in \mathcal{G}_{T_i}),$$

i.e. $\overline{\mathbf{Q}}_{T_i}$ can be regarded as the forward measure $\mathbf{Q}_{T_i}$ conditioned on survival until T_i. Once restricted to the σ-field $\mathcal{G}_t$, the defaultable forward measure becomes

$$\left.\frac{\mathrm{d}\overline{\mathbf{Q}}_{T_i}}{\mathrm{d}\mathbf{Q}_{T_i}}\right|_{\mathcal{G}_t} = \frac{B(0,T_i)}{\overline{B}(0,T_i)} \frac{\overline{B}(t,T_i)}{B(t,T_i)} \mathbf{1}_{\{\tau > t\}} = \frac{B(0,T_i)}{\overline{B}(0,T_i)} \mathbf{1}_{\{\tau > t\}} \frac{\mathbf{Q}_{T_i}(\{\tau > T_i\} | \mathcal{F}_t)}{\mathbf{Q}_{T_i}(\{\tau > t\} | \mathcal{F}_t)}.$$

The first equality follows from the fact that $\frac{B^0(\cdot,T_i)}{B(\cdot,T_i)}$ is a $\mathbf{Q}_{T_i}$-martingale, the second equality from (20).

Another very useful tool in the context of derivative pricing is the restricted defaultable forward measure, which has already been used in Bielecki and Rutkowski (2002, Section 15.2). Note that the defaultable forward measure restricted to the σ-field $\mathcal{F}_t$ is given by

$$\left.\frac{\mathrm{d}\overline{\mathbf{Q}}_{T_i}}{\mathrm{d}\mathbf{Q}_{T_i}}\right|_{\mathcal{F}_t} = \frac{B(0,T_i)}{\overline{B}(0,T_i)} \mathbf{Q}_{T_i}(\{\tau > T_i\} | \mathcal{F}_t)$$

and denote by $\mathbf{P}_{T_i}$ the restriction of $\mathbf{Q}_{T_i}$ to the σ-field $\mathcal{F}_{T_i}$. This notation differs slightly from the notation in the default-free part of the model where $\mathbf{P}_{T_i}$ was defined on $\widetilde{\mathcal{F}}_{T_i}$. However, this should not cause any confusion since $\mathcal{F}_{T_i}$ is the trivial extension of $\widetilde{\mathcal{F}}_{T_i}$.

Definition 5 *The* restricted defaultable forward (martingale) measure $\overline{\mathbf{P}}_{T_i}$ *for the settlement day* T_i *is defined on* $(\Omega, \mathcal{F}_{T_i})$ *by*

$$\frac{\mathrm{d}\overline{\mathbf{P}}_{T_i}}{\mathrm{d}\mathbf{P}_{T_i}} = \frac{B(0,T_i)}{\overline{B}(0,T_i)} \mathbf{Q}_{T_i}(\{\tau > T_i\} | \mathcal{F}_{T_i}).$$

We have an explicit expression for this density, namely

$$\frac{\mathrm{d}\overline{\mathbf{P}}_{T_i}}{\mathrm{d}\mathbf{P}_{T_i}} = \frac{B(0,T_i)}{\overline{B}(0,T_i)} e^{-\Gamma_{T_i}} = \frac{B(0,T_i)}{\overline{B}(0,T_i)} \prod_{k=0}^{i-1} \frac{1}{1 + \delta_k H(T_k, T_k)}. \tag{23}$$

Restricted to the the σ-field $\mathcal{F}_t$ this becomes (since $\prod_{k=0}^{i-1} \frac{1}{1+\delta_k H(\cdot,T_k)}$ is a $\mathbf{P}_{T_i}$-martingale)

$$\left.\frac{\mathrm{d}\overline{\mathbf{P}}_{T_i}}{\mathrm{d}\mathbf{P}_{T_i}}\right|_{\mathcal{F}_t} = \frac{B(0,T_i)}{\overline{B}(0,T_i)} \prod_{k=0}^{i-1} \frac{1}{1+\delta_k H(t,T_k)}. \tag{24}$$

We get the representation (compare Kluge (2005, (A.1)))

$$\begin{aligned}\frac{\mathrm{d}\overline{\mathbf{P}}_{T_i}}{\mathrm{d}\mathbf{P}_{T_i}} &= \mathcal{E}_{T_{i-1}}\Bigg(\int_0^{\bullet} -\sum_{l=1}^{i-1} Y_{s-}^l \sqrt{c_s}\gamma(s,T_l)\,\mathrm{d}W_s^{T_i} \\ &\quad + \int_0^{\bullet}\int_{\mathbb{R}^d} \left(\prod_{l=1}^{i-1} \left(1+Y_{s-}^l \left(e^{\langle \gamma(s,T_l),x\rangle}-1\right)\right)^{-1} - 1\right)(\mu-\nu^{T_i})(\mathrm{d}s,\mathrm{d}x)\Bigg)\end{aligned}$$

with

$$Y_s^l := \frac{\delta_l H(s,T_l)}{1+\delta_l H(s,T_l)}.$$

Hence, the two predictable processes in Girsanov's Theorem for semimartingales (see Jacod and Shiryaev (2003, Theorem III.3.24)) associated with this change of measure are

$$\begin{aligned}\beta(s) &= -\sum_{l=1}^{i-1}\left(Y_{s-}^l \gamma(s,T_l)\right) \qquad \text{and} \\ Y(s,x) &= \prod_{l=1}^{i-1}\left(1+Y_{s-}^l\left(e^{\langle\gamma(s,T_l),x\rangle}-1\right)\right)^{-1}.\end{aligned}$$

We can conclude that

$$\overline{W}_t^{T_i} := W_t^{T_i} + \int_0^t \sum_{l=1}^{i-1} Y_{s-}^l \sqrt{c_s}\gamma(s,T_l)\,\mathrm{d}s \tag{25}$$

is a $\overline{\mathbf{P}}_{T_i}$-standard Brownian motion and the $\overline{\mathbf{P}}_{T_i}$-compensator of μ is given by

$$\overline{\nu}^{T_i}(\mathrm{d}s,\mathrm{d}x) = \prod_{l=1}^{i-1}\left(1+Y_{s-}^l\left(e^{\langle\gamma(s,T_l),x\rangle}-1\right)\right)^{-1}\nu^{T_i}(\mathrm{d}s,\mathrm{d}x) =: \overline{F}_s^{T_i}(\mathrm{d}x)\,\mathrm{d}s. \tag{26}$$

Similar to the default-free part of the model, we have the following connection between restricted defaultable forward measures for different settlement days:

Lemma 6 *The defaultable Libor rate $(\overline{L}(t,T_i))_{0\le t\le T_i}$ is a $\overline{\mathbf{P}}_{T_{i+1}}$-martingale and*

$$\left.\frac{\mathrm{d}\overline{\mathbf{P}}_{T_i}}{\mathrm{d}\overline{\mathbf{P}}_{T_{i+1}}}\right|_{\mathcal{F}_t} = \frac{\overline{B}(0,T_{i+1})}{\overline{B}(0,T_i)}(1+\delta_i\overline{L}(t,T_i)) \qquad (0\le t\le T_i).$$

PROOF: From equation (13) we get

$$\begin{aligned}(1+\delta_i\overline{L}(t,T_i)) &= (1+\delta_i H(t,T_i))(1+\delta_i L(t,T_i)) \\ &= \prod_{k=0}^{i}(1+\delta_k H(t,T_k))(1+\delta_i L(t,T_i))\prod_{k=0}^{i-1}(1+\delta_k H(t,T_k))^{-1}.\end{aligned}$$

Applying equations (8) and (24) yields

$$
\begin{aligned}
(1+\delta_i\overline{L}(t,T_i)) &= \frac{B(0,T_{i+1})}{\overline{B}(0,T_{i+1})}\left.\frac{\mathrm{d}\mathbf{P}_{T_{i+1}}}{\mathrm{d}\overline{\mathbf{P}}_{T_{i+1}}}\right|_{\mathcal{F}_t}\frac{B(0,T_i)}{B(0,T_{i+1})}\left.\frac{\mathrm{d}\mathbf{P}_{T_i}}{\mathrm{d}\mathbf{P}_{T_{i+1}}}\right|_{\mathcal{F}_t}\frac{\overline{B}(0,T_i)}{B(0,T_i)}\left.\frac{\mathrm{d}\overline{\mathbf{P}}_{T_i}}{\mathrm{d}\mathbf{P}_{T_i}}\right|_{\mathcal{F}_t} \\
&= \frac{\overline{B}(0,T_i)}{\overline{B}(0,T_{i+1})}\left.\frac{\mathrm{d}\overline{\mathbf{P}}_{T_i}}{\mathrm{d}\overline{\mathbf{P}}_{T_{i+1}}}\right|_{\mathcal{F}_t}
\end{aligned}
$$

and both statements are established. □

As mentioned above, (restricted) defaultable forward measures can be used to determine prices of contingent claims. Consider a defaultable claim with a promised payoff of X at the settlement day T_i and zero recovery upon default. Then its time-t value is given by

$$\pi_t^X := \mathbf{1}_{\{\tau>t\}}B(t,T_i)\mathbb{E}_{\mathbf{Q}_{T_i}}[X\mathbf{1}_{\{\tau>T_i\}}|\mathcal{G}_t] \qquad (t\in[0,T_i]).$$

Consider the general case in which X is $\mathcal{G}_{T_i}$-measurable and the common case of an $\mathcal{F}_{T_i}$-measurable promised payoff X. The following proposition is a typo-corrected version of Bielecki and Rutkowski (2002, Proposition 15.2.3):

Proposition 7 *Assume that the promised payoff X is $\mathcal{G}_{T_i}$-measurable and integrable with respect to $\overline{\mathbf{Q}}_{T_i}$. Then*

$$\pi_t^X = \mathbf{1}_{\{\tau>t\}}\overline{B}(t,T_i)\mathbb{E}_{\overline{\mathbf{Q}}_{T_i}}[X|\mathcal{G}_t] = B^0(t,T_i)\mathbb{E}_{\overline{\mathbf{Q}}_{T_i}}[X|\mathcal{G}_t].$$

If X is $\mathcal{F}_{T_i}$-measurable, then

$$\pi_t^X = \mathbf{1}_{\{\tau>t\}}\overline{B}(t,T_i)\mathbb{E}_{\overline{\mathbf{P}}_{T_i}}[X|\mathcal{F}_t] = B^0(t,T_i)\mathbb{E}_{\overline{\mathbf{P}}_{T_i}}[X|\mathcal{F}_t].$$

Proof: The first statement can be proved along the lines of Bielecki and Rutkowski (2002, Proposition 15.2.3). For the second statement observe that

$$
\begin{aligned}
\pi_t^X &= \mathbf{1}_{\{\tau>t\}}B(t,T_i)\mathbb{E}_{\mathbf{Q}_{T_i}}[X\mathbf{1}_{\{\tau>T_i\}}|\mathcal{G}_t] \\
&= \mathbf{1}_{\{\tau>t\}}B(t,T_i)\frac{\mathbb{E}_{\mathbf{Q}_{T_i}}[X\mathbf{1}_{\{\tau>T_i\}}|\mathcal{F}_t]}{\mathbf{Q}_{T_i}\{\tau>t|\mathcal{F}_t\}} \\
&= \mathbf{1}_{\{\tau>t\}}\overline{B}(t,T_i)\frac{\mathbb{E}_{\mathbf{Q}_{T_i}}[X\mathbf{1}_{\{\tau>T_i\}}|\mathcal{F}_t]}{\mathbf{Q}_{T_i}\{\tau>T_i|\mathcal{F}_t\}} \\
&= \mathbf{1}_{\{\tau>t\}}\overline{B}(t,T_i)\frac{\mathbb{E}_{\mathbf{P}_{T_i}}[X\mathbf{Q}_{T_i}\{\tau>T_i|\mathcal{F}_{T_i}\}|\mathcal{F}_t]}{\mathbf{Q}_{T_i}\{\tau>T_i|\mathcal{F}_t\}} \\
&= \mathbf{1}_{\{\tau>t\}}\overline{B}(t,T_i)\mathbb{E}_{\overline{\mathbf{P}}_{T_i}}[X|\mathcal{F}_t].
\end{aligned}
$$

We used Bielecki and Rutkowski (2002, (5.2)) for the second equality, equation (20) for the third and the abstract Bayes rule for the last equality. □

5 Recovery rules and bond prices

In the previous sections we specified the evolution of (ratios of pre-default values of) defaultable zero coupon bonds with zero recovery. In real markets however, defaultable bonds usually have a positive recovery. In order to adapt our model to this fact, we have to incorporate suitable recovery rules for bonds. An overview on different kinds of recovery rules can be found in Bielecki and Rutkowski (2002) and Schönbucher (2003).

In default-free interest rate models, a coupon bearing bond can be considered as a portfolio of zero coupon bonds. For defaultable coupon bonds the situation is not quite as simple. A coupon bond can still be decomposed into a series of zero coupon bonds, but it does not make much sense to assume the same recovery rate π for all. The claim of a creditor on the defaulted debtor's assets is only determined by the outstanding principal and accrued interest payments of the defaulted loan or bond, any future coupon payments do not enter the consideration. We use the following recovery scheme for coupon bearing bonds:

Assumption (recovery of par). *The recovery of a defaultable coupon bond that defaults in the time interval $(T_k, T_{k+1}]$ is given by the recovery rate $\pi \in [0, 1)$ times the sum of the notional and the accrued interest over $(T_k, T_{k+1}]$. It is paid at T_{k+1}.*

Note that this assumption restricts recovery payments to the tenor dates. This restriction is not strong for a number of reasons. We refer to Schönbucher (1999, Section 6.2) for a discussion.

Let us denote by $e_k^X(t)$ the time-t value of receiving an amount of X at T_{k+1} if and only if a default occurred in the time interval $(T_k, T_{k+1}]$.

Lemma 8 *Let X be $\mathcal{F}_{T_k}$-measurable. Then, for $t \le T_k$*

$$e_k^X(t) = \mathbf{1}_{\{\tau>t\}} \overline{B}(t, T_{k+1}) \delta_k \mathrm{E}_{\overline{\mathbf{P}}_{T_{k+1}}} [X H(T_k, T_k)|\mathcal{F}_t].$$

Proof: We have

$$e_k^X(T_{k+1}) = X\mathbf{1}_{\{\tau>T_k\}} - X\mathbf{1}_{\{\tau>T_{k+1}\}}.$$

Receiving an amount of $X\mathbf{1}_{\{\tau>T_k\}}$ at T_{k+1} is equivalent to receiving an amount of $X\mathbf{1}_{\{\tau>T_k\}} B(T_k, T_{k+1})$ at T_k. Combining this fact with proposition 7 yields for $t \le T_k$

$$\begin{aligned} e_k^X(t) &= \mathbf{1}_{\{\tau>t\}} \Big(\overline{B}(t, T_k) \mathrm{E}_{\overline{\mathbf{P}}_{T_k}} [X B(T_k, T_{k+1})|\mathcal{F}_t] - \overline{B}(t, T_{k+1}) \mathrm{E}_{\overline{\mathbf{P}}_{T_{k+1}}} [X|\mathcal{F}_t] \Big) \\ &= \mathbf{1}_{\{\tau>t\}} \overline{B}(t, T_{k+1}) \\ &\quad \times \Big(\mathrm{E}_{\overline{\mathbf{P}}_{T_{k+1}}} [(1 + \delta_k \overline{L}(T_k, T_k)) X B(T_k, T_{k+1})|\mathcal{F}_t] - \mathrm{E}_{\overline{\mathbf{P}}_{T_{k+1}}} [X|\mathcal{F}_t] \Big) \\ &= \mathbf{1}_{\{\tau>t\}} \overline{B}(t, T_{k+1}) \delta_k \mathrm{E}_{\overline{\mathbf{P}}_{T_{k+1}}} [X H(T_k, T_k)|\mathcal{F}_t]. \end{aligned}$$

The second equality follows from the abstract Bayes rule, the third follows by using equation (13). □

With the help of the preceding lemma we can deduce the time-0 price of a defaultable coupon bond with m coupons of c that are promised to be paid at the dates

$T_1, \ldots, T_m$ as

$$\begin{aligned} B^{\pi}_{\text{fixed}}(0;c,m) &:= \overline{B}(0,T_m) + \sum_{k=0}^{m-1} c\overline{B}(0,T_{k+1}) + \sum_{k=0}^{m-1} \pi(1+c)e^1_k(0) \\ &= \overline{B}(0,T_m) + \sum_{k=0}^{m-1} \overline{B}(0,T_{k+1}) \Big(c + \pi(1+c)\delta_k \mathbb{E}_{\overline{\mathbf{P}}_{T_{k+1}}}[H(T_k,T_k)]\Big). \end{aligned}$$

Similarly, the price of a defaultable floating coupon bond that pays an interest rate composed of the default-free Libor rate plus a constant spread x can be obtained. In order to price defaultable fixed coupon bonds we need to evaluate $\mathbb{E}_{\overline{\mathbf{P}}_{T_{k+1}}}[H(T_k,T_k)]$: Let us use the abbreviations

$$V^i_t := \frac{\delta_i L(t,T_i)}{1+\delta_i L(t,T_i)} \qquad \text{and} \qquad Y^i_t := \frac{\delta_i H(t,T_i)}{1+\delta_i H(t,T_i)}.$$

Combining the equations (15), (38), (25), and (26) yields

$$\begin{aligned} H(t,T_k) &= H(0,T_k)\exp\bigg(\int_0^t \overline{b}^H(s,T_k,T_{k+1})\,\mathrm{d}s + \int_0^t \sqrt{c_s}\gamma(s,T_k)\,\mathrm{d}\overline{W}^{T_{k+1}}_s \\ &\qquad + \int_0^t\int_{\mathbb{R}^d} \langle\gamma(s,T_k),x\rangle(\mu - \overline{\nu}^{T_{k+1}})(\mathrm{d}s,\mathrm{d}x)\bigg), \end{aligned}$$

where

$$\begin{aligned} \overline{b}^H(s,T_k,T_{k+1}) &= -\frac{1}{2}\langle\gamma(s,T_k),c_s\gamma(s,T_k)\rangle + \sum_{l=1}^{k-1} \frac{Y^l_{s-}V^k_{s-}}{Y^k_{s-}}\langle\gamma(s,T_l),c_s\lambda(s,T_k)\rangle \\ &\quad - \int_{\mathbb{R}^d}\Big(e^{\langle\gamma(s,T_k),x\rangle} - 1 - \langle\gamma(s,T_k),x\rangle\Big)\overline{F}^{T_{k+1}}_s(\mathrm{d}x) \\ &\quad + \int_{\mathbb{R}^d}\frac{V^k_{s-}}{Y^k_{s-}}\Big(e^{\langle\lambda(s,T_k),x\rangle}-1\Big)\Big(1+Y^k_{s-}\Big(e^{\langle\gamma(s,T_k),x\rangle}-1\Big)\Big) \\ &\qquad \times\left(\prod_{l=1}^{k-1}\Big(1+Y^l_{s-}\Big(e^{\langle\gamma(s,T_l),x\rangle}-1\Big)\Big)-1\right)\overline{F}^{T_{k+1}}_s(\mathrm{d}x). \end{aligned}$$

Making use of Kallsen and Shiryaev (2002, Lemma 2.6) we get

$$\begin{aligned} &H(t,T_k) = \qquad\qquad (27) \\ &\quad H(0,T_k)\exp\left(\int_0^t\sum_{l=1}^{k-1}\frac{Y^l_{s-}V^k_{s-}}{Y^k_{s-}}\langle\gamma(s,T_l),c_s\lambda(s,T_k)\rangle\,\mathrm{d}s\right. \\ &\quad + \int_0^t\int_{\mathbb{R}^d}\frac{V^k_{s-}}{Y^k_{s-}}\Big(e^{\langle\lambda(s,T_k),x\rangle}-1\Big)\Big(1+Y^k_{s-}\Big(e^{\langle\gamma(s,T_k),x\rangle}-1\Big)\Big) \\ &\qquad \left.\times\left(\prod_{l=1}^{k-1}\Big(1+Y^l_{s-}\Big(e^{\langle\gamma(s,T_l),x\rangle}-1\Big)\Big)-1\right)\overline{\nu}^{T_{k+1}}(\mathrm{d}s,\mathrm{d}x)\right) \\ &\quad \times\mathcal{E}_t\bigg(\int_0^\bullet\sqrt{c_s}\gamma(s,T_k)\,\mathrm{d}\overline{W}^{T_{k+1}}_s + \int_0^\bullet\int_{\mathbb{R}^d}\Big(e^{\langle\gamma(s,T_k),x\rangle}-1\Big)(\mu-\overline{\nu}^{T_{k+1}})(\mathrm{d}s,\mathrm{d}x)\bigg). \end{aligned}$$

To obtain an expression for $\mathbb{E}_{\overline{\mathbf{P}}_{T_{k+1}}}[H(T_k,T_k)]$ we approximate the stochastic terms V_{s-}^i and Y_{s-}^i by their deterministic initial values V_0^i and Y_0^i. Similar approximations have been used by Brace, Gatarek, and Musiela (1997), Rebonato (1998), and Schlögl (2002). This yields

$$\begin{aligned}\mathbb{E}_{\overline{\mathbf{P}}_{T_{k+1}}}[H(T_k,T_k)] \;\approx\; & H(0,T_k)\exp\Bigg(\int_0^{T_k}\sum_{l=1}^{k-1}\frac{Y_0^l V_0^k}{Y_0^k}\langle\gamma(s,T_l),c_s\lambda(s,T_k)\rangle\,\mathrm{d}s \\ & +\int_0^{T_k}\int_{\mathbb{R}^d}\frac{V_0^k}{Y_0^k}\left(e^{\langle\lambda(s,T_k),x\rangle}-1\right)\left(1+Y_0^k\left(e^{\langle\gamma(s,T_k),x\rangle}-1\right)\right) \\ & \times\left(\prod_{l=1}^{k-1}\left(1+Y_0^l\left(e^{\langle\gamma(s,T_l),x\rangle}-1\right)\right)-1\right)\widetilde{\nu}^{T_{k+1}}(\mathrm{d}s,\mathrm{d}x)\Bigg),\end{aligned}$$

where $\widetilde{\nu}^{T_{k+1}}$ is an approximation for $\overline{\nu}^{T_{k+1}}$ given by

$$\begin{aligned}\widetilde{\nu}^{T_{k+1}}(\mathrm{d}s,\mathrm{d}x) \;=\; & \prod_{l=1}^{k}\left(1+Y_0^l\left(e^{\langle\gamma(s,T_l),x\rangle}-1\right)\right)^{-1} \\ & \times\prod_{l=k+1}^{n-1}\left(1+V_0^l\left(e^{\langle\lambda(s,T_l),x\rangle}-1\right)\right)\nu^{T^*}(\mathrm{d}s,\mathrm{d}x).\end{aligned} \tag{28}$$

6 Credit default swaps

The market for credit derivatives has increased enormously in volume since the first of these contracts have been introduced in the early 1990s. Their success is due to the fact that they allow to transfer credit risk from one party to another and therewith to manage the risk exposure. There are many publications describing various credit derivatives in detail, among which are Schönbucher (2003) and Bielecki and Rutkowski (2002). Information about the size of the credit derivatives' market as well as on the market share that different products have can be found in the credit derivatives survey of Patel (2003).

The aim of this section is to derive valuation formulae, in our model framework, for the most popular and heavily traded credit derivative: the credit default swaps. Credit default swaps are natural calibration instruments for the term structure of forward default intensities. Thus, the availability of an efficient and accurate pricing formula is of high value for the practical implementation of this model.

In the following section we also provide valuation formulae for the most popular spread volatility dependent credit derivative: The credit default swaption. While being much less liquid than the standard credit default swap, the value of this contract depends directly on the specification of the *volatility function* of the term structure of forward default intensities. Thus, given the necessary data these results can be used to calibrate the dynamics of the model to a given set of credit default swaptions. Alternatively, they serve at least as a case study regarding the pricing of an important spread volatility-dependent credit derivative.

Clearly, the employed valuation techniques can also be used to price other credit-sensitive swap contracts (e.g. total rate of return swaps and asset swaps) or other credit options (e.g. options on defaultable bonds and credit spread options). For more details we refer to Kluge (2005).

We use the notational convention that the credit derivative contract is signed between two parties A (who will usually receive a payment if a default occurs) and B (who pays in case of a default). The reference entity (e.g. a corporate bond) is issued by a third party C. If credit derivatives are traded over-the-counter, each party of the contract is exposed to the risk that the other party cannot fulfill its obligations. In the following, we assume that this counterparty risk can be neglected, i.e. only the risk that the reference entity defaults is considered.

Credit default swaps can be used to insure defaultable assets against default. The protection buyer A agrees to pay a fixed amount to the protection seller B periodically until a pre-specified credit event (e.g. the default of a bond issued by a reference party C) occurs or the contract terminates. In turn, B promises to make a specified payment to A that covers his loss if the credit event happens. There are various types of default swaps differing in the specification of the credit event as well as in the specification of the default payment.

Let us consider a standard default swap with maturity date T_m whose credit event is the default of a fixed coupon bond issued by C. The default payment is chosen such that it covers the loss of A. More precisely, A receives an amount of $1 - \pi(1+c)$ at T_{k+1} if a default happens in $(T_k, T_{k+1}]$ for $k \in \{0, \dots, m-1\}$. For this protection A pays a fee s at the dates $T_0, \dots, T_{m-1}$ until default.[1] Our goal is to determine the default swap rate, i.e. the level of s that makes the initial value of the contract equal to zero.
The time-0 value of the fee payments is

$$s \sum_{k=1}^{m} \overline{B}(0, T_{k-1}).$$

The initial value of the default payment equals

$$\sum_{k=1}^{m} (1 - \pi(1+c)) e_{k-1}^{1}(0).$$

Consequently, the default swap rate is

$$s = \frac{1 - \pi(1+c)}{\sum_{k=1}^{m} \overline{B}(0, T_{k-1})} \sum_{k=1}^{m} \Big(\overline{B}(0, T_k) \delta_{k-1} \mathbb{E}_{\overline{\mathbf{P}}_{T_k}} [H(T_{k-1}, T_{k-1})] \Big).$$

The expectations in the equation can be obtained as in the previous section.

7 Credit default swaptions

The purpose of this section is to price credit default swaptions within our model framework under the following restriction on the volatility functions:

[1] CDS with fee payments in arrears, i.e. at the dates $T_1, \dots, T_m$, can be treated similarly by adjusting indices.

Assumption ($\mathbb{DLR}.\mathbb{VOL}$). *The volatility structures factorize in the following way: for* $i \in \{1, \ldots, n-1\}$

$$\lambda(s, T_i) = \lambda_i \sigma(s) \qquad \text{and} \qquad \gamma(s, T_i) = \gamma_i \sigma(s) \qquad (0 \le s \le T_i)$$

where λ_i *and* γ_i *are positive constants and where* $\sigma : [0, T^*] \to \mathbb{R}^d_+$ *does not depend on* i.

This condition allows us to derive approximate pricing formulae that can numerically be evaluated fast. As in the previous section we neglect the counterparty risk.

A *credit default swaption* gives its holder the right to enter a credit default swap at some pre-specified time and swap rate. These options are often embedded in other credit derivatives (e.g. as an extension option in a credit default swap). For more details we refer to Schönbucher (1999).

Let us consider a credit default swaption that is knocked out at default with strike rate S and maturity T_i on a default swap that terminates at T_m ($i < m \le n$) with an underlying fixed coupon bond. Its time-T_i value is

$$\pi^{\mathrm{CDS}}_{T_i}(S, T_i, T_m) := \mathbf{1}_{\{\tau > T_i\}} \left((s(T_i; T_i, T_m) - S)^+ \sum_{k=i}^{m-1} \overline{B}(T_i, T_k) \right),$$

where $s(T_i; T_i, T_m)$ denotes the default swap rate at time T_i. Note that

$$\begin{aligned}\mathbf{1}_{\{\tau > T_i\}} s(T_i, T_i, T_m) \sum_{k=i}^{m-1} \overline{B}(T_i, T_k) = \\ \mathbf{1}_{\{\tau > T_i\}} (1 - \pi(1+c)) \sum_{k=i}^{m-1} \Big(\overline{B}(T_i, T_{k+1}) \delta_k \mathbb{E}_{\overline{\mathbf{P}}_{T_{k+1}}}[H(T_k, T_k) | \mathcal{F}_{T_i}] \Big).\end{aligned}$$

Proposition 7 yields[2]

$$\begin{aligned}\pi^{\mathrm{CDS}}_0 &:= \pi^{\mathrm{CDS}}_0(S, T_i, T_m) \\ &= \overline{B}(0, T_i) \mathbb{E}_{\overline{\mathbf{P}}_{T_i}} \Bigg[\Bigg((1 - \pi(1+c)) \sum_{k=i}^{m-1} \Big(\overline{B}(T_i, T_{k+1}) \delta_k \mathbb{E}_{\overline{\mathbf{P}}_{T_{k+1}}}[H(T_k, T_k) | \mathcal{F}_{T_i}] \Big) \\ &\qquad\qquad - S \sum_{k=i}^{m-1} \overline{B}(T_i, T_k) \Bigg)^+ \Bigg].\end{aligned}$$

As before, we approximate the stochastic terms V^i_{s-} and Y^i_{s-} in (27) by their deterministic initial values V^i_0 and Y^i_0 and obtain

$$\mathbb{E}_{\overline{\mathbf{P}}_{T_{k+1}}}[H(T_k, T_k) | \mathcal{F}_{T_i}] \approx C^{i,k} H(T_i, T_k)$$

[2] Alternatively, $\sum_{k=i}^{m-1} \overline{B}(T_i, T_k)$ can be taken as numeraire for a new probability measure, the *default swap measure* (compare Schönbucher (1999)). Whereas this can be useful for driving Brownian motions, it does not facilitate calculations for general Lévy processes.

with

$$
\begin{aligned}
C^{i,k} \quad := \quad & \exp\Bigg(\int_{T_i}^{T_k} \sum_{l=1}^{k-1} \frac{Y_0^l V_0^k}{Y_0^k} \langle \gamma(s,T_l), c_s \lambda(s,T_k)\rangle \, \mathrm{d}s \\
& + \int_{T_i}^{T_k} \int_{\mathbb{R}^d} \frac{V_0^k}{Y_0^k} \left(e^{\langle \lambda(s,T_k),x\rangle} - 1\right) \left(1 + Y_0^k \left(e^{\langle \gamma(s,T_k),x\rangle} - 1\right)\right) \\
& \times \left(\prod_{l=1}^{k-1} \left(1 + Y_0^l \left(e^{\langle \gamma(s,T_l),x\rangle} - 1\right)\right) - 1\right) \widetilde{\nu}^{T_{k+1}}(\mathrm{d}s,\mathrm{d}x) \Bigg)
\end{aligned}
$$

and $\widetilde{\nu}^{T_{k+1}}$ given by (28). Consequently,

$$
\begin{aligned}
\pi_0^{\mathrm{CDS}} &= \overline{B}(0,T_i)\mathbb{E}_{\overline{\mathbf{P}}_{T_i}}\Bigg[\Bigg((1-\pi(1+c)) \sum_{k=i}^{m-1} \left(\overline{B}(T_i,T_{k+1})\delta_k C^{i,k} H(T_i,T_k)\right) \\
&\qquad - S\sum_{k=i}^{m-1} \overline{B}(T_i,T_k) \Bigg)^+ \Bigg] \\
&= \overline{B}(0,T_i)\mathbb{E}_{\overline{\mathbf{P}}_{T_i}}\Bigg[\Bigg(\frac{(1-\pi(1+c))\delta_{m-1}C^{i,m-1}H(T_i,T_{m-1})}{\prod_{l=i}^{m-1}(1+\delta_l L(T_i,T_l))(1+\delta_l H(T_i,T_l))} \\
&\qquad + \sum_{k=i}^{m-2} \frac{(1-\pi(1+c))\delta_k C^{i,k} H(T_i,T_k) - S}{\prod_{l=i}^{k}(1+\delta_l L(T_i,T_l))(1+\delta_l H(T_i,T_l))} - S \Bigg)^+ \Bigg].
\end{aligned}
$$

To evaluate the preceding expression we use Laplace transformation methods due to Raible (2000). For this purpose, we derive a convolution representation of the swaption price first.

Combining the equations (4), (6), (15), and (38) with (9) – (12) and (25) – (26) yields for $l \in \{i,\ldots,n-1\}$

$$
\begin{aligned}
L(T_i,T_l) &= L(0,T_l)\exp\left(\int_0^{T_i} \bar{b}^L(s,T_l,T_i)\,\mathrm{d}s + \int_0^{T_i} \lambda(s,T_l)\,\mathrm{d}\overline{L}_s^{T_i} \right), \\
H(T_i,T_l) &= H(0,T_l)\exp\left(\int_0^{T_i} \bar{b}^H(s,T_l,T_i)\,\mathrm{d}s + \int_0^{T_i} \gamma(s,T_l)\,\mathrm{d}\overline{L}_s^{T_i} \right)
\end{aligned}
$$

with

$$
\overline{L}_t^{T_i} := \int_0^t \sqrt{c_s}\,\mathrm{d}\overline{W}_s^{T_i} + \int_0^t \int_{\mathbb{R}^d} x(\mu - \overline{\nu}^{T_i})(\mathrm{d}s,\mathrm{d}x)
$$

and

$$
\begin{aligned}
&\bar{b}^L(s,T_l,T_i) = \\
&\quad \left\langle \sum_{j=i}^{l} V_{s-}^j \lambda(s,T_j) - \sum_{j=1}^{i-1} Y_{s-}^j \gamma(s,T_j), c_s\lambda(s,T_l) \right\rangle - \frac{1}{2}\langle \lambda(s,T_l), c_s\lambda(s,T_l)\rangle \\
&\quad - \int_{\mathbb{R}^d} \Bigg[\left(e^{\langle \lambda(s,T_l),x\rangle} - 1\right) \prod_{j=l+1}^{n-1} \left(1 + V_{s-}^j \left(e^{\langle \lambda(s,T_j),x\rangle} - 1\right)\right) \\
&\qquad - \langle \lambda(s,T_l), x\rangle \frac{\prod_{j=i}^{n-1}\left(1 + V_{s-}^j\left(e^{\langle \lambda(s,T_j),x\rangle} - 1\right)\right)}{\prod_{j=1}^{i-1}\left(1 + Y_{s-}^j\left(e^{\langle \gamma(s,T_j),x\rangle} - 1\right)\right)} \Bigg] F_s^{T^*}(\mathrm{d}x)
\end{aligned}
$$

as well as

$$\begin{aligned}
&\overline{b}^{H}(s,T_l,T_i) = \\
&\quad \sum_{j=i}^{l} \langle Y^j_{s-}\gamma(s,T_j) + V^j_{s-}\lambda(s,T_j), c_s\gamma(s,T_l)\rangle - \frac{1}{2}\langle \gamma(s,T_l), c_s\gamma(s,T_l)\rangle \\
&\quad + \sum_{j=1}^{l-1}\left(\frac{Y^j_{s-}V^l_{s-}}{Y^l_{s-}}\langle \gamma(s,T_j), c_s\lambda(s,T_l)\rangle\right) \\
&\quad + \int_{\mathbb{R}^d}\left[\langle \gamma(s,T_l), x\rangle \frac{\prod_{j=i}^{n-1}\left(1+V^j_{s-}\left(e^{\langle \lambda(s,T_j),x\rangle}-1\right)\right)}{\prod_{j=1}^{i-1}\left(1+Y^j_{s-}\left(e^{\langle \gamma(s,T_j),x\rangle}-1\right)\right)}\right. \\
&\qquad\qquad \left. - \left(e^{\langle \gamma(s,T_l),x\rangle}-1\right)\frac{\prod_{j=l+1}^{n-1}\left(1+V^j_{s-}\left(e^{\langle \lambda(s,T_j),x\rangle}-1\right)\right)}{\prod_{j=1}^{l}\left(1+Y^j_{s-}\left(e^{\langle \gamma(s,T_j),x\rangle}-1\right)\right)}\right] F^{T^*}_s(\mathrm{d}x) \\
&\quad + \int_{\mathbb{R}^d}\left[\frac{V^l_{s-}}{Y^l_{s-}}\left(e^{\langle \lambda(s,T_l),x\rangle}-1\right)\prod_{j=l+1}^{n-1}\left(1+V^j_{s-}\left(e^{\langle \lambda(s,T_j),x\rangle}-1\right)\right)\right. \\
&\qquad\qquad \left. \times\left(1-\prod_{j=1}^{l-1}\left(1+Y^j_{s-}\left(e^{\langle \gamma(s,T_j),x\rangle}-1\right)\right)^{-1}\right)\right] F^{T^*}_s(\mathrm{d}x).
\end{aligned}$$

Again, we approximate the stochastic terms V^k_{s-} and Y^k_{s-} in the drift terms $\overline{b}^{L}(s,T_l,T_i)$ and $\overline{b}^{H}(s,T_l,T_i)$ by their initial values and call the resulting (deterministic) drifts $\overline{b}^{L}_0(s,T_l,T_i)$ and $\overline{b}^{H}_0(s,T_l,T_i)$ respectively. Then, due to the assumption on the volatility structure, we get

$$\begin{aligned}
L(T_i,T_l) &\approx L(0,T_l)\exp\left(\frac{\lambda_l}{\sigma_{\text{sum}}}X_{T_i} + B^L_l\right), \\
H(T_i,T_l) &\approx H(0,T_l)\exp\left(\frac{\gamma_l}{\sigma_{\text{sum}}}X_{T_i} + B^H_l\right)
\end{aligned}$$

with

$$\begin{aligned}
\sigma_{\text{sum}} &:= \sum_{l=i}^{m-1}(\lambda_l+\gamma_l), \\
X_{T_i} &:= \int_0^{T_i}\sum_{l=i}^{m-1}(\lambda(s,T_l)+\gamma(s,T_l))\,\mathrm{d}\overline{L}^{T_i}_s = \sigma_{\text{sum}}\int_0^{T_i}\sigma(s)\,\mathrm{d}\overline{L}^{T_i}_s,
\end{aligned}$$

and

$$B^L_l := \int_0^{T_i}\overline{b}^{L}_0(s,T_l,T_i)\,\mathrm{d}s, \qquad B^H_l := \int_0^{T_i}\overline{b}^{H}_0(s,T_l,T_i)\,\mathrm{d}s.$$

The price of the credit default swaption now depends on the distribution of one random variable only, namely on the distribution of X_{T_i} with respect to $\overline{\mathbf{P}}_{T_i}$. Assume that this distribution possesses a Lebesgue-density φ (we refer to Eberlein and Kluge (2006) for a discussion on this assumption). The option price can then be written as a convolution, namely

$$\pi^{\text{CDS}}_0 = \overline{B}(0,T_i)\int_{\mathbb{R}} g(-x)\varphi(x)\,\mathrm{d}x = \overline{B}(0,T_i)(g*\varphi)(0) \tag{29}$$

with $g(x) := (v(x))^+$ and

$$
\begin{aligned}
v(x) \quad := \quad & \frac{(1-\pi(1+c))\delta_{m-1}C^{i,m-1}H(0,T_{m-1})\exp\big(-\frac{\gamma_{m-1}}{\sigma_{\text{sum}}}x+B_{m-1}^H\big)}{\prod_{l=i}^{m-1}\big(1+\delta_l L(0,T_l)\exp\big(-\frac{\lambda_l}{\sigma_{\text{sum}}}x+B_l^L\big)\big)} \\
& \times \frac{1}{\prod_{l=i}^{m-1}\big(1+\delta_l H(0,T_l)\exp\big(-\frac{\gamma_l}{\sigma_{\text{sum}}}x+B_l^H\big)\big)} \\
& + \sum_{k=i}^{m-2}\Bigg(\frac{(1-\pi(1+c))\delta_k C^{i,k}H(0,T_k)\exp\big(-\frac{\gamma_k}{\sigma_{\text{sum}}}x+B_k^H\big)-S}{\prod_{l=i}^{k}\big(1+\delta_l L(0,T_l)\exp\big(-\frac{\lambda_l}{\sigma_{\text{sum}}}x+B_l^L\big)\big)} \\
& \times \frac{1}{\prod_{l=i}^{k}\big(1+\delta_l H(0,T_l)\exp\big(-\frac{\gamma_l}{\sigma_{\text{sum}}}x+B_l^H\big)\big)}\Bigg) - S.
\end{aligned}
$$

The next step is to determine for which values the bilateral Laplace transform of g exists. Note that v is continuous, tends to $-S$ as $x \to -\infty$ and to $-(m-i)S$ as $x \to \infty$. Consequently, g has compact support and the bilateral Laplace transform of g exists for all $z \in \mathbb{C}$. In a numerical evaluation of v, for large values of $m-i$, we can save computational time by applying the multiplication scheme

$$
\sum_{k=i}^{m-2} c_k \prod_{l=i}^{k} d_l = d_i(c_i + d_{i+1}(c_{i+1} + d_{i+1}(\ldots(c_{m-3} + d_{m-2}c_{m-2})))).
$$

Putting pieces together, we obtain the following formula for the price of the credit default swaption:

Proposition 9 *Suppose that the distribution of X_{T_i} possesses a Lebesgue-density. Denote by $\overline{M}_{T_i}^{X_{T_i}}$ the $\overline{\mathbf{P}}_{T_i}$-moment generating function of X_{T_i}. Choose an $R \in \mathbb{R}$ such that $\overline{M}_{T_i}^{X_{T_i}}(-R) < \infty$ (e.g. $R = 0$). Then the price of the credit default swaption is approximately given by*

$$
\pi_0^{CDS}(K, T_i, T_m) = \overline{B}(0, T_i)\frac{1}{\pi}\int_0^\infty \Re\Big(L[g](R + \mathrm{i}u)\overline{M}_{T_i}^{X_{T_i}}(-R - \mathrm{i}u)\Big)\,\mathrm{d}u, \tag{30}
$$

where $L[g]$ denotes the bilateral Laplace transform of g. Furthermore, we have for $z \in \mathbb{C}$ with real part $\Re z = -R$

$$
\begin{aligned}
\overline{M}_{T_i}^{X_{T_i}}(z) \quad \approx \quad & \exp\Bigg(z^2\sigma_{sum}^2\int_0^{T_i}\langle\sigma(s), c_s\sigma(s)\rangle\,\mathrm{d}s \qquad (31) \\
& + \int_0^{T_i}\int_{\mathbb{R}^d}\Big(e^{z\sigma_{sum}\langle\sigma(s),x\rangle} - 1 - z\sigma_{sum}\langle\sigma(s),x\rangle\Big)\widetilde{\nu}^{T_i}(\mathrm{d}s,\mathrm{d}x)\Bigg)
\end{aligned}
$$

with

$$
\widetilde{\nu}^{T_i}(\mathrm{d}s,\mathrm{d}x) := \frac{\prod_{l=i}^{n-1}\big(1 + V_0^l\big(e^{\langle\lambda(s,T_l),x\rangle} - 1\big)\big)}{\prod_{l=1}^{i-1}\big(1 + Y_0^l\big(e^{\langle\gamma(s,T_l),x\rangle} - 1\big)\big)}\nu^{T^*}(\mathrm{d}s,\mathrm{d}x). \tag{32}
$$

Proof: Using the convolution representation (29) and performing Laplace and inverse Laplace transformations (compare the proof of Theorem 4.1 in Eberlein and

Kluge (2006)), we arrive at (30). It remains to derive an expression for the moment generating function:

$$\begin{aligned}\overline{M}_{T_i}^{X_{T_i}}(z) &= \mathbb{E}_{\overline{\mathbf{P}}_{T_i}}\left[\exp\left(z\sigma_{\text{sum}}\int_0^{T_i}\sigma(s)\,\mathrm{d}\overline{W}_s^{T_i}\right.\right.\\ &\qquad \left.\left.+z\sigma_{\text{sum}}\int_0^{T_i}\int_{\mathbb{R}^d}\langle\sigma(s),x\rangle(\mu-\overline{\nu}^{T_i})(\mathrm{d}s,\mathrm{d}x)\right)\right]\end{aligned}$$

with

$$\overline{\nu}^{T_i}(\mathrm{d}s,\mathrm{d}x)=\frac{\prod_{l=i}^{n-1}\left(1+V_{s-}^l\left(e^{\langle\lambda(s,T_l),x\rangle}-1\right)\right)}{\prod_{l=1}^{i-1}\left(1+Y_{s-}^l\left(e^{\langle\gamma(s,T_l),x\rangle}-1\right)\right)}\nu^{T^*}(\mathrm{d}s,\mathrm{d}x).$$

We approximate the random compensator $\overline{\nu}^{T_i}$ by the non-random compensator given in (32). Expression (31) then follows (e.g. by using Kluge (2005, Proposition 1.9)). □

A Specification of the drift

We specify the drift recursively starting with $b(\cdot,T_1,T_2)$. More precisely, we look for a process $b(\cdot,T_1,T_2)$ such that $\left((1+\delta_1\widehat{H}(t,T_1))^{-1}\right)_{0\le t\le T_1}$ becomes a $\mathbf{Q}_{T_2}$-martingale. Next, $b(\cdot,T_2,T_3)$ is specified such that $\left(\left((1+\delta_1\widehat{H}(t,T_1))(1+\delta_2\widehat{H}(t,T_2))\right)^{-1}\right)_{0\le t\le T_2}$ becomes a $\mathbf{Q}_{T_3}$-martingale, and so on. Let us begin with two lemmata:

Lemma 10 *Let X be a real-valued semimartingale with $X_0=0$ and $\Delta X>-1$. Then*

$$(\mathcal{E}(X))^{-1}=\mathcal{E}\left(-X+\langle X^c,X^c\rangle+\left(\frac{1}{1+x}-1+x\right)*\mu^X\right).$$

Proof: Use Lemma 2.6 in Kallsen and Shiryaev (2002) twice plus the fact that $(\exp X)^{-1}=\exp(-X)$. □

Lemma 11 *For $k\in\{2,\dots,n\}$ and $i\in\{1,\dots,k-1\}$*

$$\begin{aligned}\widehat{H}(t,T_i) &= H(0,T_i)\mathcal{E}_t\Big(\int_0^\bullet a(s,T_i,T_k)\,\mathrm{d}s+\int_0^\bullet\sqrt{c_s}\gamma(s,T_i)\,\mathrm{d}W_s^{T_k}\\ &\qquad +\int_0^\bullet\int_{\mathbb{R}^d}\left(e^{\langle\gamma(s,T_i),x\rangle}-1\right)(\mu-\nu^{T_k})(\mathrm{d}s,\mathrm{d}x)\Big),\end{aligned}$$

where

$$\begin{aligned}a(s,T_i,T_k) &:= b^H(s,T_i,T_k)+\frac{1}{2}\langle\gamma(s,T_i),c_s\gamma(s,T_i)\rangle\\ &\qquad +\int_{\mathbb{R}^d}\left(e^{\langle\gamma(s,T_i),x\rangle}-1-\langle\gamma(s,T_i),x\rangle\right)F_s^{T_k}(\mathrm{d}x).\end{aligned}\tag{33}$$

and $b^H(s,T_i,T_k)$ is given by (34).

PROOF: From the default-free part of the model we know that

$$W_t^{T_{i+1}} = W_t^{T_k} - \int_0^t \sqrt{c_s} \left(\sum_{l=i+1}^{k-1} \alpha(s, T_l, T_{l+1}) \right) \mathrm{d}s$$

and

$$\nu^{T_{i+1}}(\mathrm{d}t, \mathrm{d}x) = \left(\prod_{l=i+1}^{k-1} \beta(s, x, T_l, T_{l+1}) \right) \nu^{T_k}(\mathrm{d}t, \mathrm{d}x)$$

with α and β given by (10) and (12). Consequently, equation (15) implies

$$\begin{aligned} \widehat{H}(t, T_i) &= H(0, T_i) \exp\Bigg(\int_0^t b^H(s, T_i, T_k)\, \mathrm{d}s + \int_0^t \sqrt{c_s} \gamma(s, T_i)\, \mathrm{d}W_s^{T_k} \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \langle \gamma(s, T_i), x \rangle (\mu - \nu^{T_k})(\mathrm{d}s, \mathrm{d}x) \Bigg), \end{aligned}$$

where

$$\begin{aligned} b^H(s, T_i, T_k) &:= b^H(s, T_i, T_{i+1}) \\ &\quad - \left\langle \gamma(s, T_i), c_s \left(\sum_{l=i+1}^{k-1} \alpha(s, T_l, T_{l+1}) \right) \right\rangle \\ &\quad - \int_{\mathbb{R}^d} \langle \gamma(s, T_i), x \rangle \left(\prod_{l=i+1}^{k-1} \beta(s, x, T_l, T_{l+1}) - 1 \right) F_s^{T_k}(\mathrm{d}x). \end{aligned} \tag{34}$$

The claim now follows from Kallsen and Shiryaev (2002, Lemma 2.6). □

Proposition 12 $\left(\frac{1}{1+\delta_1 \widehat{H}(t, T_1)} \right)_{0 \le t \le T_1}$ *is a* $\mathbf{Q}_{T_2}$*-martingale if for* $s \in [0, T_1]$

$$\begin{aligned} b^H(s, T_1, T_2) &= \left(Y_{s-}^1 - \frac{1}{2} \right) \langle \gamma(s, T_1), c_s \gamma(s, T_1) \rangle \\ &\quad + \int_{\mathbb{R}^d} \left(\langle \gamma(s, T_1), x \rangle - \frac{e^{\langle \gamma(s, T_1), x \rangle} - 1}{1 + Y_{s-}^1 \left(e^{\langle \gamma(s, T_1), x \rangle} - 1 \right)} \right) F_s^{T_2}(\mathrm{d}x), \end{aligned} \tag{35}$$

where $Y_s^1 := \frac{\delta_1 \widehat{H}(s, T_1)}{1+\delta_1 \widehat{H}(s, T_1)}$.

PROOF: Lemma 11 gives us

$$\begin{aligned} \widehat{H}(t, T_1) &= H(0, T_1) \mathcal{E}_t \Bigg(\int_0^\bullet a(s, T_1, T_2)\, \mathrm{d}s + \int_0^\bullet \sqrt{c_s} \gamma(s, T_1)\, \mathrm{d}W_s^{T_2} \\ &\quad + \int_0^\bullet \int_{\mathbb{R}^d} \left(e^{\langle \gamma(s, T_1), x \rangle} - 1 \right) (\mu - \nu^{T_2})(\mathrm{d}s, \mathrm{d}x) \Bigg) \end{aligned}$$

with

$$\begin{aligned} a(s, T_1, T_2) &= b^H(s, T_1, T_2) + \frac{1}{2} \langle \gamma(s, T_1), c_s \gamma(s, T_1) \rangle \\ &\quad + \int_{\mathbb{R}^d} \left(e^{\langle \gamma(s, T_1), x \rangle} - 1 - \langle \gamma(s, T_1), x \rangle \right) F_s^{T_2}(\mathrm{d}x). \end{aligned} \tag{36}$$

Consequently, for $X_t^1 := 1 + \delta_1 \widehat{H}(t, T_1)$ we have

$$\begin{aligned} \mathrm{d}X_t^1 &= \delta_1 \, \mathrm{d}\widehat{H}(t, T_1) \\ &= X_{t-}^1 \Big(Y_{t-}^1 a(t, T_1, T_2) \, \mathrm{d}t + Y_{t-}^1 \sqrt{c_t} \gamma(t, T_1) \, \mathrm{d}W_t^{T_2} \\ &\qquad + \int_{\mathbb{R}^d} Y_{t-}^1 \left(e^{\langle \gamma(t,T_1), x \rangle} - 1 \right) (\mu - \nu^{T_2})(\mathrm{d}t, \mathrm{d}x) \Big). \end{aligned}$$

Lemma 10 implies

$$\begin{aligned} (X_t^1)^{-1} &= (X_0^1)^{-1} \mathcal{E}_t \Big(\int_0^\bullet A(s, T_2) \, \mathrm{d}s - \int_0^\bullet Y_{s-}^1 \sqrt{c_s} \gamma(s, T_1) \, \mathrm{d}W_s^{T_2} \\ &\quad + \int_0^\bullet \int_{\mathbb{R}^d} \left(\left(1 + Y_{s-}^1 \left(e^{\langle \gamma(s,T_1), x \rangle} - 1 \right) \right)^{-1} - 1 \right) (\mu - \nu^{T_2})(\mathrm{d}s, \mathrm{d}x) \Big), \end{aligned}$$

where

$$\begin{aligned} A(s, T_2) &:= - Y_{s-}^1 a(s, T_1, T_2) + (Y_{s-}^1)^2 \langle \gamma(s, T_1), c_s \gamma(s, T_1) \rangle \\ &\quad + \int_{\mathbb{R}^d} Y_{s-}^1 \left(e^{\langle \gamma(s,T_1), x \rangle} - 1 - \frac{e^{\langle \gamma(s,T_1), x \rangle} - 1}{1 + Y_{s-}^1 \left(e^{\langle \gamma(s,T_1), x \rangle} - 1 \right)} \right) F_s^{T_2}(\mathrm{d}x). \end{aligned} \tag{37}$$

Thus, $(1+\delta_1 \widehat{H}(\cdot, T_1))^{-1}$ is a $\mathbf{Q}_{T_2}$-local martingale if $A(\cdot, T_2) \equiv 0$. In this case it is also a martingale since it is bounded by 0 and 1 and therefore of class $[D]$ (compare Jacod and Shiryaev (2003, I.1.47c)). Combining $A(\cdot, T_2) \equiv 0$ with (36) and (37) yields (35). □

More generally, we get the following proposition:

Proposition 13 *$\left(\prod_{i=1}^{k-1} \frac{1}{1+\delta_i \widehat{H}(t,T_i)} \right)_{0 \le t \le T_{k-1}}$ is a martingale with respect to $\mathbf{Q}_{T_k}$ for $k \in \{2, \dots, n\}$ if for all $i \in \{1, \dots, k-1\}$ and $s \in [0, T_i]$*

$$\begin{aligned} & b^H(s, T_i, T_{i+1}) = \\ & \quad \sum_{j=1}^{i} Y_{s-}^j \langle \gamma(s, T_j), c_s \gamma(s, T_i) \rangle - \frac{1}{2} \langle \gamma(s, T_i), c_s \gamma(s, T_i) \rangle \\ & \quad + \sum_{j=1}^{i-1} \left(\frac{Y_{s-}^j}{Y_{s-}^i} \langle \gamma(s, T_j), c_s \alpha(s, T_i, T_{i+1}) \rangle \right) \\ & \quad + \int_{\mathbb{R}^d} \left(\langle \gamma(s, T_i), x \rangle - \frac{e^{\langle \gamma(s,T_i), x \rangle} - 1}{\prod_{j=1}^{i} \left(1 + Y_{s-}^j \left(e^{\langle \gamma(s,T_j), x \rangle} - 1 \right) \right)} \right) F_s^{T_{i+1}}(\mathrm{d}x) \\ & \quad + (Y_{s-}^i)^{-1} \int_{\mathbb{R}^d} (\beta(s, x, T_i, T_{i+1}) - 1) \\ & \qquad \times \left(1 - \prod_{j=1}^{i-1} \left(1 + Y_{s-}^j \left(e^{\langle \gamma(s,T_j), x \rangle} - 1 \right) \right)^{-1} \right) F_s^{T_{i+1}}(\mathrm{d}x), \end{aligned} \tag{38}$$

where $Y_s^i := \frac{\delta_i \widehat{H}(s,T_i)}{1+\delta_i \widehat{H}(s,T_i)}$.

PROOF: The proposition can be proved similarly as the previous one. Since this proof is calculationally intense, we omit it and refer to Kluge (2005, Proposition 4.8) for a complete proof. □

Note that we cannot just define $b^H(s, T_i, T_{i+1})$ by (38) since the term on the right hand side involves Y_s^i which depends on $\widehat{H}(s, T_i)$ and thus on $b^H(\cdot, T_i, T_{i+1})$ itself. In other words, we have to deal with a stochastic differential equation. Suppose that for every $i \in \{1, \ldots, k-1\}$ there is a unique solution to the SDE

$$\begin{aligned} h(t, T_i) &= h(0, T_i) + \int_0^t f^i(s, h(s-, T_i))\,\mathrm{d}s + \int_0^t \sqrt{c_s}\gamma(s, T_i)\,\mathrm{d}W_s^{T_{i+1}} \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \langle \gamma(s, T_i), x\rangle (\mu - \nu^{T_{i+1}})(\mathrm{d}s, \mathrm{d}x) \end{aligned} \tag{39}$$

with

$$h(0, T_i) := \log H(0, T_i)$$

and

$$f^i(s, x) := f_1^i(s) + f_2^i(s, x) + f_3^i(s, x) + f_4^i(s, x),$$

where

$$\begin{aligned} f_1^i(s) &:= \sum_{j=1}^{i-1} \frac{\delta_j e^{h(s-,T_j)}}{1+\delta_j e^{h(s-,T_j)}} \langle \gamma(s, T_j), c_s\gamma(s, T_i)\rangle - \frac{1}{2}\langle \gamma(s, T_i), c_s\gamma(s, T_i)\rangle \\ &\quad - \int_{\mathbb{R}^d} \left(e^{\langle \gamma(s,T_i), y\rangle} - 1 - \langle \gamma(s, T_i), y\rangle \right) F_s^{T_{i+1}}(\mathrm{d}y), \\ f_2^i(s, x) &:= \frac{\delta_i e^x}{1+\delta_i e^x} \langle \gamma(s, T_i), c_s\gamma(s, T_i)\rangle \\ &\quad + \frac{1+\delta_i e^x}{\delta_i e^x} \sum_{j=1}^{i-1} \left(\frac{\delta_j e^{h(s-,T_j)}}{1+\delta_j e^{h(s-,T_j)}} \langle \gamma(s, T_j), c_s\alpha(s, T_i, T_{i+1})\rangle \right) \\ f_3^i(s, x) &:= \frac{1+\delta_i e^x}{\delta_i e^x} \int_{\mathbb{R}^d} (\beta(s, y, T_i, T_{i+1}) - 1) \\ &\quad \left(1 - \prod_{j=1}^{i-1} \left(1 + \frac{\delta_j e^{h(s-,T_j)}}{1+\delta_j e^{h(s-,T_j)}} \left(e^{\langle \gamma(s,T_j), y\rangle} - 1 \right) \right)^{-1} \right) F_s^{T_{i+1}}(\mathrm{d}y), \end{aligned} \tag{40}$$

and

$$\begin{aligned} f_4^i(s, x) &:= \int_{\mathbb{R}^d} \left(e^{\langle \gamma(s,T_i), y\rangle} - 1 \right) \left(1 - \left(1 + \frac{\delta_i e^x}{1+\delta_i e^x} \left(e^{\langle \gamma(s,T_i), y\rangle} - 1 \right) \right)^{-1} \right. \\ &\quad \left. \times \prod_{j=1}^{i-1} \left(1 + \frac{\delta_j e^{h(s-,T_j)}}{1+\delta_j e^{h(s-,T_j)}} \left(e^{\langle \gamma(s,T_j), y\rangle} - 1 \right) \right)^{-1} \right) F_s^{T_{i+1}}(\mathrm{d}y). \end{aligned}$$

Then $\widehat{H}(s, T_i) := \exp h(s, T_i)$ satisfies (15) with drift term $b^H(s, T_i, T_{i+1})$ given by (38). In this case Proposition 13 yields that $\left(\prod_{i=1}^{k-1} \frac{1}{1+\delta_i \widehat{H}(t,T_i)} \right)_{0 \le t \le T_{k-1}}$ is a $\mathbf{Q}_{T_k}$-martingale.

To prove that there is a unique solution to (39) we make use of the following theorem which is a direct consequence of Protter (1992, Theorem V.7) (see also Protter (1992, Theorem V.6)):

Theorem 14 *Assume a (one-dimensional) semimartingale Z with $Z_0 = 0$ on a complete stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P})$ to be given and let $f : \mathbb{R}_+ \times \Omega \times \mathbb{R} \to \mathbb{R}$ be such that*

1. *for fixed $x \in \mathbb{R}$, $(t, \omega) \mapsto f(t, \omega, x)$ is an adapted càglàd process, i.e. it has left-continuous paths that admit right-hand limits.*
2. *there exists a finite random variable K such that for all $t \in \mathbb{R}_+$*

$$|f(t, \omega, x) - f(t, \omega, y)| \leq K(\omega)|x - y|.$$

Then the stochastic differential equation

$$X_t = X_0 + Z_t + \int_0^t f(s, \cdot, X_{s-})\,\mathrm{d}s,$$

where X_0 is a constant, has a unique (strong) solution. This solution is a semimartingale.

Unfortunately, the functions f_2^i and f_3^i in (39) are not globally Lipschitz, i.e. they do not satisfy condition 2 of the previous theorem. However, for the SDE in consideration we can weaken this condition by assuming that f is locally Lipschitz and satisfies a growth condition, as the following proposition shows:

Proposition 15 *Assume we are given a d-dimensional special semimartingale $S := \int_0^\bullet \sqrt{c_s}\,\mathrm{d}W_s + \int_0^\bullet \int_{\mathbb{R}^d} x(\mu - \nu)(\mathrm{d}s, \mathrm{d}x)$ on a complete stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P})$, where W is a standard Brownian motion, c is deterministic, and μ is the random measure associated with the jumps of S with (possibly non-deterministic) compensator $\nu(\mathrm{d}s, \mathrm{d}x) = F_s(\mathrm{d}x)\,\mathrm{d}s$. Suppose that $\sigma : \mathbb{R}_+ \to \mathbb{R}^d$ is a bounded càglàd function and let $f : \mathbb{R}_+ \times \Omega \times \mathbb{R} \to \mathbb{R}$ be such that*

1. *for fixed $x \in \mathbb{R}$, $(t, \omega) \mapsto f(t, \omega, x)$ is an adapted càglàd process.*
2. *for all $r > 0$ there is a real number K_r such that for all (t, ω) and all $x, y \in \mathbb{R}$ with $|x|, |y| \leq r$*

$$|f(t, \omega, x) - f(t, \omega, y)| \leq K_r|x - y| \qquad \textit{and} \qquad |f(t, \omega, x)| \leq K_r.$$

3. *there is a constant B_1 such that for all (t, ω) and all $x \in \mathbb{R}$*

$$xf(t, \omega, x) \leq B_1(1 + x^2).$$

Suppose further that there is a constant B_2 such that for all (t, ω)

$$\langle \sigma(t), c_t \sigma(t) \rangle + \int_{\mathbb{R}^d} \langle \sigma(t), y \rangle^2 F_s(\mathrm{d}y) \leq B_2. \tag{41}$$

Then the stochastic differential equation

$$X_t = X_0 + \int_0^t f(s,\cdot,X_{s-})\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}S_s, \tag{42}$$

where X_0 is a constant, has a unique (non-exploding) solution which is a semimartingale.

PROOF: The proof uses ideas of the proofs of Theorems 2.2.3 and 2.3.3 in Reiss (2003), where a similar statement is established for a deterministic function f and a driving Brownian motion.

Let us show uniqueness of a solution first. Suppose X^1 and X^2 are two solutions. To show that they are indistinguishable, it is enough to show that they are modifications of each other since their paths are right continuous. Fix a $t \in \mathbb{R}_+$ and define for $n \in \mathbb{N}$

$$\tau_n^1 := \inf\{s \geq 0 : |X_s^1| \geq n\}, \qquad \tau_n^2 := \inf\{s \geq 0 : |X_s^2| \geq n\}.$$

Since the usual conditions hold, τ_n^1 and τ_n^2 are stopping times. Consequently, $\tau_n := \min(\tau_n^1, \tau_n^2)$ is a stopping time that converges to infinity almost surely as $n \to \infty$. Hence,

$$\begin{aligned}
\mathbb{E}\left[|X^1_{t\wedge\tau_n} - X^2_{t\wedge\tau_n}|\right] &= \mathbb{E}\left[\left|\int_0^{t\wedge\tau_n} f(s,\cdot,X^1_{s-}) - f(s,\cdot,X^2_{s-})\,\mathrm{d}s\right|\right] \\
&\leq K_n \mathbb{E}\left[\int_0^{t\wedge\tau_n} |X_s^1 - X_s^2|\,\mathrm{d}s\right] \\
&= K_n \int_0^t \mathbb{E}\left[\mathbf{1}_{\{\tau_n \geq s\}} |X_s^1 - X_s^2|\right]\mathrm{d}s \\
&\leq K_n \int_0^t \mathbb{E}\left[|X^1_{s\wedge\tau_n} - X^2_{s\wedge\tau_n}|\right]\mathrm{d}s.
\end{aligned}$$

We can apply Gronwall's Lemma and conclude $\mathbb{E}\left[|X^1_{t\wedge\tau_n} - X^2_{t\wedge\tau_n}|\right] = 0$. Thus, $X^1_{t\wedge\tau_n} = X^2_{t\wedge\tau_n}$ almost surely for all n. Letting $n \to \infty$ yields $X_t^1 = X_t^2$ almost surely.

To prove the existence statement, we use the previous theorem together with a suitable cut-off scheme. For any $R > 0$ define

$$f_R(s,\omega,x) := \begin{cases} f(s,\omega,x) & \text{for } |x| \leq R \\ \left(2 - \frac{x}{R}\right) f(s,\omega,R) & \text{for } r < x < 2R \\ \left(2 + \frac{x}{R}\right) f(s,\omega,-R) & \text{for } -2R < x < R \\ 0 & \text{for } |x| \geq 2R. \end{cases}$$

Then f_R satisfies the conditions of Theorem 14 with $K(\omega) := \max\left(K_R, \frac{K_R}{R}\right) =: \overline{K}_R$. Denote by X^R the (by Theorem 14) unique solution of the SDE

$$\begin{aligned}
X_t &= X_0 + \int_0^t f_R(s,\cdot,X_{s-})\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}S_s \\
&= X_0 + \int_0^t f_R(s,\cdot,X_{s-})\,\mathrm{d}s + \int_0^t \sqrt{c_s}\sigma(s)\,\mathrm{d}W_s + \int_0^t \int_{\mathbb{R}^d} \langle\sigma(s),x\rangle(\mu-\nu)(\mathrm{d}s,\mathrm{d}x).
\end{aligned}$$

Introduce the stopping time $\tau_R := \inf\{t \geq 0 : |X_t^R| \geq R\}$ and define

$$X_t^\infty := X_t^R \qquad \text{for } t \leq \tau_R.$$

To check that X^∞ is well defined let $0 < R_1 < R_2$ and $\tau := \min(\tau_{R_1}, \tau_{R_2})$, then (similarly as in the proof of uniqueness)

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq s\leq t\wedge\tau} |X_s^{R_1} - X_s^{R_2}|\right] &= \mathbb{E}\left[\sup_{0\leq s\leq t\wedge\tau} \left|\int_0^s f_{R_1}(u,\cdot,X_{u-}^{R_1}) - f_{R_2}(u,\cdot,X_{u-}^{R_2})\,\mathrm{d}u\right|\right] \\
&\leq \mathbb{E}\left[\int_0^{t\wedge\tau} |f_{R_1}(u,\cdot,X_{u-}^{R_1}) - f_{R_2}(u,\cdot,X_{u-}^{R_2})|\,\mathrm{d}u\right] \\
&\leq \overline{K}_{R_2}\mathbb{E}\left[\int_0^{t\wedge\tau} |X_u^{R_1} - X_u^{R_2}|\,\mathrm{d}u\right] \\
&\leq \overline{K}_{R_2}\int_0^t \mathbb{E}\left[\sup_{0\leq s\leq u\wedge\tau} |X_s^{R_1} - X_s^{R_2}|\right]\mathrm{d}u.
\end{aligned}$$

Again, we can apply Gronwall's Lemma and conclude

$$\mathbb{E}\left[\sup_{0\leq s\leq t\wedge\tau} |X_s^{R_1} - X_s^{R_2}|\right] = 0 \qquad \text{for all } t.$$

Hence, $X_t^{R_1}$ and $X_t^{R_2}$ coincide almost surely for $t \leq \min(\tau_{R_1}, \tau_{R_2})$ and X^∞ is well defined. It remains to show that $\lim_{R\to\infty} \tau_R = \infty$ almost surely, since in this case X^∞ is a solution to (42) and therefore a semimartingale.

Let $h(x,t) := e^{-Bt}(1+x^2)$ with $B := 2B_1 + B_2$ and $Y_t^R := h(X_t^R, t)$, then by Itô's formula

$$\begin{aligned}
Y_t^R - Y_0^R &= e^{-Bt}(1 + (X_t^R)^2) - (1 + (X_0)^2) \\
&= \int_0^t 2X_{s-}^R e^{-Bs}\,\mathrm{d}X_s^R - B\int_0^t e^{-Bs}\big(1 + (X_{s-}^R)^2\big)\,\mathrm{d}s \\
&\quad + \int_0^t e^{-Bs}\langle\sigma(s), c_s\sigma(s)\rangle\,\mathrm{d}s + \int_0^t\int_{\mathbb{R}^d} e^{-Bs}\langle\sigma(s), x\rangle^2\mu(\mathrm{d}s,\mathrm{d}x) \\
&= \text{local martingale} \\
&\quad + \int_0^t e^{-Bs}\Big(2X_{s-}^R f_R(s,\cdot,X_{s-}^R) - B\big(1 + (X_{s-}^R)^2\big) \\
&\qquad\qquad + \langle\sigma(s), c_s\sigma(s)\rangle + \int_{\mathbb{R}^d}\langle\sigma(s), x\rangle^2 F_s(\mathrm{d}x)\Big)\,\mathrm{d}s.
\end{aligned}$$

From condition 3 of the prerequisites and (41) we know that there is a localizing sequence $(T_n)_{n\geq1}$ such that the stopped process $(Y^R)^{T_n}$ is a supermartingale for all n. By the optional stopping theorem, the stopped process $(Y^R)^{T_n\wedge\tau_R}$ is a supermartingale for all n. Hence,

$$\begin{aligned}
1 + (X_0)^2 &\geq \mathbb{E}\left[e^{-B(t\wedge T_n\wedge\tau_R)}\left(1 + (X_{t\wedge T_n\wedge\tau_R}^R)^2\right)\right] \\
&\geq e^{-Bt}(1 + R^2)\mathbf{P}(\{\tau_R \leq t\} \cap \{\tau_R \leq T_n\}).
\end{aligned}$$

Taking the limes inferior (over n) on both sides and using Fatou's Lemma we obtain

$$1 + (X_0)^2 \geq e^{-Bt}(1 + R^2)\mathbf{P}(\{\tau_R \leq t\}).$$

From $\lim_{R\to\infty} 1 + R^2 = \infty$ we get $\lim_{R\to\infty} \mathbf{P}(\{\tau_R \leq t\}) = 0$. Since for $R_1 < R_2$ we have $\{\tau_{R_2} \leq t\} \subset \{\tau_{R_1} \leq t\}$, there exists for $\mathbf{P}$-almost every ω and all $t > 0$ a constant R_0 (which may depend on ω and t) such that $\tau_R(\omega) \geq t$ for all $R \geq R_0$. This is equivalent to $\lim_{R\to\infty} \tau_R = \infty$ almost surely. □

We can use the previous proposition to check that, at least in case the driving process L is one-dimensional (d=1), the SDE (39) admits a unique non-exploding solution:

Proposition 16 *Assume $d = 1$. Suppose that the characteristics of L^{T^*} are chosen in such a way that $f_1^i(\cdot,\cdot,x), \ldots, f_4^i(\cdot,\cdot,x)$ have càglàd paths for each $x \in \mathbb{R}$. Then the stochastic differential equation (39) admits a unique (non-exploding) solution for each $i \in \{1, \ldots, n-1\}$.*

Proof: We use Proposition 15 with

$$\begin{aligned} (\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P}) &:= (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{F}}, \mathbf{P}_{T_{i+1}}), \\ S_t &:= \int_0^t \sqrt{c_s}\, \mathrm{d}W_s^{T_{i+1}} + \int_0^t \int_{\mathbb{R}} x(\mu - \nu^{T_{i+1}})(\mathrm{d}s, \mathrm{d}x), \end{aligned}$$

and $\sigma(s) := \gamma(s, T_i)$. Assumptions (1), (2), and ($\mathbb{DLR}$.1) imply (41). It remains to verify the conditions 1-3 of proposition 15 for f^i. Condition 1 is satisfied by assumption. Conditions 2 and 3 can be checked separately for $f_1^i, \ldots, f_4^i$. Again, (1), (2), and ($\mathbb{DLR}$.1) yield that condition 2 holds for $f_1^i, \ldots, f_4^i$ and that condition 3 holds for f_1^i and f_4^i. It remains to show that condition 3 is also satisfied for f_2^i and f_3^i. For this purpose, it is sufficient to prove that there are constants C_2, C_3 such that for all (t, ω) and all $x \in \mathbb{R}$

$$0 \leq \widetilde{f}_j^i(t, \omega, x) \leq C_j \qquad (j \in \{2, 3\}),$$

where $\widetilde{f}_j^i(t, \omega, x) := \frac{\delta_i e^x}{1+\delta_i e^x} f_j^i(t, \omega, x)$. The existence of the upper bound once again follows from (1), (2), and ($\mathbb{DLR}$.1). Moreover, $\widetilde{f}_2^i$ and $\widetilde{f}_3^i$ are nonnegative since $\alpha(\cdot, T_i, T_{i+1})$ and the integrand in (40) are nonnegative (at this point, the assumption $d = 1$ is needed). □

Remark: To prove the existence of a solution to (39) for $d > 1$ we have to put further restrictions on the characteristics of L to meet the growth condition (condition 3) of proposition 15. For example, in the case of a multivariate Gaussian model (i.e. $F_s = 0$ for all $s \in [0, T^*]$) assuming that $\langle \gamma(s, T_j), c_s \lambda(s, T_i) \rangle$ is nonnegative for all $1 \leq j < i \leq n-1$ will do the job (since it implies that $\widetilde{f}_2^i$ is nonnegative and the proof of Proposition 16 therefore applies). The easiest way to satisfy this assumption is to choose all entries in the matrices c_s as nonnegative numbers.

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